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ERRATUM.

At page 98 of the Volume for 1855, in Archdeacon Pratt's paper "On the Attraction of the Himalayas on the Plumb-Line in India," lines 7 and 5 from the bottom, *read*

$$\alpha = -0{\cdot}0019208 + 0{\cdot}0059576\,u - 0{\cdot}0014564\,v$$

and

$$= a(0{\cdot}9980797 + 0{\cdot}0059576\,u - 0{\cdot}0014564\,v).$$

This correction affects none of the results of the paper.

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Adjudication of the Medals of the Royal Society for the year 1857 by the President and Council.

The Copley Medal to Michel Eugène Chevreul, for his Researches in Organic Chemistry, particularly on the Composition of the Fats, and for Researches on the Contrast of Colours.

A Royal Medal to Dr. Edward Frankland, for the Isolation of the Organic Radicals of the Alcohols, and for his Researches on the Metallic Derivatives of Alcohol.

A Royal Medal to Dr. John Lindley, for his numerous Researches and Works on all branches of Scientific Botany, and especially for his "Vegetable Kingdom" and his "Genera and Species of Orchideæ."

The Bakerian Lecture was delivered by Michael Faraday, Esq., D.C.L., F.R.S., and entitled "Experimental Relations of Gold (and other Metals) to Light."

The Croonian Lecture was delivered by James Paget, Esq., F.R.S., and entitled "On the Cause of the Rhythmic Motion of the Heart."

ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the Council-books and Journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgement of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the

thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

The Meteorological Journal hitherto kept by the Assistant Secretary at the Apartments of the Royal Society, by order of the President and Council, and published in the Philosophical Transactions, has been discontinued. The Government, on the recommendation of the President and Council, has established at the Royal Observatory at Greenwich, under the superintendence of the Astronomer Royal, a Magnetical and Meteorological Observatory, where observations are made on an extended scale, which are regularly published. These, which correspond with the grand scheme of observations now carrying out in different parts of the globe, supersede the necessity of a continuance of the observations made at the Apartments of the Royal Society, which could not be rendered so perfect as was desirable, on account of the imperfections of the locality and the multiplied duties of the observer.

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The *fifty* Foreign Members of the Royal Society.

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PHILOSOPHICAL TRANSACTIONS.

I. *On the Evidence of the existence of the Decennial Inequality in the Solar-diurnal Magnetic Variations, and its non-existence in the Lunar-diurnal Variation, of the Declination at Hobarton. By Major-General* EDWARD SABINE, *of the Royal Artillery, Treas. and V.P. of the Royal Society.*

Received November 17,—Read December 18, 1856.

IN a report presented to the British Association at Liverpool in September 1854, entitled "On some of the results obtained at the British Colonial Magnetic Observatories," I stated that, as far as my examination of the observations had then gone, I had found in the Lunar-diurnal magnetic variation no trace of the *decennial period* which is so distinctly marked in all the variations connected with the Sun. And in a subsequent communication to the Royal Society in June 1856, "On the Lunar-diurnal Variation at Toronto," in which the moon's influence on each of the three magnetic elements was examined, the conclusion arrived at was to the same effect, viz. that the observations at Toronto "showed no appearance of the decennial period which constitutes so marked a feature in the solar-diurnal variations."

Since these statements were made, I have read M. KREIL's memoir "On the Influence of the Moon on the horizontal component of the Magnetic Force," presented to the Imperial Academy of Sciences at Vienna in 1852 and printed in 1853, from which I learn (pp. 45, 46) that M. KREIL is of opinion that the observations of different years at Milan and Prague, when combined, would rather favour the supposition that the same decennial period which exists in the solar variation affects also the lunar magnetic influence. The question is one of such manifest importance in its theoretical bearing, that I considered it desirable to lose no time in re-examining it by the aid of the Hobarton observations, which, as it appeared to me, were particularly suitable for the purpose, inasmuch as they consist of eight consecutive years of hourly observation (from January 1841 to December 1848 inclusive), made with one and the same set of instruments, and by a uniform system of observation. The results of this examination have been, as it appears to me, so decidedly confirmatory of the conclusions drawn from the Toronto observations, both as regards the existence of the decennial period in the

two classes of solar-diurnal variation (viz. in the mean diurnal variation occasioned by the disturbances of large amount, and in what may be termed the more regular solar-diurnal variation), and the non-existence of a similar decennial period in the case of the lunar-diurnal variation, that I have been induced to make these results the subject of a communication to the Royal Society.

The observations at Hobarton comprise, so far as I am aware, the most extensive consecutive series of magnetic observations that have hitherto been made. The *hourly* system was adopted there from the first commencement of the Observatory. We owe it to the zeal and clear-sightedness of Captain (now Admiral) Sir James Clark Ross, by whom the Hobarton Observatory was established, that this improvement upon the two-hourly system proposed by the Royal Society was introduced. Its superiority for many of the contemplated objects of the Observatories was speedily recognized, and the example followed at the Ordnance Colonial Observatories, by the employment of one additional non-commissioned officer at each, thus enabling the number of daily observations to be doubled at the small extra expense of about £24 a-year for each observatory. The duties of "Observer," which at the Ordnance establishments were performed by non-commissioned officers of the Royal Artillery, devolved at Hobarton upon the officers of the Royal Navy who formed the *personnel* of that Observatory: and I gladly avail myself of this opportunity of placing on record, that to the steady and unremitting zeal of Commander Joseph Henry Kay during the whole period of eight years, of Lieutenants Peter Scott and Joseph Dayman for the first four years, and of Lieutenants Alexander Smith and Francis Simpkinson during the last four years, are primarily due whatever results may now or hereafter be derived from the Hobarton observations*.

The observations of each month, arranged in the same forms as those of the Ordnance establishments, were transmitted by the Admiralty to my office at Woolwich, and have been printed. For the purpose of this examination they have been treated precisely as those of Toronto, as described in the Philosophical Transactions for 1856, Art. XVI. and Art. XXII. The aggregate number of the observations of the Declination at Hobarton which have been passed through those processes of reduction is 56,202; of these, 4207 were found to differ from the monthly normal at the same hour by an amount which equalled or exceeded four scale-divisions, or $2'{\cdot}84$ in arc value, and were accordingly separated from the rest, for the purpose of deducing from them the periodical laws of the class of phenomena to which they belong. The monthly normals, derived from the great body of the observations after the exclusion of the disturbances of four scale-divisions and upwards, have been employed to show the variation in amount in the different years of the mean diurnal solar influence, forming what is generally understood as the regular solar-diurnal variation; and the same observations, 51,995 in number, rearranged according to the lunar hours to which they respectively most nearly approximate, have supplied the mean lunar-diurnal variation in the different years treated of in this communication.

* In mentioning the names of the observers, that of Mr. Jeffrey, for several years assistant to Captain Kay, should not be omitted.

Disturbance-variation.—Before, however, we enter on the lunar question, I wish to show the accordance of the two classes of the solar-diurnal phenomena at Hobarton with the decennial period, as it has been inferred from observations in other parts of the globe; and I shall commence with the *disturbance-variation*, because it was from this branch of the phenomena that the decennial connexion between the solar-magnetic variations and those of the solar spots was first inferred; and because, on account of the labour which is required in the investigation, the periodical laws of the disturbances have hitherto been made out at no other observatories than those of the British Colonies. I gladly avail myself therefore of the opportunity of showing, that when the disturbances, occurring at Hobarton during eight years, are broken into four distinct and equal portions, each of two years' duration, each such portion manifests, under a suitable mode of investigation, the same periodical law of diurnal variation, almost identical in the principal features of direction and turning hours, and differing only in the magnitude of the variation in different years, in which difference it conforms strictly to the decennial period as indicated elsewhere, having a minimum in 1843—1844, and a maximum five years later.

As the existence of a law of diurnal variation regulating the occurrence of the disturbances of large amount rests hitherto, I believe, only on investigations made by myself, I may be excused for here calling attention to the fact, that each of the four portions, into which the disturbances at Hobarton have been divided, bears its testimony to the existence of this law;—which may be accounted a general one, since it has been found to prevail at stations so widely distant from each other as Toronto, St. Helena, and Hobarton.

The following Table exhibits, for each of the two-yearly periods, the excess at the several hours of mean solar time at the station, of the aggregate easterly differences from the monthly normal at the same hour over the aggregate westerly differences (or westerly over easterly as the case may be), divided by the number of days of observation in each two-yearly period: it is therefore strictly a table of the mean diurnal variation due to the disturbances that have been separated for the purpose of investigating the periodical laws of this class of the phenomena.

TABLE I.

Hours.	1841 and 1842.	1843 and 1844.	1845 and 1846.	1847 and 1848.	Hours.	1841 and 1842.	1843 and 1844.	1845 and 1846.	1847 and 1848.
Noon.	0·00	+0·03	+0·10	+0·09	12	−0·53	−0·38	−0·45	−0·65
1	+0·10	+0·11	+0·08	+0·03	13	−0·30	−0·23	−0·39	−0·54
2	0·00	+0·11	+0·03	−0·02	14	−0·20	−0·14	−0·35	−0·36
3	+0·05	+0·12	+0·07	+0·20	15	−0·17	−0·05	−0·19	−0·33
4	+0·15	+0·06	+0·07	+0·23	16	+0·05	+0·03	−0·05	−0·28
5	+0·13	+0·03	+0·03	+0·32	17	+0·08	+0·09	+0·01	+0·03
6	−0·04	−0·05	−0·05	+0·20	18	+0·13	+0·09	+0·04	+0·10
7	−0·17	−0·17	−0·11	−0·10	19	+0·20	+0·09	+0·17	+0·22
8	−0·43	−0·30	−0·22	−0·26	20	+0·27	+0·11	+0·10	+0·23
9	−0·42	−0·43	−0·37	−0·39	21	+0·14	+0·06	+0·15	+0·10
10	−0·55	−0·36	−0·41	−0·64	22	+0·05	+0·06	+0·09	+0·19
11	−0·59	−0·43	−0·46	−0·82	23	+0·01	+0·05	+0·12	+0·13

The values are decimals of a minute of arc. The sign + corresponds to Easterly and − to Westerly disturbance.

These variations may be represented by the following formulæ:—

1841—1842 . . $\Delta_x = -0'\cdot085 - 0'\cdot299 \sin(a + 294°\cdot7) + 0'\cdot178 \sin(2a + 298°\cdot4)$

1843—1844 . . $\Delta_x = -0\cdot062 - 0\cdot231 \sin(a + 294\cdot1) + 0\cdot126 \sin(2a + 319\cdot7)$

1845—1846 . . $\Delta_x = -0\cdot083 - 0\cdot273 \sin(a + 283\cdot9) + 0\cdot107 \sin(2a + 280\cdot2)$

1847—1848 . . $\Delta_x = -0\cdot097 - 0\cdot447 \sin(a + 300\cdot4) + 0\cdot227 \sin(2a + 278\cdot6)$

Δ being the variation at x, the hour required, and a (corresponding to x) counted in hours from noon and multiplied by 15°.

The curves computed by these formulæ are shown to the eye in seconds of arc in Plate I. fig. 1. It will at once be seen how thoroughly their inflections accord in character, and at the same time how decidedly and systematically they differ in amount. The inflections are least in the curve derived from the years 1843—1844, and greatest in that derived from the years 1847—1848. The hour of greatest westerly deflection is about 11 P.M., and that of greatest easterly deflection about 7 A.M. The difference between the extreme deflections at these hours, or the range of the variation in twenty-four hours in seconds of arc, is given by the formulæ as follows:—

	1841—1842.	1843—1844.	1845—1846.	1847—1848.
Easterly extreme	12″·0	6″·8	8″·8	23″·7
Westerly extreme	33·5	24·7	27·5	44·2
Sum	45·5	31·5	36·3	67·9

Taking the differences between the actual results of the observations at the same hours, or the range of the variation in twenty-four hours, from direct observation, we have as follows:—

	1841—1842.	1843—1844.	1845—1846.	1847—1848.
Easterly extreme	12″·0	5″·4	11″·2	13″·2
Westerly extreme	35·4	25·8	27·6	49·2
Sum	47·4	31·2	38·8	62·4

Thus it will be seen, that if we estimate the variation in the mean effect of the disturbances, in the course of the decennial period, by the difference in their mean diurnal range respectively in 1843—1844 and 1847—1848, we should infer that their mean effect is at least twice as great in the year of maximum as in the year of minimum.

It is probable that if either a higher or a lower arbitrary standard than four scale-divisions were taken as constituting a "large disturbance," the inequality between the years of maximum and minimum might be rendered somewhat greater or somewhat less, but still not materially so; the result is of too decided a character to doubt the evidence it affords of a subsisting substantial difference between the years of maximum and minimum.

Direct Solar-diurnal Variation.—Table II. exhibits the mean solar-diurnal variation for the same four periods, each of two years' duration, derived from the observations after the exclusion of the larger disturbances. The sign + signifies Easterly and — Westerly differences from the mean or normal direction in the twenty-four hours.

TABLE II.

Hours.	1841 and 1842.	1843 and 1844.	1845 and 1846.	1847 and 1848.	Hours.	1841 and 1842.	1843 and 1844.	1845 and 1846.	1847 and 1848.
Noon.	+1·61	+1·29	+1·20	+1·00	12	−1·01	−0·87	−1·04	−0·89
1	+3·46	+3·19	+3·42	+3·79	13	−0·77	−0·68	−0·86	−0·65
2	+4·42	+4·03	+4·54	+5·25	14	−0·56	−0·41	−0·62	−0·49
3	+4·10	+3·85	+4·46	+5·01	15	−0·33	−0·26	−0·42	−0·36
4	+3·04	+2·96	+3·44	+3·83	16	−0·41	−0·33	−0·46	−0·43
5	+1·82	+1·82	+2·08	+2·23	17	−0·76	−0·59	−0·86	−0·95
6	+0·99	+0·98	+1·11	+1·37	18	−1·30	−1·07	−1·26	−1·40
7	+0·36	+0·40	+0·46	+0·80	19	−2·13	−1·85	−1·83	−2·53
8	−0·16	−0·11	−0·07	+0·27	20	−3·04	−2·88	−3·00	−3·74
9	−0·72	−0·55	−0·52	−0·25	21	−3·36	−3·29	−3·61	−4·51
10	−1·03	−0·92	−0·91	−0·64	22	−2·46	−2·71	−2·92	−3·92
11	−1·11	−0·95	−1·09	−0·90	23	−0·63	−0·98	−1·19	−1·91

In all the four periods the greatest easterly deflection occurs at 2 P.M., and the greatest westerly at 9 A.M. The extreme differences are,—

	1841—1842.	1843—1844.	1845—1846.	1847—1848.
2 P.M. Easterly	4·42	4·03	4·54	5·25
9 A.M. Westerly	3·36	3·29	3·61	4·51
Sums . . .	7·78	7·32	8·15	9·76

Here also, if we estimate the decennial variation by the difference in the range of the diurnal movement, we find 1843—1844 the years of minimum and 1847—1848 those of maximum; although the inequality between the extremes is not so great as in the disturbance-variation.

We obtain this result from the eight years in which the observations were made hourly; but we have also observations in the six following years made with the same instruments at the hours of 2 P.M. and 6 A.M., and we are thus enabled to trace the decennial inequality through a period of fourteen consecutive years. In so doing we have to employ throughout (*i. e.* in the eight years as in the six years) the mean positions at 2 P.M. and 6 A.M. only. The range in the different years is here strictly comparative, but it is not of course the extreme range. Table III. shows these particulars.

TABLE III. Hobarton Declination.

Years.	Mean scale reading at		Differences.—2 P.M.—6 A.M.	
	2 P.M.	6 A.M.	Scale-divisions.	Arc.
1841	61·5	53·6	7·9	5·61
1842	83·3	75·9	7·4	5·25
1843	83·8	76·4	7·4	5·25
1844	81·6	74·4	7·2	5·11
1845	85·0	76·5	8·5	6·03
1846	87·1	79·2	7·9	5·61
1847	90·0	79·7	10·3	7·31
1848	91·5	80·3	11·2	7·95
1849	91·8	81·6	10·2	7·24
1850	91·1	82·3	8·8	6·25
1851	90·7	81·9	8·8	6·25
1852	90·3	80·7	9·6	6·82
1853	81·5	74·0	7·5	5·32
1854	76·8	68·4	8·4	5·96

This Table affords a striking illustration of the systematic character of the decennial inequality; and with the evidence previously adduced in reference to the mean diurnal effect of the greater magnetic disturbances in different years, shows that in both classes of solar-diurnal variation, (the daily and constant, and that due to disturbances of occasional occurrence,) the decennial inequality is as distinctly and decidedly marked in the Hobarton observations as it has been found to be elsewhere.

Lunar-diurnal Variation.—Table IV. exhibits the lunar-diurnal variation in the four periods, each of two years' duration, obtained from the 51,995 observations remaining after the separation of the disturbances of large amount, rearranged according to lunar hours and treated in the manner described in the Philosophical Transactions for 1856, Art. XXII. The sign + signifies Easterly, and the sign — Westerly deflections from the normal direction.

TABLE IV.

One scale-division = 0′·71.

Lunar hours.	1841 and 1842.	1843 and 1844.	1845 and 1846.	1847 and 1848.	Lunar hours.	1841 and 1842.	1843 and 1844.	1845 and 1846.	1847 and 1848.
	Sc. div.	Sc. div.	Sc. div.	Sc. div.		Sc. div.	Sc. div.	Sc. div.	Sc. div.
0	+0·14	+0·18	+0·12	+0·27	12	+0·14	+0·15	+0·16	+0·02
1	+0·08	+0·24	+0·14	+0·24	13	+0·26	+0·27	+0·25	+0·15
2	+0·07	+0·16	+0·25	+0·23	14	+0·19	+0·28	+0·25	+0·09
3	−0·06	+0·21	+0·28	+0·10	15	+0·13	+0·22	+0·14	−0·02
4	−0·12	+0·13	+0·26	+0·08	16	+0·09	+0·14	+0·05	−0·01
5	−0·14	−0·09	+0·11	−0·09	17	+0·08	−0·08	−0·09	−0·10
6	−0·27	−0·14	−0·04	−0·13	18	−0·14	−0·17	−0·19	−0·15
7	−0·20	−0·22	−0·04	−0·18	19	−0·13	−0·25	−0·30	−0·26
8	−0·16	−0·25	−0·13	−0·06	20	−0·12	−0·27	−0·40	−0·15
9	−0·06	−0·09	−0·09	−0·06	21	−0·03	−0·23	−0·30	−0·15
10	−0·04	−0·03	−0·06	+0·01	22	+0·02	−0·01	−0·24	+0·09
11	−0·01	+0·11	+0·07	+0·08	23	+0·08	+0·03	−0·05	+0·18

The lunar-diurnal variation for these periods may therefore be respectively represented by the following formulæ, in which the coefficients are expressed in seconds of arc, and a is reckoned in hours (multiplied by 15°) from the epoch of the moon's superior culmination.

1841—1842 . . . $\Delta_x = -0''\cdot35 - 3''\cdot61 \sin(a + 21^h\cdot8) + 6''\cdot69 \sin(2a + 61^h\cdot8)$.
1843—1844 . . . $\Delta_x = +0\cdot51 + 1\cdot08 \sin(a + 284\cdot4) + 10\cdot7 \sin(2a + 43\cdot6)$.
1845—1846 . . . $\Delta_x = +0\cdot26 + 5\cdot14 \sin(a + 339\cdot2) + 10\cdot4 \sin(2a + 27\cdot6)$.
1847—1848 . . . $\Delta_x = +0\cdot32 + 2\cdot7 \sin(a + 55\cdot1) + 7\cdot7 \sin(2a + 65\cdot0)$.
Mean of the eight years $\Delta_x = +0\cdot25 + 0\cdot98 \sin(a + 318\cdot4) + 8\cdot75 \sin(2a + 45\cdot8)$.

We may at once perceive, from the relative magnitude of the coefficient of principal significance (that of sin $2a$), that the differences in the lunar-diurnal variation corresponding to different years show no conformity to the inequality manifested in those of the solar-diurnal variations.

The curves severally corresponding to these formulæ are shown in Plate I. fig. 2. for each of the four two-yearly periods. They exhibit such differences as are to be expected from periods of such limited duration; but the differences present no appearance whatsoever of a systematic variation corresponding to the decennial inequality of the solar-diurnal variations. There are in each curve four extremes nearly equidistant from each other, two of which are easterly and two westerly extremes: if in each period we take the sum of the extremes, not indeed as an exact measure, but as furnishing an approximate indication of the relative magnitudes of the respective variations, the periods arrange themselves in the following order:—

1. 1843—1844. Sum of extremes . . . . . 41''·8
2. 1845—1846. Sum of extremes . . . . . 41·6
3. 1847—1848. Sum of extremes . . . . . 30·6
4. 1841—1842. Sum of extremes . . . . . 28·1

If we should prefer as a criterion of the magnitude of the variation, the sum of all the deflections at the different hours in the respective periods, we should have the succession as follows, taking the deflections as calculated by the formulæ:—

1. 1845—1846. Sum of deflections . . . . 169''·9
2. 1843—1844. Sum of deflections . . . . 165·5
3. 1847—1848. Sum of deflections . . . . 120·5
4. 1841—1842. Sum of deflections . . . . 113·4

Or taking the deflections as actually observed at the different hours:—

1. 1845—1846. Sum of deflections . . . . 170''·8
2. 1843—1844. Sum of deflections . . . . 168·3
3. 1847—1848. Sum of deflections . . . . 123·5
4. 1841—1842. Sum of deflections . . . . 117·6

In both classes of *solar* variations the years 1841, 1846, 1847 and 1848 are the years of greatest range, and the years 1842, 1843, 1844 and 1845 those of least range. If we unite in one curve the *lunar* results in the four first-named years, and in a second curve the results in the four last-named years, and compare these two curves, we are so far from finding in the lunar variation an inequality corresponding to that in the solar variation, that the difference between the curves, such as it is, is in fact in the opposite direction. These curves are drawn in Plate I. fig. 3, and are there shown in comparison with a third curve, which is that of the mean lunar-diurnal variation in the entire period of eight years.

The Tables I. to IV. were prepared, under Mr. MAGRATH's superintendence, by the Non-Commissioned Officers of the Royal Artillery employed in my office, each entry having been verified by two independent computers. The observations themselves are printed in the first three volumes of the 'Hobarton Observations.'

II. *On the Stability of Loose Earth.* *By* W. J. Macquorn Rankine, *F.R.S.*

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§ 1. *General Principle.*

The subject of this paper is,—the mathematical theory of that kind of stability, which, in a mass composed of separate grains, arises wholly from the mutual friction of those grains, and not from any adhesion amongst them.

Previous researches on this subject are based (so far as I am acquainted with them) on some mathematical artifice or assumption, such as Coulomb's "wedge of least resistance." Researches so based, although leading to true solutions of many special problems, are both limited in the application of their results, and unsatisfactory in a scientific point of view. I propose, therefore, to investigate the mathematical theory of the frictional stability of a granular mass, without the aid of any artifice or assumption, and from the following sole

Principle.

The resistance to displacement by sliding along a given plane in a loose granular mass, is equal to the normal pressure exerted between the parts of the mass on either side of that plane, multiplied by a specific constant.

The specific constant is the *coefficient of friction* of the mass, and is regarded as the tangent of an angle called the *angle of repose*. Let P denote the normal pressure per unit of area of the plane in question; F the resistance to sliding (per unit of area also); ϕ the angle of repose; then the symbolical expression of the above principle is as follows:—

$$\frac{F}{P} = \tan \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

This principle forms the basis of every investigation of the stability of earth. The peculiarity of the present investigation consists in its deducing the laws of that stability from the above principle alone, without the aid of any other special principle. It will in some instances be necessary to refer to Mr. Moseley's "Principle of the Least Resistance;" but this must be regarded not a special principle, but as a general principle of statics.

§ 2. *Corollary as to Limit of Obliquity of Pressure.*

It is necessary to the stability of a granular mass, that the direction of the pressure between the portions into which it is divided by any plane should not at any point make with the normal to the plane an angle exceeding the angle of repose.

That is to say, symbolically, let R be the total pressure, per unit of area, at any point of the given plane, making with the normal to the plane the angle of obliquity θ; let P be the normal and Q the tangential component of R; so that

$$P = R\cos\theta; \qquad Q = R\sin\theta;$$

$$\frac{Q}{P} = \tan\theta;$$

then it is necessary to stability that

$$Q \leq F = P\tan\phi, \quad \ldots\ldots\ldots\

Then the magnitude and direction of the pressure exerted at the plane normal to On, are given by the following equations:—

$$\left.\begin{aligned} R_n &= \sqrt{M^2 + D^2 + 2MD\cos 2\psi} \\ \tan\theta &= \frac{D\sin 2\psi}{M + D\cos 2\psi}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (5.)$$

or otherwise by the following:—

$$\left.\begin{aligned} P_n &= M + D\cos 2\psi \\ Q_n &= D\sin 2\psi. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (6.)$$

The *maximum* value Θ of the obliquity θ, and the corresponding position of the normal On, are given by the following equations:—

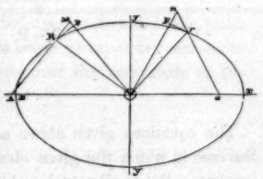
$$\left.\begin{aligned} \Psi &= \frac{\pi}{4} + \frac{1}{2}\sin^{-1}\frac{D}{M} \\ \Theta &= \sin^{-1}\cdot\frac{D}{M}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (7.)$$

to which correspond the following pressures, total, normal, and tangential:—

$$\left.\begin{aligned} R(\Psi) &= \sqrt{M^2 - D^2} \\ P(\Psi) &= M\left(1 - \frac{D^2}{M^2}\right) \\ Q(\Psi) &= D\sqrt{1 - \frac{D^2}{M^2}}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (8.)$$

The following geometrical construction represents the theorems expressed symbolically by the equations (5.), (6.), (7.), (8.) (fig. 1). Let xOx, yOy be the axes of greatest and least pressure at the point O. Given the half-sum M, and half-difference D, of those pressures, it is required to find:—

Fig. 1.



First, the direction and magnitude of the pressure at a plane normal to On, which line is in the plane xy, and makes with the axis of greatest pressure the given angle $xOn = \psi$. Make $On = M$. On Ox take the point a so that $na = On$. On na take $nr = D$. Then will $Or = R_n$ represent the pressure required, and $rOn = \theta$ will be its angle of obliquity.

Also, let fall $rp \perp On$; then $Op = P_n$, $pr = Q_n$.

Secondly, to find the plane for which the obliquity θ is greatest. Make $xON = \Psi =$ half a right angle + half the angle whose sine is $\frac{D}{M}$. Then ON will be the normal to a plane having the required property. There is obviously a pair of such planes, whose normals make equal angles at either side of Ox.

The remainder of the construction is to be proceeded with as before, to find the total

pressure $\overline{OR}$, and its normal and tangential components $\overline{OP}$, $\overline{PR}$. It is obvious that ORN is a right angle.

The locus of the points r is an ellipse, whose semiaxes $Ox=P_x$, $Oy=P_y$ represent the greatest and least pressures at the point O.

§ 4. *Additional Lemmata, as to the Transformation of Stress in the plane of greatest and least pressure.*

By the plane of greatest and least pressure at a given point O, is meant the plane containing the axes Ox and Oy of greatest and least pressure. Let there be taken any two new rectangular axes in that plane, Ox′ and Oy′, and let

$$\angle xOx' = \angle yOy' = \psi, \text{ so that } \angle xOy' = \frac{\pi}{2} + \psi.$$

Also let $P_{x'}$ and $P_{y'}$ be the normal pressures at planes perpendicular to the axes Ox′, Oy′, respectively, and Q′ the tangential pressure on either of those planes. Then from equations (6.) it appears that

$$\left.\begin{aligned} P_{x'} &= M + D\cos 2\psi \\ P_{y'} &= M - D\cos 2\psi \\ Q' &= D\sin 2\psi. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9.)$$

Consequently, if the elementary stresses $P_{x'}$, $P_{y'}$, Q′, at any pair of planes at right angles to each other and to the plane of greatest and least pressure be given, the greatest and least pressures, and the positions of their axes, are given by the following equations:—

$$\left.\begin{aligned} \frac{P_x + P_y}{2} &= M = \frac{P_{x'} + P_{y'}}{2} \\ \frac{P_x - P_y}{2} &= D = \sqrt{\left\{\frac{(P_{x'} - P_{y'})^2}{4} + Q'^2\right\}} \\ \tan 2\psi &= \frac{2Q'}{P_{x'} - P_{y'}}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (10.)$$

The equations given above solve a particular case only of the general problem, viz. the case in which the given elementary stresses act in the plane of greatest and least pressure. But in all actual problems respecting the stability of earth, the plane of greatest and least stress is known; and it is therefore unnecessary to apply to that subject the general problem as to the finding of the axes of pressure in space of three dimensions; a problem which requires the solution of a cubic equation.

§ 5. *Lemmata as to Conjugate Planes and Pressures.*

It is a well-known theorem in the theory of the elasticity of solids, that if the pressure on a given plane at a given point be parallel to a second plane, the pressure on the second plane at the same point must be parallel to the first plane. Such planes are said to be *conjugate* to each other, with respect to the pressures on them; the pressures also are said to be conjugate.

To adapt this theorem to the present question, the first step is to transform the equations (5.), so as to make the obliquity of the pressure, θ, the given angle, instead of the angle of direction ψ of the normal to the plane. Thus are obtained the following equations, from which, when the greatest and least pressures at a point are given, there may be found the position of a plane perpendicular to the plane of greatest and least pressure, on which the obliquity of the pressure shall be equal to a given angle θ; and also the amount R_θ of the pressure corresponding to such obliquity.

$$\left.\begin{aligned} R_\theta &= M\cos\theta \mp \sqrt{D^2 - M^2\sin^2\theta} \\ 2\psi &= \frac{\pi}{2} + \theta \pm \cos^{-1}\frac{M\sin\theta}{D}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (11.)$$

Hence it appears, that for each value of the obliquity θ, there are two values of ψ and two of R, the less value of ψ corresponding to the greater value of R, and conversely.

Let ψ_u be the *less*, and ψ_v the *greater* value of ψ, R_u the greater, and R_v the less value of R. Let the two normals Ou, Ov be drawn at opposite sides of the axis of greatest pressure Ox; then the angle between them is

$$\left.\begin{aligned} uOv &= \psi_u + \psi_v = \frac{\pi}{2} + \theta; \\ \text{and the angle between the planes to which they are normal is } & \\ \pi - \psi_u - \psi_v &= \frac{\pi}{2} - \theta; \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (12.)$$

therefore those two planes are *conjugate*.

Problem.—The positions of a pair of conjugate planes, both perpendicular to the plane of greatest and least pressure, and the pressures on them, being given, it is required to find the position of the axes of greatest and least pressure, and the magnitude of the greatest and least pressures.

From the equations (11.) it is easily deduced, that

$$\left.\begin{aligned} M &= \frac{R_u + R_v}{2\cos\theta} \\ D &= M\sqrt{\left\{\sin^2\theta + \left(\frac{R_u - R_v}{R_u + R_v}\cos\theta\right)^2\right\}} \\ &= \frac{1}{2}\cdot\sqrt{\{(R_u + R_v)^2\tan^2\theta + (R_u - R_v)^2\}} \\ 2\psi &= \frac{\pi}{2} + \theta \mp \cos^{-1}\frac{1}{\sqrt{\left\{1 + \left(\frac{R_u - R_v}{R_u + R_v}\operatorname{cotan}\theta\right)^2\right\}}}. \end{aligned}\right\} \quad \ldots\ldots \quad (13.)$$

The axis of greatest pressure will be found in the obtuse angle between the normals Ou, Ov, and nearer to Ou, the normal to the plane on which the pressure is the greater.

When $R_u = R_v$, then $\theta = \Theta$, the angle of greatest obliquity. In this case let

$$R_u = R_v = R(\Psi),$$

of which the value, in terms of M and D, has already been given. Then it appears that

$$\left.\begin{aligned} M &= \frac{R(\Psi)}{\cos\Theta} \\ D &= M\sin\Theta = R(\Psi)\tan\Theta \\ 2\Psi &= \frac{\pi}{2} + \Theta, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (14.)$$

in accordance with the equations (7.) and (8.).

From the equations (13.) and (14.) it is easily deduced that the ratio of a pair of conjugate pressures has the following value:—

$$\frac{R_\pi}{R_\pi} = \frac{\cos\theta - \sqrt{\sin^2\Theta - \sin^2\theta}}{\cos\theta + \sqrt{\sin^2\Theta - \sin^2\theta}} \quad \ldots\ldots\ldots \quad (15.)$$

§ 6. *Lemmata as to the Internal Equilibrium of a Solid Mass.*

Let Ox', Oy', Oz' be rectangular axes, of which Ox' is vertical, and Oz' is vertical downwards; and let G be the weight of unity of volume of a solid mass. Then the well-known conditions of the internal equilibrium of such a mass are the following:—

$$\left.\begin{aligned} \frac{dP_{x'}}{dx'} + \frac{dQ_{x'}}{dy'} + \frac{dQ_{y'}}{dz'} &= G; \\ \frac{dQ_{x'}}{dx'} + \frac{dP_{y'}}{dy'} + \frac{dQ_{z'}}{dz'} &= 0; \\ \frac{dQ_{y'}}{dx'} + \frac{dQ_{z'}}{dy'} + \frac{dP_{z'}}{dz'} &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (16.)$$

In all actual problems respecting the stability of earth, the plane of greatest and least pressures is vertical, and there is one horizontal direction, normal to that plane, along which the state of stress of the earth does not vary. It will be sufficient, therefore, to restrict the above equations to two dimensions, by making $Q_{x'} = 0$; $Q_{y'} = 0$; $\frac{d}{dz'} = 0$; and putting Q simply for $Q_{z'}$. Then we have the two equations,—

$$\left.\begin{aligned} \frac{dP_{x'}}{dx'} + \frac{dQ}{dy'} &= G; \\ \frac{dQ}{dx'} + \frac{dP_{y'}}{dy'} &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (17.)$$

§ 7. *Surfaces of Uniform Horizontal Thrust.*

The following is a peculiar transformation of these differential equations, suited to the subject of the present investigation. OX, fig. 2, being vertical, and OY horizontal, and in the plane of greatest and least pressure, conceive the mass to be subdivided into prismatic molecules by an indefinite number of vertical planes perpendicular to the plane XY, such as $y_1x_{1,1}x_{2,1}$, $y_2x_{1,2}x_{2,2}$, and by an indefinite number of surfaces, also perpendicular to the plane XY, such as a_1b_1, a_2b_2, and of such a figure, that the tangent

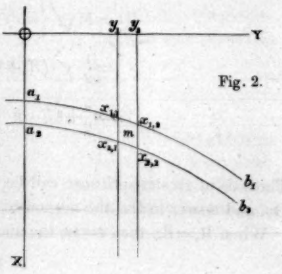


Fig. 2.

plane at each point of each of those surfaces is conjugate to a vertical plane, in the sense explained in section 5. Let

$$a = Oa$$

be the vertical ordinate of any one of those surfaces corresponding to $y=0$; then

$$x = f(a, y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.)$$

will be its equation. By the definition of conjugate planes, the pressure on each element of such a surface is vertical. Let R_s be its amount per unit of area of the surface, and R_v the pressure on unity of area of a vertical plane, which pressure is parallel to a tangent to ab; the angle of obliquity being given by the equation

$$\tan \theta = \frac{dx}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.)$$

Let X be the vertical pressure on a given element of a surface ab, per unit of area of the *projection of that surface on a horizontal plane*; then

$$X = \frac{R_s}{\cos \theta} = R_v \sqrt{1 + \frac{dx^2}{dy^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20.)$$

It is evident that the equations of the equilibrium of a prismatic element m are the following:—

$$\left.\begin{aligned} &\frac{d}{da}(X - Gx) + \frac{d}{dy}\left(R_v \cdot \sin\theta \cdot \frac{dx}{da}\right) = 0; \\ &\frac{d}{dy}\left(R_v \cos\theta \frac{dx}{da}\right) = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.)$$

The second of these equations being integrated, gives

$$R_v \cos\theta \frac{dx}{da} = \frac{R_v \frac{dx}{da}}{\sqrt{1 + \frac{dx^2}{dy^2}}} = F(a), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.)$$

which value being introduced into the first equation, gives the following,

$$\frac{d}{da}(X - Gx) + F(a) \cdot \frac{d^2x}{dy^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23.)$$

Now let

$$H = \int_0^a F(a)da = \int_0^a R_v \cos\theta \frac{dx}{da} da \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24.)$$

be the *total horizontal thrust* of the solid mass from its upper surface, down to the surface under consideration. This quantity, being independent of y, may be used as an independent variable instead of a; that is to say, dividing equation (23.) by $F(a)$, we obtain the following:—

$$\frac{d}{dH}(Gx - X) = \frac{d^2x}{dy^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25.)$$

which is the differential equation of a *Surface of Uniform Thrust*.

§ 8. *Surfaces of Uniform Thrust and Uniform Vertical Pressure.*

With an exception to be described in the next section, the only case in which the equation (25.) becomes linear with respect to x, and capable of being satisfied by an indefinite number of arbitrary forms of surface, is that in which each surface of uniform thrust is also a surface of uniform vertical pressure; that is to say, when

$$X = F(H). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26.)$$

In this case, the integral of equation (25.), as found by the method of FOURIER, is capable of being expressed in various forms, of which the following is the most comprehensive:—

$$x = \frac{F(H)}{G} + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-m^2} f\left(y + 2m\sqrt{\frac{H}{G}}\right) dm; \quad . \quad . \quad . \quad . \quad . \quad . \quad (27.)$$

the function f being such, that neither

$$e^{-m^2} f, \text{ nor } e^{-m^2} f',$$

shall become infinite for any value of the argument, how great soever, nor increase indefinitely with the argument, and that they shall both vanish at the limits of integration. This function is determined by the following condition at the upper surface of the mass:—

$$x_0 = f(y). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28.)$$

In all those cases in which the upper surface of the mass deviates alternately above and below an inclined or horizontal plane by deviations which recur periodically in each horizontal distance 2B, the integral of the differential equation (25.) is capable of being expressed in the following form:—

$$x = \frac{F(H)}{G} + Ay + \Sigma . e^{-\frac{n^2 \pi^2 H}{G B^2}} \left(C_n \sin \frac{n\pi y}{B} + C'_n \cos \frac{n\pi y}{B} \right), \quad . \quad . \quad . \quad . \quad (29.)$$

where A is the tangent of the average declivity, above and below which the surfaces of equal thrust deviate periodically, and C_n and C'_n are determined by the following formulæ:—

$$\left.\begin{aligned} C_n &= \frac{2}{B} \int_{-\frac{B}{2}}^{\frac{B}{2}} (x_0 - Ay) \sin \frac{n\pi y}{B} . dy \\ C'_n &= \frac{2}{B} \int_{-\frac{B}{2}}^{\frac{B}{2}} (x_0 - Ay) \cos \frac{n\pi y}{B} . dy, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30.)$$

Such are the integrals of the equations of internal equilibrium in two dimensions in a vertical plane, in those cases in which each surface of uniform thrust is also a surface of uniform vertical pressure; a condition realized in those cases in which the horizontal thrust is caused by the vertical pressure.

The relation $X = F(H)$ between the horizontal thrust and the vertical pressure, still remains to be determined by the physical conditions of each particular problem.

NOTE. *Equilibrium of Arched Ribs.*

Although foreign to the immediate subject, the fact may here be noted, that by making $X=F(H)=0$ in equations (27.) and (29.), they furnish the means of determining the form of the intrados of any equilibrated arched rib loaded solely with its own weight, or proportionally to the weight of each of its parts, when the form of the extrados is either represented by equation (28.), or developed by means of the formulæ (30.); care being taken, in the latter case, to include the entire rib within the limits $y=\pm\frac{1}{2}B$. If the rib be divided into voussoirs, their joints ought theoretically to be normal to the series of curves given by equation (27.) or equation (29.).

§ 9. *Extrinsic and Intrinsic Vertical Pressures.*

The exception referred to at the beginning of the last section is as follows. Besides the vertical pressure

$$X_H=F(H),$$

which is uniform for each surface of equal thrust, there may also be a vertical pressure, uniform in each vertical line, and varying proportionally to an arbitrary function of the horizontal coordinate, viz.—

$$X_y=\Phi(y). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31.)$$

This last pressure does not enter into the differential equation (25.), and its amount is therefore to be determined in each problem by conditions distinct from the general conditions of internal equilibrium. It may be called the *Extrinsic Vertical Pressure*; and the vertical pressure depending on the horizontal thrust, the *Intrinsic Vertical Pressure*; so that the surfaces determined in the last section are surfaces of *uniform intrinsic vertical pressure*, the extrinsic vertical pressure being independent of their form and position*.

§ 10. *Limiting Relations of the Pressures, at a point in a Mass of loose Earth.*

The seven preceding sections, from § 3 to § 9 inclusive, contain principles which are common to every case of the internal equilibrium of a solid mass, in what manner soever constituted.

In the present section, those common principles will be combined with the peculiar principle of the stability of an incoherent granular mass, explained in §§ 1 and 2.

The symbol $\lessgtr$ will be used in the following sense:—

$$a \lessgtr b \pm c$$

means, "a is *not less* than $b-c$, and *not greater* than $b+c$." The symbols $\leq$ and $\geq$ will be used in their ordinary sense of "not greater than" and "not less than."

φ being, as in § 2, the angle of repose of a given kind of earth, and Θ, as in § 3, the

* In the sequel, when vertical pressures are not expressly designated as extrinsic, they are to be understood as intrinsic.

greatest obliquity of the pressure at any point in any plane traversing that mass, it appears from equation (2.) that the condition of stability of the mass is

$$\Theta \leq \varphi. \qquad (32.)$$

From this condition the following propositions are deduced:—

THEOREM I. *At each point in a mass of earth, the ratio of the difference of the greatest and least pressures to their sum cannot exceed the sine of the angle of repose.*

This theorem* follows from the second of the equations (7.), its symbolical expression being

$$\frac{P_x - P_y}{P_x + P_y} = \frac{D}{M} \leq \sin\varphi. \qquad (33.)$$

THEOREM II. *The following is the expression of the condition of stability of a mass of earth, in terms of the pressures at a point, referred to any pair of rectangular axes,* Ox′, Oy′, *in the plane of greatest and least pressures:—*

$$\frac{(P_{x'} - P_{y'})^2 + 4Q'^2}{(P_{x'} + P_{y'})^2} = \frac{D^2}{M^2} \leq \sin^2\varphi. \qquad (34.)$$

This follows from the equations (10.).

THEOREM III. *The following is the expression of the condition of the stability of a mass of earth, in terms of the ratio of a pair of conjugate pressures in the plane of greatest and least pressures:—*

Let R_x, R_y be the two conjugate pressures, θ their common angle of obliquity; then

$$\frac{R_x}{R_y} \gtrless \frac{\cos\theta \mp \sqrt{\cos^2\theta - \cos^2\varphi}}{\cos\theta \pm \sqrt{\cos^2\theta - \cos^2\varphi}}. \qquad (35.)$$

This follows from equation (15.).

THEOREM IV. *The positions of a pair of conjugate planes being given, the following are the limits, consistent with stability, of the angle which the axis of greatest pressure can make with either of the normals to the conjugate planes:—*

$$\psi \gtrless \frac{\pi}{4} + \frac{\theta}{2} \pm \frac{1}{2}\cos^{-1}\frac{\sin\theta}{\sin\varphi}. \qquad (36.)$$

This follows from the second of the equations (11.).

THEOREM V. *The amount* R *and obliquity* θ *of the pressure on a given plane being given, the following are the limits, consistent with stability, of the half sum* M *and half difference* D, *of the greatest and least principal pressures:—*

$$\left.\begin{aligned} M &\gtrless \frac{R}{\cos\theta \mp \sqrt{\cos^2\theta - \cos^2\varphi}} \\ D = M\sin\theta &\gtrless \frac{R\sin\varphi}{\cos\theta \mp \sqrt{\cos^2\theta - \cos^2\varphi}} \end{aligned}\right\} \qquad (37.)$$

This follows from the first of the equations (11.).

The greater values of M and D given by this equation correspond to the greater values of the angle ψ given by equation (36.).

* Already published in the 'Proceedings of the Royal Society' for the 6th of March, 1856.

Corollary to the Theorems III., IV., and V.—When the angle of obliquity θ is equal to the angle of repose ϕ, the quantities given by the equations (35.), (36.), (37.), have each but one value, without any limits of deviation, viz.—

$$\left.\begin{aligned} &\frac{R_y}{R_x}=1\,;\ \Psi=\frac{\pi}{4}+\frac{\phi}{2}\,;\\ &M=R\sec\phi\,;\ D=R\tan\phi. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (38.)$$

§ 11. *Planes of Rupture.*

The angle Ψ, above given, is that made on either side of the axis of greatest pressure by the normals to the pair of planes along which the tendency of the earth to give way by sliding is greatest. The angle made by these planes themselves on either side of the axis of greatest pressure is therefore

$$\Psi-\phi=\frac{\pi}{4}-\frac{\phi}{2} \quad \ldots\ldots\ldots\ldots \quad (39.)$$

Those planes are called *Planes of Rupture*. Their position, in the particular case of a horizontal bank, where the axis of greatest pressure is vertical, was determined by COULOMB by the aid of his ideal *wedge of least resistance*.

§ 12. *Application of* Mr. MOSELEY'S *Principle of Least Resistance to the Stability of Earth.*

This principle may be stated as follows:—The forces which balance each other in or upon a given body or structure being distinguished into two systems, called respectively *active* and *passive*, which stand to each other in the relation of cause and effect, then will the passive forces be the least which are capable of balancing the active forces, consistently with the physical condition of the body or structure.

In a mass of earth, the active forces are the vertical pressures produced by the gravitation of its parts; the passive forces are the pressures conjugate to those vertical pressures, whereby the earth is prevented from spreading. The pressures conjugate to the vertical pressures will therefore be the least which are at once consistent with the conditions of internal equilibrium given in §§ 6, 7, 8, and 9, and with the conditions of stability at each point, given in § 10.

§ 13. *Statement of the General Problem of the Stability of a Mass of Earth under its own Weight.*

The upper and free surface of the mass of earth, at which the intrinsic vertical pressure $X_n=0$, is supposed to be curved in one coordinate plane only, that of greatest and least pressures; and the form of the section of that surface by the vertical plane of greatest and least pressures is supposed to be given by an equation of the form (28.).

The specific gravity G and angle of repose ϕ of the earth being given, it is required

to determine the form and position of the surface at which the intrinsic vertical pressure has any given value.

In order that the equations (27.), or (29.) and (30.) may furnish the complete solution of this problem, it is necessary now to determine, from the conditions of stability in § 10, the relation $X_H = F(H)$, between the intrinsic vertical pressure and the horizontal thrust.

The case in which the upper surface of the earth is an indefinitely extended plane, horizontal or inclined, is the only case which admits of an exact solution. It will therefore be solved first, and its solution used to facilitate the solution of the more complex case, which is solved approximately by the integral (27.), or by infinite series of the form (29.).

§ 14. PROBLEM I. *Surfaces of Equal Pressure and Thrust in the case of a horizontal or uniformly sloping bank.*

In this case, equation (28.), giving the form of the free surface, becomes

$$x_0 = Ay = y \tan \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40.)$$

Equation (27.) gives for the form and position of any surface of uniform intrinsic vertical pressure,

$$x = \frac{X_H}{G} + Ay; \text{ or } x - x_0 = \frac{X_H}{G}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41.)$$

Hence the surfaces of equal intrinsic vertical pressure are planes parallel to the free surface, and the vertical pressure is simply the weight of a column of earth of unity of area of base, and of the height $x - x_0$. At each point, a vertical plane, and a plane parallel to the free surface, are conjugate to each other; that is, the pressure on a plane parallel to the surface is vertical, and the pressure on a vertical plane is parallel to the surface; and the angle of slope, θ, is the common angle of obliquity of those conjugate pressures.

Equation (20.) gives for the vertical pressure per unit of area of a plane parallel to the surface,

$$R_0 = X \cos \theta = G(x - x_0) \cos \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42.)$$

Now from the principle of least resistance, it follows that the pressure at any point on a vertical plane, in a direction parallel to the slope, must have the least value consistent with the equation of stability (35.); that is to say,

$$R_0 = R_0 \cdot \frac{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \varphi}}{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \varphi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43.)$$

For brevity's sake, let

$$\cos^2 \theta \cdot \frac{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \varphi}}{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \varphi}} = k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44.)$$

Then the horizontal component of R_0 has the following value:—

$$\frac{dH}{dx} = R_0 \cos \theta = kX = kG(x - x_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45.)$$

And the total horizontal thrust, from the surface down to a given depth $x-x_0$, is

$$H=\frac{kG(x-x_0)^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46.)$$

The relation, then, between the vertical pressure and the total horizontal thrust down to a given surface of uniform thrust and pressure, is expressed by the equation

$$H=\frac{kX^2}{2G}, \text{ or } X=\sqrt{\frac{2GH}{k}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47.)$$

Equation (36.) gives, for the angle made by the axis of greatest pressure at each point with the vertical,

$$\frac{\pi}{2}-\psi=\frac{\pi}{4}-\frac{\theta}{2}-\frac{1}{2}\cos^{-1}.\frac{\sin\theta}{\sin\varphi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48.)$$

This axis lies in the acute angle between the slope of the surface and the vertical.

The two planes of rupture at each point make with this axis the angle given by equation (39.); that is to say, they make with the vertical the following angles respectively at opposite sides:—

$$\left.\begin{aligned} &\frac{\pi}{2}-\frac{\theta}{2}-\frac{1}{2}\cos^{-1}\frac{\sin\theta}{\sin\varphi}-\frac{\varphi}{2}, \text{ and} \\ &\frac{\theta}{2}+\frac{1}{2}\cos^{-1}\frac{\sin\theta}{\sin\varphi}-\frac{\varphi}{2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49.)$$

§ 15. *Extreme cases of this Problem.*

The two extreme cases of this problem are respectively, when the free surface is horizontal, or $\theta=0$, and when it slopes at the angle of repose, or $\theta=\varphi$.

The following are the results in these two cases.

Case First. $\theta=0$:

$$\left.\begin{aligned} &R_u=X;\ R_v=X\frac{1-\sin\varphi}{1+\sin\varphi};\ k=\frac{1-\sin\varphi}{1+\sin\varphi}; \\ &\frac{\pi}{2}-\psi=0, \text{ or the axis of greatest pressure is vertical, and the planes} \\ &\text{of rupture make the angles } \frac{\pi}{4}-\frac{\varphi}{2} \text{ on either side of the vertical.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (50.)$$

Case Second. $\theta=\varphi$:

$$\left.\begin{aligned} &R_u=X\cos\varphi;\ R_v=R_u=X\cos\varphi;\ k=\cos^2\varphi; \\ &\frac{\pi}{2}-\psi=\frac{\pi}{4}-\frac{\varphi}{2}, \text{ or the axis of greatest pressure bisects the acute angle} \\ &\text{between the vertical and the slope of the surface, and the planes of} \\ &\text{rupture are respectively vertical, and parallel to the surface.} \end{aligned}\right\} \quad . \quad . \quad . \quad (51.)$$

§ 16. *Revêtement Walls.* PROBLEM II. *To determine the magnitude and direction of the resultant pressure on the base of the revêtement wall of an earthen bank, having a horizontal or uniformly sloping upper surface, the angle of repose of the earth upon the masonry being not less than that of the earth upon itself.*

Case 1. When the back of the wall does *not* overhang the base. (See fig. 3.)

Let ABDE be the section of the wall. Through A draw AF vertical ($=x-x_0$), cutting the upper surface of the bank in F. If the back of the wall is vertical, F coincides with B.

Fig. 3.

Through C, the centre of gravity of the mass of masonry and earth AFBDE, draw CW vertical. Take $\overline{AH}=\frac{1}{3}\overline{AF}$, and draw HR parallel to the surface of the bank, cutting CW in O; this will be the position and direction of the resultant pressure on the plane AF. To represent that resultant pressure, take

$$\overline{OR}=\frac{H}{\cos\theta}=\frac{kG.\overline{AF}^2}{2\cos\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52.)$$

Also take $\overline{OW}$ to represent the weight of the mass of earth and masonry AFBDE; then will the resultant $\overline{OS}$ of $\overline{OR}$ and $\overline{OW}$ represent in magnitude and direction the resultant pressure required on the base AE.

$$\left.\begin{aligned} \overline{OS}&=\sqrt{\left\{\overline{OR}^2+\overline{OW}^2+2\overline{OR}.\overline{OW}.\sin\theta\right\}};\\ \sin\angle WOS&=\frac{H}{OS}: \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53.)$$

$\sin\theta$ is to be taken as positive when the bank slopes down towards the wall.

Case 2. When the back of the wall overhangs, as represented, for example, by AT.

Proceed in all respects as above, except that in finding the centre of gravity C, and the weight $\overline{OW}$, the mass AFBDE alone is to be treated as masonry, and the prism AFT is to be treated as if its specific gravity were merely the *excess* of the specific gravity of the masonry above that of the earth. This is because a pressure equal to the weight of the earth which the prism AFT could contain, is sustained by the earth vertically below it, leaving only the excess of the weight of masonry over that of earth to add to the stability of the wall. As this excess is in general very small, it follows that there is in general little or no advantage in building revêtement walls so as to overhang behind. For the same reason it is evident, that if a straight line be drawn from A to U, where the line of slope cuts the face of the wall, the masonry behind this line contributes little to the stability of the wall.

§ 17. *General Case:—its ambiguity.*

The application of the principle of least resistance to the case in which the vertical section of the upper surface of the earth is of any form (limited only by the condition that the slope shall nowhere exceed the angle of repose), is attended with this difficulty;—that at certain portions of each layer of equal thrust, viz. the lower portion, it is difficult, if not impossible, to determine which of the two conjugate pressures, the vertical pressure and the horizontal or inclined pressure, is to be regarded as *cause* and which as *effect*; and thus the solution becomes ambiguous.

There is one case, however, which is not affected by such ambiguity; viz. that in which the steepest declivity of each surface of equal thrust is the angle of repose ϕ; for in that

case there is but one relation, consistent with stability, between the vertical pressure and horizontal thrust, without any limits of deviation.

The relation between those forces which is *the only solution* in the case specified, is *one of the solutions* consistent with stability in every other case, as will appear from the following

LEMMA. *The intensity of the horizontal thrust,* $R_v \cos\theta = X\cos^3\varphi$, *which corresponds with a given vertical pressure,* X, *for a declivity sloping at the angle of repose,* φ, *lies between the limits consistent with stability for every declivity sloping at a less angle.*

For the condition of stability deduced from equation (35.) is

$$\frac{R_v\cos\theta}{X}=\frac{R_v}{R_u}\cos^3\theta\gtrless\cos^2\theta\cdot\frac{1\mp\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}}{1\pm\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}}\lessgtr\frac{\cos^2\varphi}{\left(1\pm\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}\right)^2}; \quad . \quad . \quad . \quad . \quad . \quad (54.)$$

and $\cos^3\varphi$ is always within the limits fixed by this equation.

§ 18. PROBLEM III. *The vertical section of the upper surface of a mass of earth, curved in the plane of section only, being given, it is required to find the form and position of the surfaces of uniform vertical pressure, when the greatest declivity of each of them is the angle of repose.*

It appears from the preceding lemma that the solution of this problem is always *one of the solutions* even when the greatest declivity is less than the angle of repose.

Let the equation of the upper surface be expressed by the equation (28.), or developed by the formulæ (30.). Then the *form* of the surfaces of equal pressure is given by the transcendental parts of the equations (27.) or (29.); and it only remains to determine their *position*, by finding the relation between the vertical pressure X and the horizontal thrust H.

The condition of stability gives

$$X=\frac{R_v\cos\theta}{\cos^2\varphi}=\frac{1}{\cos^2\varphi\,\frac{dx}{dH}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55.)$$

From the general differential equation (25.), it appears that

$$\frac{dx}{dH}=\frac{1}{G}\left(\frac{dX}{dH}+\frac{d^2x}{dy^2}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56.)$$

but at the points of greatest declivity of each surface

$$\frac{d^2x}{dy^2}=0,$$

consequently equation (55.) becomes

$$X=\frac{G}{\cos^2\varphi\,\frac{dX}{dH}},$$

which being integrated, gives

$$H=\frac{\cos^2\varphi\,.\,X^2}{2G};\quad X=\frac{\sqrt{2GH}}{\cos\varphi}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57.)$$

being the solution required.

The introduction of the above value of the thrust into the equations (27.) and (29.) gives the following results, showing at once the relative position and the form of the surfaces of equal pressure:—

$$\left.\begin{matrix}\text{General}\\ \text{Solution}\end{matrix}\right\} x=\frac{X}{G}+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-m^2} f\left(y+\frac{\sqrt{2}\cdot mX\cos\varphi}{G}\right)dm. \quad . \quad . \quad . \quad . \quad . \quad (58.)$$

$$\left.\begin{matrix}\text{Periodical}\\ \text{Solution}\end{matrix}\right\} x=\frac{X}{G}+Ay+\Sigma\cdot e^{-\frac{1}{2}\left(\frac{nX\cos\varphi}{GA}\right)^2}\left(C_n\sin\frac{ny}{B}+C_n'\cos\frac{ny}{B}\right). \quad . \quad . \quad (59.)$$

In using the first of the above equations, it is to be observed that

$$\int_{-\infty}^{+\infty} e^{-m^2}dm=\sqrt{\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60.)$$

§ 19. *Approximate Description of Surfaces of Equal Pressure by Graphic Construction.*

The following method of approximately determining the forms of the surfaces of equal pressure or equal thrust, is analogous to that employed by Professor WILLIAM THOMSON in some recent researches on Electricity, for the purpose of finding the successive forms of the curve showing the electric tension in a conductor.

Let $x=\frac{X}{G}+x_y$. Then, by the general equations (25.), (26.), we have

$$G\frac{dx_y}{dH}=\frac{d^2x_y}{dy^2}.$$

Substituting finite differences for differentials, it follows that the subjoined equation is approximately true:—

Fig. 4.

$$\frac{\Delta_H x_y}{\Delta_y^2 x_y}=\frac{\Delta H}{G\Delta y^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61.)$$

Δ_H denoting differences caused by varying H, and Δ_y those caused by varying y.

Now let the vertical section of any surface of equal thrust be given; and let it be intersected by equidistant vertical ordinates, their uniform interval in a horizontal direction being Δy. Join by a straight chord any two of the points of intersection whose distance apart is two intervals, or $2\Delta y$. The vertical sagitta intercepted upon the intermediate ordinate between the chord and the given line of section will be $\frac{1}{4}\Delta_y^2 x_y$; and if a second surface of equal thrust be conceived to pass through the middle of the chord, the same sagitta will be $\Delta_H x_y$. Consequently, for the pair of surfaces which respectively traverse the ends and the middle of the chord in question,

$$\frac{\Delta_H x_y}{\Delta_y^2 x_y}=\frac{\Delta H}{G\Delta y^2}=\frac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62.)$$

Hence we have the following construction. Inscribe a polygon in the vertical section of the first surface of equal thrust by joining the points where the surface is intersected by alternate ordinates: the points where the intermediate ordinates bisect the sides of the

polygon will lie very nearly in a curve parallel to a surface of equal thrust at which the total thrust is greater than at the first surface by the amount $\frac{1}{2}G\Delta y^2$. Inscribe a polygon in the second curve by joining the points found in it; the points of bisection of the sides of this second polygon will indicate a third curve, and so on; the constant increment of the thrust from any curve to the next being still $\frac{1}{2}G\Delta y^2$.

Thus are drawn approximately the *forms* of a series of surfaces of equal thrust. Their *distances apart* are to be computed by equation (57.), as before.

Let the first surface be the upper surface of the mass of earth, where $H=0$; then the successive inscription of the number of polygons denoted by p will show the form of the surface at which the total horizontal thrust is

$$H=\frac{p}{2}G\Delta y^2; \quad \ldots\ldots\ldots\ldots \quad (63.)$$

and the mean depth of this surface below the upper surface (that is to say, the depth to which the form of each surface is to be transferred vertically downwards, after having been drawn by means of the inscription of polygons,) will be given by the formula

$$\frac{X}{G}=\sec\varphi\sqrt{\frac{2H}{G}}=\sqrt{p}.\sec\varphi.\Delta y. \quad \ldots\ldots\ldots \quad (64.)$$

Fig. 4 shows the result of applying this process to an outline of upper surface marked $p=0$, consisting partly of two horizontal planes, and partly of a declivity sloping at an angle of 30°, which is also supposed to be the angle of repose.

§ 20. *Of the Limits of Extrinsic Pressure and the Stability of Foundations in Earth.*

At any point on a surface of uniform thrust and uniform vertical pressure, where the declivity θ is less than the angle of repose φ, it appears from the equations (35.) and (54.) that there are two limits to the ratio of the vertical to the horizontal pressure, given by the equation

$$\begin{aligned} X\frac{dx}{dH} &\lessgtr \frac{1}{\cos^2\theta}\cdot\frac{1\pm\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}}{1\mp\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}} \\ &\lessgtr \frac{1}{\cos^2\varphi}\cdot\left(1\pm\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}\right)^2. \quad \ldots\ldots\ldots \quad (65.) \end{aligned}$$

Should, therefore, the intrinsic vertical pressure X_H due to the weight of the earth be less than that fixed by the upper limit of the above expression, an *extrinsic vertical pressure* may be applied at the given point of the given layer (by means, for example, of the foundation of a building) not exceeding the amount whereby the intrinsic pressure falls short of the limit; that is to say,—

$$X_y\leq\frac{1}{\frac{dx}{dH}\cos^2\varphi}\cdot\left(1+\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}\right)^2-X_H. \quad \ldots\ldots \quad (66.)$$

In the case to which Problem III. refers, in which

$$X_H = \frac{1}{\frac{dx}{dH}\cos^2\varphi},$$

equation (66.) becomes

$$X_y \leq \frac{1}{\frac{dx}{dH}\cos^2\varphi} \cdot \left\{1 - \frac{\cos^2\varphi}{\cos^2\theta} + 2\sqrt{1 - \frac{\cos^2\varphi}{\cos^2\theta}}\right\}. \quad \ldots\ldots\ldots \quad (67.)$$

The most important application of these principles relates to the power of a mass of earth whose upper surface is horizontal, to sustain the weight of a building founded at a given depth.

In this case, the condition of stability expressed by the equation (65.) becomes

$$X\frac{dx}{dH} > \frac{1+\sin\varphi}{1-\sin\varphi}. \quad \ldots\ldots\ldots\ldots \quad (68.)$$

Now the horizontal pressure of the earth may be so increased by ramming, as to have the maximum amount consistent with the intrinsic vertical pressure due to the weight of the earth, in which case

$$X_H\frac{dx}{dH} = \frac{1-\sin\varphi}{1+\sin\varphi}. \quad \ldots\ldots\ldots\ldots \quad (69.)$$

And consequently, the extrinsic vertical pressure, due to the excess of the weight of a building above that of the earth which it displaces, is limited by the equation

$$X_y \leq \frac{4\tan\varphi}{\frac{dx}{dH}\cos\varphi} \leq X_H \cdot \frac{4\sin\varphi}{(1-\sin\varphi)^2}; \quad \ldots\ldots\ldots \quad (70.)$$

and the *ratio of the weight of a building, to the weight of the earth displaced by its foundation*, is limited by the equation

$$\frac{X_H + X_y}{X_H} \leq \left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)^2. \quad \ldots\ldots\ldots\ldots \quad (71.)$$

When the angle of repose, for example, is 30°, the limit of that ratio is 9.

§ 21. *Negative Extrinsic Pressure. Resistance of Screw Piles to Extraction.*

When the intrinsic vertical pressure exceeds the lower limit given by the equation (65.), the earth will resist a *Negative Extrinsic Pressure*, or upward tension, not exceeding such excess. To avoid the use of negative signs, let

$$-X_y = T_y;$$

then

$$T_y \leq X_H - \frac{1}{\frac{dx}{dH}\cos^2\varphi} \cdot \left(1 - \sqrt{1 - \frac{\cos^2\varphi}{\cos^2\theta}}\right)^2. \quad \ldots\ldots\ldots \quad (72.)$$

Let the surface of the earth be horizontal, and let the horizontal pressure have the least amount consistent with stability. Then

$$X_H \frac{dx}{dH} = \frac{1+\sin\varphi}{1-\sin\varphi}, \quad \ldots \quad (73.)$$

and

$$T_v \leq \frac{4\tan\varphi}{\frac{dx}{dH}\cos\varphi} \leq X_H \cdot \frac{4\sin\varphi}{(1+\sin\varphi)^2}. \quad \ldots \quad (74.)$$

The last expression shows the ratio borne to the superincumbent load, by the upward tension necessary to extract from the earth a body with a nearly horizontal upper surface, like the thread of a screw pile. When the angle of repose, for example, is 30°, that ratio is $\frac{8}{9}$.

§ 22. *Resistance of Earth to the Horizontal Thrust of a Building.*

The limit of the resistance of a horizontal stratum of earth to the horizontal thrust of the foundation of a building, is the upper limit of the horizontal thrust of the stratum, consistent with its weight; that is to say,—

$$H \leq \frac{X^2}{2G} \cdot \frac{1+\sin\varphi}{1-\sin\varphi} = \frac{G(x-x_0)^2}{2} \cdot \frac{1+\sin\varphi}{1-\sin\varphi}. \quad \ldots \quad (75.)$$

III. *On the Geometrical Isomorphism of Crystals.*

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Crystals, as it is well known, are solids bounded by plane surfaces termed faces, which vary in shape and in their relative positions. They occur generally in small imperfect fragments, and all that we can be said to know about them, is the shapes and positions of their faces, and the angles which those faces make with each other.

All the crystals at present known have been classed in six groups or systems, each group consisting of a series of elementary solids, or as they will be termed primary forms, and of other forms, termed secondary, derived from the primary according to laws to be afterwards explained. These different groups or systems are distinguished from each other by the shapes and positions of some of their faces, and the crystals of the different minerals comprised in each group or system, except the cubic, are distinguished by the angles which particular faces make with each other.

The six systems, and the elementary solids proper to each, are as follows:—

The cubic, consisting of cubes which are bounded by six square faces.

The pyramidal, consisting of prisms having their terminal faces square, and their lateral faces rectangular.

The rhombohedral, consisting of solids bounded by six equal rhombs.

The prismatic, consisting of solids contained within three pairs of parallel rectangular faces at right angles to each other.

The oblique, consisting of solids, also contained within three pairs of parallel faces, of which only two intersect at right angles, the third being an oblique pair.

The anorthic. It is said that in the few solids comprised in this system, no two faces intersect each other at a right angle; and hence the name of the system.

The accompanying Tables of the pyramidal and rhombohedral systems contain lists of the crystallized minerals comprised in each, arranged in a manner new, as the writer believes, to crystallography; presenting that science under a new aspect, and affording some altogether unexpected results, which it is the object of this paper to communicate to the Society.

Notation.—Previously, however, to stating these results, it is necessary to explain briefly the crystallographic language and notation that will be employed in describing them.

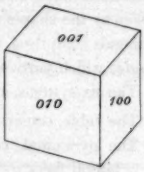
The term *form* is used in crystallography in two different senses: in one of these it simply means shape; in the other—a sense peculiar to this science—it denotes particular sets of faces that may or may not enclose a finite solid. Thus, in some instances, single pairs of parallel faces constitute crystallographic forms.

The *angles* quoted in the accompanying Tables, and throughout this paper, are those *between normals* to the several faces, although, to avoid the frequent repetition of the words "between normals," they are expressed as angles between faces. The angle between any two faces, as given by most other authors, and the angle between normals to those faces, are supplements to each other, so that, if either is given, the other can be readily found.

A *zone* in crystallography is a series of faces on any crystal which intersect each other in parallel lines, or would do so if sufficiently enlarged.

It has been found that the secondary faces of crystals are not produced as it were at random, but that they cut the primary edges in definite proportions measured along those edges in particular directions; and it is by means of the differences of those proportions, and of the directions in which they are measured, that secondary faces can be denoted and distinguished from each other. This system of notation was first proposed by the Rev. Dr. WHEWELL, in a paper read before the Royal Society and published in the Philosophical Transactions for 1825, and has been employed by Professor MILLER of Cambridge in a work on Crystallography published in 1839, and in the new edition of the Treatise on Mineralogy, by the late WILLIAM PHILLIPS.

Fig. 1.



According to this system, the faces of a primary form, as they are usually shown in engraved figures, are expressed by the symbols $1\,0\,0$, $0\,1\,0$, $0\,0\,1$, placed in the relative positions shown in fig. 1; and the faces respectively parallel to these are denoted by $\bar{1}\,0\,0$, $0\,\bar{1}\,0$, $0\,0\,\bar{1}$.

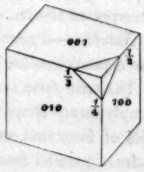
These numbers, taken separately, and regarding 0 or zero as a number, are termed *indices*, and those with a line over them are termed *negative* or *minus*.

Fig. 2.

It is seen in the skeleton (fig. 2) of the primary form, that the edge between the faces $0\,0\,1$ and $0\,1\,0$, and the three edges parallel to these, have an m at one end and $\bar{m}$ at the other; that the edge between $0\,0\,1$ and $1\,0\,0$ and the three parallel edges have n and $\bar{n}$ on each edge; and that the edge between $1\,0\,0$ and $0\,1\,0$, and the three parallel edges, have s and $\bar{s}$ on each edge.

By means of these indices an appropriate symbol can be assigned to every possible face of every crystal, an advantage not afforded by any other system of notation.

Fig. 3.



Suppose a secondary face to occur, as in fig. 3, on the solid angle surrounded by the letters $m\,n\,s$, and to be such as would cut away $\frac{1}{3}$ of the edge m, $\frac{1}{2}$ of n, and $\frac{1}{4}$ of s, measured along each respective edge from the point where m, n, and s meet. This face is denoted by the symbol $3\,2\,4$; the separate indices being *the reciprocals* of the proportions supposed to be cut away from the respective edges by this particular face.

If a face truncates that solid angle of fig. 3 which is in fig. 2

surrounded by $\overline{m}$, n, s, and intersects $\overline{m}\,n$ at $\frac{1}{3}$ of its length measured from the point where $\overline{m}$, n, s meet, and if it theoretically removes $\frac{1}{3}$ of the edge n, and a $\frac{1}{4}$ of s, measured from the same point, its symbol would be $\overline{3}\,2\,4$.

An analogous face on the solid angle $\overline{m}$, $\overline{n}$, s would have for its symbol $\overline{3}\,\overline{2}\,4$, and another similar face on the solid angle m, $\overline{n}$, s would have the symbol $3\,\overline{2}\,4$; and if we suppose a similar face to truncate the solid angle $\overline{m}$, $\overline{n}$, $\overline{s}$, its symbol would be $\overline{3}\,\overline{2}\,\overline{4}$.

A face that would cut away equal proportions of each of the edges m, n, s, would have for its symbol $1\,1\,1$, and the face parallel to it $\overline{1}\,\overline{1}\,\overline{1}$; and the symbol of a face parallel to any given face is denoted by simply changing the positive indices of the given face to negative, and the negative indices to positive.

The faces which replace the edges of crystals being always parallel to those edges, do not anywhere intersect them, and in consequence the index relating to the replaced edge is always 0.

Thus in fig. 4 the edge m is entirely removed, while $\frac{1}{3}$ of n and $\frac{1}{4}$ of s are theoretically cut away by the truncating face, whence its symbol becomes $(0\,2\,4)$. The symbol of the face in fig. 5 is similarly $3\,0\,4$, and of that in fig. 6 is $3\,2\,0$.

Fig. 4.

Fig. 5.

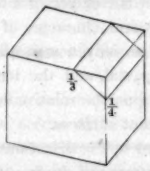
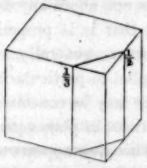


Fig. 6.



Thus recollecting that the symbols of primary faces have two zeros, and of faces on the edges one, the position of any face denoted by any symbol may be readily perceived and easily remembered.

It may be noticed, that the order in which the indices stand in each of the preceding symbols is that of the letters m, n, s; and in the symbol of any face whatever, the first index always denotes the reciprocal of the proportion supposed to be cut away from one of the edges m; the second index, the reciprocal of the proportion removed from one of the edges n; and the third, the reciprocal of the proportion removed from one of the edges s.

Faces may be expressed by their symbols, as face $1\,0\,0$ or $1\,1\,1$.

An angle between two faces may be expressed by means of their symbols, as the angle between $1\,0\,0$ and $1\,1\,1$, or simply the angle $1\,0\,0, 1\,1\,1$.

A zone may be expressed by the symbols of any two faces in it, as $(1\,0\,0, 1\,1\,1)$.

There are in each of the systems, except the cubic, angles between particular faces by which the crystals belonging to the different minerals contained respectively in each are distinguished from each other. These will be termed *elementary angles*.

The crystals of the different minerals comprised in the cubic system constitute geometrically a single isomorphous group, and hence the elementary angle is the same throughout the group.

In the pyramidal system, the elementary angle is between a face 0 0 1 and some other face assumed to be denoted by the symbol 1 0 1.

In the rhombohedral system, the elementary angle is the angle between the face 1 1 1 and some other face denoted by the symbol 1 0 0.

It is thus that at the very outset of our investigations of the forms and angles of crystals an arbitrary assumption of certain elements, not indicated by the crystals themselves, is forced upon us. And hence it is not surprising that such an apparently capricious choice, as will be afterwards pointed out, of faces to be represented by particular symbols should have been exercised by different observers.

But, it may be asked, is the assumption of particular faces to represent those of the primary and other forms as arbitrary as is here supposed?

Cleavage.—It is well known that HAÜY regarded cleavage as a natural indication of primary form, and the same opinion is probably entertained by some crystallographers of the present day. It is therefore necessary, before we proceed, to show that cleavage does not afford any definite rule for the choice of a form to be regarded as the primary, but that it is properly only a physical character of the same nature as that of slaty structure generally. It seems to be only a separation of plates of crystalline matter parallel to particular faces of the crystal, and the tendency to separate parallel to such faces may be conceived to depend upon the relative degrees of cohesion of the crystalline particles in planes parallel to different surfaces.

This view appears to be supported by the directions of the cleavage in different crystals belonging to the cubic system. In some of these, as in galena, the particles cohere with less force in planes parallel to the cubic surfaces than to those of any other of its forms. In fluor the cubic surfaces cohere so firmly as to be inseparable when the crystals are broken, but the octahedral surfaces are held together with so small a force as to separate easily when the crystalline mass is broken. In blende the dodecahedral faces are those of easy separation, while the cohesion at the surfaces of the cubic and octahedral forms is so perfect as not to show a trace of cleavage in those directions. In many cases there are crystals without any perceptible cleavage, and in others there is no cleavage parallel to the assumed primary faces, although there are cleavages in other directions; and in other instances there are two or more sets of cleavages, either of which might be taken to indicate the primary form.

It does not appear that this greater or less degree of cohesive force at the surfaces of particular faces has received any satisfactory explanation, and it is unnecessary here to consider it further; the present object being only to show that the direction of the cleavage may be parallel to the faces of any one or more of the many forms that a crystal belonging to any system may present, and cannot therefore be a certain guide to the choice of an elementary or primary form.

Magnitude of Indices.—It has also been suggested that the numerical values of the indices of secondary faces should in some degree influence the choice of the form to be regarded as the primary. For as it has been observed that in the cubic system the highest numerical index is seldom greater than 6, it has been inferred as a sort of law that all indices ought to be expressible in low numbers, and that any form which allows the other forms of the crystal to be denoted by the lowest numbers, is on that account entitled to a preference as a primary form.

But on looking through the lists of indices in the new edition of William Phillips's 'Mineralogy,' several instances occur of indices as high as 23, and there does not appear theoretically to be any reason why still higher numbers might not be so employed, as they occasionally are in the accompanying Tables, and in a recent paper on Quartz by M. Descloizeaux, in which indices of observed *direct* faces are as high as 32, and of *direct* faces corresponding to observed *inverse* faces as high as 80.

The terms *direct* and *inverse* faces relate only to rhombohedral crystals, and may be explained as follows:—

Let $h\,k\,k$ be the symbol of any direct face in the quadrant 1 1 1, 1 0 0 of the zone 1 1 1 1 0 0, and let $p\,q\,q$ be that of a face in the same zone lying on the opposite side of the principal axis of the crystal and making an equal angle with the face 1 1 1. Then $p\,q\,q$ is termed an inverse face.

It frequently happens that direct faces and the corresponding inverse faces occur in pairs upon the same crystal.

Sometimes, however, a face occurs in a direct position without a corresponding inverse face, and sometimes an inverse face occurs without any corresponding direct face. In the cases where the inverse faces occur alone, the symbols of the corresponding direct faces have been computed and entered in the accompanying Tables, R 1 and R 2, as those of direct faces.

But it is necessary to consider the subject of high indices rather more in detail.

The indices of faces do not express absolute numbers high or low, but only the ratios of the numbers so employed. Thus the symbol of 1 0 2 is equivalent to, and denotes geometrically, the same face as 2 0 4, 3 0 6, 10 0 20, 99 0 198, or any other numbers in the same ratio.

It is usual to express indices in their lowest terms in order to avoid writing figures unnecessarily, and to save time and trouble in the calculations of indices and angles. But it has the disadvantage of masking, and thus keeping out of sight, the true relations among such indices as are now under consideration.

Let it, for example, be proposed to ascertain what faces can occur in the pyramidal system between the faces 1 0 1 and 1 0 3. The face 1 0 2 first presents itself as the intermediate face. But let 1 0 1 be expressed by 2 0 2, and 1 0 3 by 2 0 6, and 2 0 3, 2 0 4, and 2 0 5 appear as intermediate faces, and present to the eye a relation which is not so immediately perceived if we state the numbers as 2 0 3, 1 0 2, 2 0 5. If the expressions 1 0 1 and 1 0 3 become 3 0 3 and 3 0 9, the symbols 3 0 4, 3 0 5, 3 0 6 (=1 0 2), 3 0 7, 3 0 8

appear to be those of possible intermediate faces. And when higher multiples are employed, the number of possible intermediate faces is increased.

If the multiplier is 20 the original symbols become 20 0 20 and 20 0 60, showing thirty-nine possible intermediate faces between 1 0 1 and 1 0 3, as in the following list:—

20 0 20 ..... = 1 0 1
20 0 21
20 0 22 = 10 0 11
20 0 23
20 0 24 = 10 0 12 = 5 0 6
20 0 25 ..... = 4 0 5
20 0 26 = 10 0 13
20 0 27
20 0 28 = 10 0 14 = 5 0 7
20 0 29
20 0 30 = 10 0 15 = 4 0 6 = 2 0 3
20 0 31
20 0 32 = 10 0 16 = 5 0 8
20 0 33
20 0 34 = 10 0 17
20 0 35 ..... = 4 0 7
20 0 36 = 10 0 18 = 5 0 9
20 0 37
20 0 38 = 10 0 19
20 0 39
20 0 40 = 10 0 20 = 4 0 8 = 2 0 4 = 1 0 2
20 0 41
20 0 42 = 10 0 21
20 0 43
20 0 44 = 10 0 22 = 5 0 11
20 0 45 ..... = 4 0 9
20 0 46 = 10 0 23
20 0 47
20 0 48 = 10 0 24 = 5 0 12
20 0 49
20 0 50 = 10 0 25 = 4 0 10 = 2 0 5
20 0 51
20 0 52 = 10 0 26 = 5 0 13
20 0 53
20 0 54 = 10 0 27
20 0 55 ..... = 4 0 11
20 0 56 = 10 0 28 = 5 0 14
20 0 57
20 0 58 = 10 0 29
20 0 59
20 0 60 = 10 0 30 = 5 0 15 ..... = 1 0 3

It is apparent from this list, that when the number of faces in a zone increases, one or more of the indices by which the additional faces can be expressed must become larger.

It is equally clear, that when the number of faces in a zone increases, the angles between adjacent faces must become less, and hence when the angle between any two adjacent faces in a zone becomes less, one or more of the indices required to denote one or both of such faces must become larger. To limit therefore the magnitude of indices would virtually be to limit the number of possible faces in a zone.

We have before shown that the indices of faces are the reciprocals of the proportions of the primary edges conceived to be cut away by such faces, and in order to provide an exact expression of these proportions, in the cases of high indices, we have only to imagine the primary edges divided into a sufficiently large number of equal parts to allow of such an exact expression.

It is evident that the symbols in the preceding list accurately denote separate faces. But it may be asked, are all these faces equally possible, and if so, are they equally probable; or is 20 0 40 more probable than 20 0 39, because the indices 20 0 40 can be divided by 20 without a remainder, and because those of 20 0 39 cannot? There does not appear to be any theoretical reason why any one of these faces should be more probable than any other.

But it may be said that the face 20 0 40 has been frequently observed and expressed by the symbol 1 0 2, while there is no published record of the faces 20 0 39 or 20 0 41 having ever been noticed, and that nature has therefore shown a preference of one of

these faces to either of the others, and has thus established a sort of probability that the face so preferred will occur more frequently than the others.

Is it, however, true that the others have not been observed, although the observations have not been recorded; or is it not equally probable that such faces have been frequently seen and measured, but from the nearness of their angles with other faces, to the angles of the face 1 0 2 with the same other faces, they have been taken as imperfect instances of the face 1 0 2?

If the three faces above referred to should occur on the same crystal they might doubtless be distinguished, but if they occur on different crystals there is only one natural test for ascertaining whether they are really different, or are only imperfect examples of the same face.

This test is, to find whether the face on one of the crystals is common to any two zones, and whether the face on the other crystal is common to the same two zones. If the faces are common to the same two zones on both crystals, the faces are similar, and if they are not, they are different faces. But from the occurrence of crystals generally in only small fragments, it does not often happen that any of these are sufficiently perfect to present clearly the faces of any two required zones, and hence the difficulty of ascertaining with certainty the identity or otherwise of such nearly corresponding faces.

But although there may be no theoretical limit to the magnitude of indices, there is a practical limit, beyond which faces denoted by high indices would not be distinguishable with certainty from each other. This limit occurs when the difference of the angles of any two faces with some other face is within the probable range of error of ordinary measurement, arising from imperfection of faces, or from imperfect observation. It is difficult to state the probable amount of error which may be ascribed to these causes, but perhaps about half a degree to a degree may not be an improbable quantity.

The following are the several angles between the face 0 0 1 and the first and last ten faces in the preceding list:—

20 0 21 = 43° 11′	20 0 26 = 37° 9′	20 0 51 = 21° 8′	20 0 56 = 19° 23′
20 0 22 = 41 51	20 0 27 = 36 7	20 0 52 = 20 45	20 0 57 = 19 4
20 0 23 = 40 35	20 0 28 = 35 8	20 0 53 = 20 24	20 0 58 = 18 46
20 0 24 = 39 23	20 0 29 = 34 12	20 0 54 = 20 3	20 0 59 = 18 28
20 0 25 = 38 15	20 0 30 = 33 18	20 0 55 = 19 42	20 0 60 = 18 11

It appears from the small differences in the angles in this statement, that some of the adjacent faces in the preceding list might not be distinguishable from each other by ordinary measurement, and that the zone test already alluded to might here become necessary.

But besides these possible sources of error in our crystallographic investigations, another element of disagreement occurs in practical crystallography.

The ordinary method of examining a crystal is, first, to determine, from the character and relative positions of its faces, the system to which it belongs.

With regard to the cubic, pyramidal, and rhombohedral systems, this process is

attended with little difficulty; but in the cases of the prismatic, oblique, and anorthic systems it is far from definite.

Having determined the system to which the crystal belongs, the next operation is to measure the angles between the several faces, and to select from among these the angle which is to be regarded as elementary. And as the faces themselves afford no assistance in fixing upon a proper angle, the selection is left to the arbitrary choice of the observer.

After having fixed upon the elementary angle, the symbols of the faces which at their intersection produce that angle are determined, and from these and the known angles between other faces, other symbols and angles may be found.

It is usual to measure as large a number of angles as the crystals in our possession will allow. And when the observed angles between apparently similar pairs of faces are slightly different, these slight differences have been hitherto ascribed to the imperfection of the measured faces, and a mean of the measured angles has in this case been assumed as the probable angle between the faces. But as it seldom if ever happens that the mean of any number of measured angles agrees exactly with the geometric angle corresponding to the assumed symbols of the faces yielding that angle, it is usual to *adjust*, as it is termed, the symbols and angles in question, by increasing or diminishing the value of the measured angles, and by altering, if necessary, the indices until both the angles and indices are in conformity with the elementary faces and angles of the crystals.

This process of adjustment is the element of disagreement above alluded to, and is really the adapting or fitting to a particular primary form the angles and symbols of the other faces of the crystal, so that if a new primary form has to be assumed, the new symbols may become expressible in only very high numbers, and a readjustment of angles and symbols has then to take place. This process will, however, be governed by the number and greater or less degree of perfection of the observed faces, and by the care with which the angles have been measured.

In nearly all the hitherto published works on mineralogy the faces of crystals have, for the convenience of writing and printing, been distinguished by letters. But it appears to the writer that the use of WHEWELL'S symbols for this purpose instead of letters, is so much to be preferred, on account of its presenting at once to the eye the geometrical relations of the several faces, as to more than counterbalance the inconvenience of having three or more figures to write instead of one letter; and particularly as these letters are frequently accompanied by one or more conventional marks of distinction which are troublesome both to write and to remember.

Construction of Tables.—We have now to explain the construction of the accompanying Tables, to consider the facts they present, and their results.

These Tables do not contain all the observed angles of the crystals belonging to the different minerals, but only such as have been, or might have been used in each case as elementary angles, these being sufficient for the purposes of this paper.

No table is given of the *cubic system*, as that system consists of only a single strictly isomorphous group.

In the Tables of the two other systems the names of the minerals at present comprised respectively in each, are given in an upper horizontal line in the order in which they occur in the new edition of the work of WILLIAM PHILLIPS.

In the *pyramidal system*, Table P 1 (Plate II.), the first vertical column below the name of each mineral contains the angles between the face 0 0 1 and the observed faces in the quadrant (0 0 1....1 0 0).

The third column contains the angles between the face 0 0 1 and the observed faces in the quadrant (0 0 1....1 1 0).

The second and fourth columns contain the symbols hitherto assigned to the several faces which make with the face 0 0 1 the angles shown in the first and third columns.

In the *rhombohedral system*, Table R 1 (Plate IV.), the first vertical column under each mineral contains the angles between the face 1 1 1 and the observed direct faces in the quadrant (1 1 1...1 0 0), and also the angles between the face 1 1 1 and some observed inverse faces in the quadrant (1 1 1...$\bar{2}$ 1 1).

Facts presented by these Tables.—The most important of these are the horizontal ranges of nearly equal angles, as shown in each system, and the general disagreement in the symbols of the faces which make with some other face those nearly agreeing angles.

With regard to these facts no difference of opinion can arise, unless the sources from which they have been derived (some of the most recent works on mineralogy) are incorrect. And it is possible that from inaccurate transcription, or error in printing, some of the angles may not be correctly given, and hence perhaps may have arisen some of the anomalies presented by the Tables.

But differences of opinion may be entertained relative to the interpretation of these facts.

The interpretation to which the writer inclines, is that the near agreement in angle between two corresponding faces is not simply accidental, but that it is the effect of some natural relation, not hitherto noticed, among all the crystals in each respective system; and hence that where the angles between particular faces nearly agree, there ought to be a corresponding agreement in the forms of their symbols.

It would seem from these disagreements between angles and symbols that a tacit impression has existed in the minds of crystallographers, that the crystals of different minerals are always to be regarded as individual and isolated cases of crystallization, except in the instances of isomorphous groups.

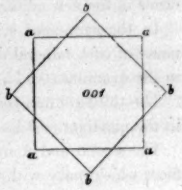
The idea that first suggested itself to the mind of the writer as to the nature of these newly observed relations was, that as in each system the angles between particular faces nearly agree, there might also be a near agreement in the elementary angles of all the crystals belonging to each system; and an examination of the crystals in the pyramidal and rhombohedral systems with a view of ascertaining how far this idea might be well founded, has led to the very unexpected result, that a geometrical isomorphism can be shown to exist throughout each of these two systems, and consequently that similar relations may be imagined to exist in the others.

The following remarks will further illustrate and explain the relations in question.

Pyramidal tables, P 1, P 2 (Plates II. and III).—In the pyramidal system there are in the

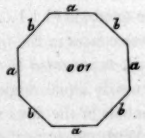
case of each mineral two possible prisms, *a* and *b*, of equal dimensions, but related to each other in position in the manner shown in the annexed sketch, fig. 7, which is a projection of the two intersecting prisms on a plane parallel to their common terminal face 0 0 1. When these prisms occur together in natural crystals, they present the outline shown in fig. 8. From which it is evident, that if the prism *a* be taken as the primary prism, the faces of *b* will appear to truncate the lateral edges of *a*; and conversely, if *b* is taken as the primary. It is also clear that any face which truncates a terminal edge of *a* would be a truncation of a solid angle of *b*, and conversely.

Fig. 7.



It does not appear that natural crystals afford any indication to govern our choice of one or the other of these prisms as the primary form of any mineral; and as relates to the crystals of any single mineral, it is indifferent which prism we select. But it is not so with respect to the crystals of any group of two or more different minerals which have afterwards to be compared with each other.

Fig. 8.



In these cases the same prism, *a* or *b*, should be assumed as the primary prism throughout the group. And as the position of *a* is that in which pyramidal crystals are usually drawn, the faces of all the crystals exhibited in the Tables P 1 and P 2, are referred to elementary prisms assumed to be in the position of *a*.

It has hence been necessary where *b* has been given in previous works, and in Table P 1 (Plate II.), as the elementary prism of some of the minerals, to substitute *a* for *b* in Table P 2 (Plate III.), and to make the necessary changes in the symbols of the other faces consequent upon the change of the elementary prism.

It appears in Table P 1 (Plate II.), that the faces of different minerals making equal or nearly equal angles with the face 0 0 1 have been referred by different observers to a prism in the position of *a* in some cases, and to a prism *b* in others.

Thus a face which in Towanite makes an angle of 54° 20′ with the face 0 0 1, has for its symbol 1 1 1, denoting a face replacing a solid angle; while in Braunite, a face making the same angle with 0 0 1, has for its symbol 0 1 1, denoting a face on an edge, and showing that if the prism *a* has been used for the primary of Towanite, the prism *b* has been used for that of Braunite.

Again, it will be seen in Table P 1 (Plate II.), that two of the faces of Scheelite make angles of 36° 34′ and 46° 22′ respectively with the face 0 0 1, and that two faces making the same angles with 0 0 1 occur in Wulfenite. Yet, instead of regarding these as angles between corresponding faces in the two minerals, they are referred in Scheelite to the prism *a*, and to *b* in Wulfenite. And as if further to complicate and conceal the identity of these crystalline forms, the symbols assigned to their corresponding faces also disagree.

Having reduced the elementary prisms throughout the system to corresponding positions, the next point for consideration has been the angle which might be selected as a proximate elementary angle of the group.

A guide to this selection presents itself in the crystals of Apophyllite, the angles of which scarcely differ from those of a cube, and therefore admit an elementary angle of exactly 45°, which angle has been assumed as a variable type of the elementary angles throughout the system.

Whether this angle of 45° is accidental, or is the effect of any at present unknown relation between the cube and the square prism, the writer is not aware. But it is a rather curious fact that there should also be in the rhombohedral system a series of elementary angles of about 55°, not differing much from angles between two corresponding faces of a cube.

On referring to Table P 1 (Plate II.), it will be seen that the crystals of Somervillite, Faujasite and Romeine are represented by single faces only, to each of which the symbol 1 0 1 has been hitherto applied. But as the occurrence of only those particular faces on the three crystals must be regarded as purely accidental, there is no reason for applying that particular symbol to either of them. And as the symbols of faces are not in any manner indicated by the faces themselves, we are at liberty to assign to those or to any other faces any symbols which are consistent with the geometrical relations of the faces in question. And hence have arisen some of the differences of symbols shown in the Tables P 1 and P 2 (Plates II. and III.).

The elementary angle of the crystals of Towanite being 44° 35′, and as those crystals present as great a number of observed faces as those of any other mineral in this group, and as the position of the primary form may be assumed to be that of the prism *a*, the symbols and angles of this mineral have been used as standards for comparing and regulating the symbols of the remainder of the group, as shown in Table P 2 (Plate III.).

The method of reducing the series of minerals in each of the pyramidal and rhombohedral systems to single groups, has been to form in the first instance in each case small groups, and then to associate these into larger ones.

Thus in Table P 1 (Plate II.), there appears in

Towanite . an angle of 19° 13′, corresponding to the symbol 1 1 4
Anatase . . an angle of 19 34, corresponding to the symbol 1 1 7
Apophyllite an angle of 19 30, corresponding to the symbol 1 1 5,

forming a small geometrical group, to each member of which in Table P 2 the symbol 1 1 4 of Towanite has been assigned as a common and connecting symbol of the group.

Again, in Table P 1 (Plate II.), in

Towanite . an angle of 33° 18′ has the symbol 2 0 3
Rutile . . an angle of 32 47 has the symbol 1 0 1
Ox. Tin . an angle of 33 55 has the symbol 1 0 1
Zircon . . an angle of 32 38 has the symbol 1 0 1
Somervillite, an angle of 32 51 has the symbol 1 0 1,

producing another analogous group under the connecting symbol 2 0 3 of Towanite, and thus directly associating four other minerals with Towanite.

But this direct method has not in all cases been practicable, and it has consequently been necessary in some instances to associate some small groups indirectly; that is, to form small groups not immediately connected with Towanite, and then to connect them with Towanite by means of some intermediate minerals.

Thus it is already shown, that Anatase and Apophyllite may be regarded as plesiomorphous with Towanite.

But Anatase may also be associated with Uranite by the angles—

Anatase 39° 57′, Uranite . 39° 53′;

with Calomel and Matlockite by the angles—

Anatase 60° 38′, Calomel . 60° 9′,
Matlockite 60 36,

thus connecting indirectly Uranite, Calomel, and Matlockite, with Towanite. And by proceeding in this manner, it has been found that the entire series of pyramidal crystals can be brought to agree very nearly in their elements with those of Towanite.

In Table P 1 the series of elementary angles range from 21° 5′ to 61° 38′. In P 2 the new series, chiefly of angles which, although probable to occur, have not yet been observed, appears as a horizontal line of nearly agreeing angles, the greater part of which have been computed from observed faces.

Table P 2 also exhibits the changes of symbols consequent upon the adoption of these angles as elementary, and upon the changes of the fundamental prism from *b* to *a* where required. The entire series of pyramidal crystals is thus presented as a single geometrical plesiomorphous group.

Rhombohedral Tables, R 1, R 2 (Plates IV. and V.).—After the explanations already given relative to the pyramidal tables, only a few additional remarks will be necessary concerning those of the rhombohedral system.

The construction of Table R 1 has been already explained, and the distinction between direct and inverse faces has been pointed out.

On referring to the Table R 1 (Plate IV.), we may perceive the same horizontal ranges of nearly equal angles, and the same disparity between angles and symbols, as in the pyramidal system; and we may also observe, that the rhombohedral elementary angles range in this Table from 20° 50′ in Millerite to 81° 20′ in Parisite. It will also be seen in this system, as in the pyramidal, that where a mineral is represented by a single face, the angle of that face with the face 1 1 1 is assumed to be the elementary angle of that mineral, disregarding altogether its apparent relation to the nearly equal angles between the corresponding faces of other minerals.

In the construction of Table R 2, the first point to be considered was the angle which should become the elementary type of the system.

It appears in R 1, that there is one principal horizontal range of nearly agreeing angles of about 37° or 38°; another, of about 43° to 46°; and another, of about 55° to 58°, either of which might supply this elementary type: but as there are more instances of

agreeing angles in the range between 55° and 58° than in the others, the elementary angle of 57° 30′ of Hematite has been adopted as the elementary type.

Now Hematite may be associated with Quartz by the angles—

Hematite 75° 42′, Quartz 75° 18′,
Hematite 82 44, Quartz 82 31,

and with Calcite by the angles—

Hematite 38° 7′, Calcite 38° 17′,
Hematite 44 27, Calcite 44 37,

forming a small plesiomorphous group, with which other individuals and other groups have been successively connected, so as to reduce the entire system to a single plesiomorphous group of nearly equal angles, as shown in Table R 2.

To exhibit the effect of this grouping upon Hematite, Quartz, and Calcite, the Table T has been framed, by first assuming new primary forms for Quartz and Calcite, then transferring to each of the three minerals the symbols of the other two, and then calculating in each case the angles corresponding to the transferred symbols.

Symbols common to	T. Corresponding Angles of Hematite.	Quartz.	Calcite.	Symbols common to	T. Corresponding Angles of Hematite.	Quartz.	Calcite.
6 5 5	° ′ 5 36	° ′ 5 26	° ′ 5 15	10 $\bar{1}$ $\bar{1}$	° ′ 65 8	° ′ 64 30	° ′ 63 50
10 7 7	11 6	10 47	10 29	68 $\bar{7}$ $\bar{7}$	65 22	64 43	64 3
8 5 5	14 40	14 15	13 52	8 $\bar{1}$ $\bar{1}$	66 59	66 23	65 45
7 4 4	17 26	16 57	16 29	13 $\bar{2}$ $\bar{2}$	69 5	68 31	67 55
2 1 1	21 25	20 52	20 18	5 $\bar{1}$ $\bar{1}$	72 20	71 50	71 20
11 5 5	24 9	23 32	22 55	17 $\bar{4}$ $\bar{4}$	74 44	74 18	73 51
5 2 2	27 37	26 56	26 15	4 $\bar{1}$ $\bar{1}$	75 42	75 18	74 53
3 1 1	32 6	31 22	30 37	19 $\bar{5}$ $\bar{5}$	76 34	76 11	75 47
22 7 7	33 11	32 25	31 40	82 $\bar{23}$ $\bar{23}$	77 41	77 19	76 57
17 5 5	34 54	34 7	33 20	7 $\bar{2}$ $\bar{2}$	78 0	77 40	77 18
4 1 1	38 7	37 19	36 30	10 $\bar{3}$ $\bar{3}$	78 54	78 35	78 15
13 3 3	39 35	38 44	37 55	23 $\bar{7}$ $\bar{7}$	79 11	78 52	78 32
31 7 7	39 56	39 10	38 17	3 $\bar{1}$ $\bar{1}$	80 57	80 41	80 25
6 1 1	44 27	43 37	42 46	122 $\bar{43}$ $\bar{43}$	82 5	81 51	81 37
20 3 3	45 45	44 54	44 3	17 $\bar{6}$ $\bar{6}$	82 7	81 53	81 38
7 1 1	46 18	45 28	44 37	11 $\bar{4}$ $\bar{4}$	82 44	82 31	82 18
13 1 1	51 28	50 39	49 49	35 $\bar{13}$ $\bar{13}$	83 11	82 59	82 47
14 1 1	51 54	51 5	50 15	8 $\bar{3}$ $\bar{3}$	83 24	83 12	82 59
16 1 1	52 36	51 47	50 58	21 $\bar{8}$ $\bar{8}$	83 44	83 33	83 19
20 1 1	53 35	52 47	51 57	76 $\bar{29}$ $\bar{29}$	83 46	83 35	83 24
1 0 0	57 30	56 44	55 57	19 $\bar{8}$ $\bar{8}$	85 53	85 50	85 42
11 $\bar{1}$ $\bar{1}$	64 27	63 48	63 7	116 $\bar{49}$ $\bar{49}$	86 1	85 54	85 47

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The assumed new primary form of Quartz has been observed by M. DESCLOIZEAUX as an inverse and as a direct face, and gives 56° 44′ as the angle between the new 1 0 0 and 1 1 1.

The new primary form of Calcite corresponds to the old direct form $8\,\bar{1}\,\bar{1}$, deduced from the old inverse form $\bar{4}\,5\,5$, and gives 1 0 0, 1 1 1=55° 57′.

Table T also contains many additional symbols with their corresponding angles, to show the relations between greatly disagreeing symbols and nearly agreeing angles.

Thus the angle of 1 1 1 with the face denoted by $82\,\overline{23}\,\overline{23}$, differs but 19′ from the nearly corresponding angle of 1 1 1 with the face denoted by $7\,\bar{2}\,\bar{2}$. The difference between the angle 1 1 1, $122\,\overline{43}\,\overline{43}$, and 1 1 1, $17\,\bar{6}\,\bar{6}$ is only 2′, rendering the distinction of these faces by ordinary measurement impracticable.

This Table is open to other remarks of the same nature, but after the previous explanations it appears unnecessary to pursue the subject further.

These new elementary angles, as well as those in the pyramidal system, will doubtless undergo alterations as more numerous and better crystals present themselves for examination.

IV. *On the Equation of* LAPLACE'S *Functions, &c.* *By* W. F. DONKIN, *M.A., F.R.S., F.R.A.S., Savilian Professor of Astronomy in the University of Oxford.*

Received December 3,—Read December 11, 1856.

SECTION I.

THE equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0,$$

transformed to polar coordinates by putting

$$x=r\sin\theta\cos\varphi,\quad y=r\sin\theta\sin\varphi,\quad z=r\cos\theta,$$

becomes, as is well known,

$$\frac{d^2u}{d\theta^2}+\cot\theta\frac{du}{d\theta}+\frac{1}{(\sin\theta)^2}\frac{d^2u}{d\varphi^2}+r^2\frac{d^2u}{dr^2}+2r\frac{du}{dr}=0,$$

which may be written in the form

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\varphi}\right)^2+(\sin\theta)^2r\frac{d}{dr}\left(r\frac{d}{dr}+1\right)\right\}u=0; \quad \ldots\ldots \quad (1.)$$

and if it be assumed that

$$u=u_0+u_1r+u_2r^2+\ldots\ldots,$$

then by substituting this value of u in either of the two equations last written, we find that u_n is a function of θ and φ satisfying the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\varphi}\right)^2+n(n+1)(\sin\theta)^2\right\}u_n=0, \quad \ldots\ldots \quad (2.)$$

which, under a slightly different form, is commonly called the *Equation of* LAPLACE'S *Functions*.

This equation was first solved in finite terms by Mr. HARGREAVE*, but in a form very inconvenient for practical applications. A solution free from this objection was afterwards obtained by Professor BOOLE†, by a method explained in his memoir "On a General Method in Analysis," Philosophical Transactions, 1844. Lastly, in the second volume of the Journal referred to, the same mathematician gave two more solutions, one of which however is reducible, as he states, to Mr. HARGREAVE'S form; the other, though much more convenient than this, is still for most purposes probably less useful than that given in the first volume, from which it differs essentially in form, as well as in the method by which it is deduced.

* Philosophical Transactions, 1841.

† Cambridge and Dublin Mathematical Journal, vol. i. p. 10.

In the following pages I have treated the equation (2.) by a very simple method*. The result bears a general resemblance to Professor BOOLE's first solution, and I conceive that the two forms must be capable of being identified by the assumption of a proper relation between the arbitrary functions; but I am not able at present to show this identity.

Some further investigations are added, which it is unnecessary to notice beforehand.

1. Putting k for $\frac{d}{d\phi}$ in equation (2.), we have

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+k^2+n(n+1)(\sin\theta)^2\right\}u_n=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

In the case of $n=0$, this becomes

$$\left(\left(\sin\theta\frac{d}{d\theta}\right)^2+k^2\right)u_0=0, \text{ or } \frac{d^2u_0}{dt^2}+k^2u_0=0$$

$\left(\text{where } t=\log\tan\frac{\theta}{2}\right)$. Hence

$$u_0=C_1\left(\tan\frac{\theta}{2}\right)^{k\sqrt{-1}}+C_2\left(\tan\frac{\theta}{2}\right)^{-k\sqrt{-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

If, however, $k=0$, we should have

$$u_0=C_1+C_2\log\tan\frac{\theta}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

* It may be worth while to give a preliminary illustration of this method (which contains no novelty except in detail), by applying a similar process to the well-known equation

$$\frac{d^2u}{dx^2}+\left(k^2-\frac{n(n+1)}{x^2}\right)u=0.$$

This may be written in the form

$$\left\{\left(D-\frac{n}{x}\right)\left(D+\frac{n}{x}\right)+k^2\right\}u=0, \quad \left(D=\frac{d}{dx}\right);$$

now

$$\left(D+\frac{n}{x}\right)\left(D-\frac{n}{x}\right)=\left(D-\frac{n-1}{x}\right)\left(D+\frac{n-1}{x}\right);$$

hence if we put $u=\left(D-\frac{n}{x}\right)v$, and then operate on each side with $\left(D-\frac{n}{x}\right)^{-1}$, we have

$$\left\{\left(D-\frac{n-1}{x}\right)\left(D+\frac{n-1}{x}\right)+k^2\right\}v=\left(D-\frac{n}{x}\right)^{-1}0.$$

Assuming 0 as the value of the right-hand member, and then putting $v=\left(D-\frac{n-1}{x}\right)w$, and so on successively, it is obvious that we shall ultimately have

$$u=\left(D-\frac{n}{x}\right)\left(D-\frac{n-1}{x}\right)\cdots\left(D-\frac{1}{x}\right)z,$$

$$(D^2+k^2)z=0;$$

hence, observing that $D-\frac{i}{x}=x^iDx^{-i}$, we obtain finally,

$$u=x^n\left(D\frac{1}{x}\right)^n(C_1\sin kx+C_2\cos kx),$$

which agrees with the result obtained by Professor BOOLE's general method (Philosophical Transactions, 1844).

which is the complete solution of (2.) in the particular case in which u_n does not contain φ, and $n=0$.

But putting $\frac{d}{d\varphi}$ for k in (4.), and arbitrary functions of φ instead of C_1, C_2, we have

$$u_0=f\left(\varphi+\sqrt{-1}.\log\tan\tfrac{\theta}{2}\right)+\mathrm{F}\left(\varphi-\sqrt{-1}.\log\tan\tfrac{\theta}{2}\right); \quad . \quad . \quad . \quad (6.)$$

instead of which we may evidently write

$$u_0=f\left(e^{\varphi\sqrt{-1}}\tan\tfrac{\theta}{2}\right)+\mathrm{F}\left(e^{-\varphi\sqrt{-1}}\tan\tfrac{\theta}{2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

as well as other forms. Either of these is the complete value of u_n in the case of $n=0$.

2. Returning to the general case, if we put for shortness

$$\left(\sin\theta\frac{d}{d\theta}\right)^2+n(n+1)(\sin\theta)^2=\varrho_n,$$

it is evident that

$$\varrho_{-n}=\varrho_{n-1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

and also, since

$$\left(\sin\theta\frac{d}{d\theta}+n\cos\theta\right)\left(\sin\theta\frac{d}{d\theta}-n\cos\theta\right)=\left(\sin\theta\frac{d}{d\theta}\right)^2+n(n+1)(\sin\theta)^2-n^2,$$

if we put

$$\varpi_n=\sin\theta\frac{d}{d\theta}+n\cos\theta,$$

we have

$$\varpi_n\varpi_{-n}+n^2=\varrho_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

from which, changing n into $-n$, and observing (8.), we get

$$\varpi_{-n}\varpi_n+n^2=\varrho_{n-1};$$

on the other hand, changing n into $n-1$ in (9.),

$$\varpi_{n-1}\varpi_{-(n-1)}+(n-1)^2=\varrho_{n-1}.$$

Comparing the last two results, we see that the operation ϖ_n possesses the following property, namely,

$$\varpi_{-n}\varpi_n+n^2=\varpi_{n-1}\varpi_{-(n-1)}+(n-1)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

Now the equation to be solved (3.), art. 1, is

$$(\varpi_n\varpi_{-n}+n^2+k^2)u_n=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.)$$

Let $u_n=\varpi_n v$, and let the operation ϖ_n^{-1} be performed on each side; assuming for the present that $\varpi_n^{-1}0$ may be put $=0$ without ultimate loss of generality, we obtain

$$(\varpi_{-n}\varpi_n+n^2+k^2)v=0;$$

and this, by virtue of (10.), becomes

$$(\varpi_{n-1}\varpi_{-(n-1)}+(n-1)^2+k^2)v=0;$$

which, compared with (11.), gives the relation $v=u_{n-1}$, and consequently, since $u_n=\varpi_n v$,

$$u_n=\varpi_n u_{n-1},$$

or

$$u_n=\left(\sin\theta\frac{d}{d\theta}+n\cos\theta\right)u_{n-1}.$$

This relation is true whatever be the value of n. But as we are now supposing n a positive integer, it is evident that we have

$$u_n = \varpi_n \varpi_{n-1} \ldots \varpi_2 \varpi_1 u_0, \quad \ldots \quad (12.)$$

where u_0 is given by (6.) or (7.) of the last article in the general case, and by (5.) in the case in which u_n is a function of θ without φ.

This solution contains two arbitrary functions in the former case, and two arbitrary constants in the latter. It is therefore, *in general*, the complete solution. (See, however, arts. 8, 9, 17, &c.)

3. The operative symbol $\varpi_n \varpi_{n-1} \ldots \varpi_2 \varpi_1$, written at length, is

$$\left(\sin\theta\frac{d}{d\theta}+n\cos\theta\right)\left(\sin\theta\frac{d}{d\theta}+(n-1)\cos\theta\right)\ldots\left(\sin\theta\frac{d}{d\theta}+\cos\theta\right). \quad \ldots \quad (13.)$$

Now it is evident that

$$\sin\theta\frac{d}{d\theta}+n\cos\theta=\frac{1}{(\sin\theta)^{n-1}}\frac{d}{d\theta}(\sin\theta)^n$$

(the subject of operation being of course omitted on both sides); and if this be put in the form

$$\frac{1}{(\sin\theta)^n}\sin\theta\frac{d}{d\theta}\sin\theta(\sin\theta)^{n-1},$$

it will be easily seen that the expression (13.) is equivalent to

$$\frac{1}{(\sin\theta)^n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n, \quad \ldots \quad (14.)$$

which may also be put in the less symmetrical form

$$\frac{1}{(\sin\theta)^{n-1}}\left((\sin\theta)^n\frac{d}{d\theta}\right)^n\sin\theta.$$

Either of these forms has the peculiarity, as compared with (13.), of being intelligible without supposing n a positive integer (setting aside the difficulties of general differentiation).

4. The results of the preceding articles may be summed up as follows:—

The complete solution of the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+n(n+1)(\sin\theta)^2\right\}u_n=0 \text{ is}$$

$$u_n=(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(C_1+C_2\log\tan\frac{\theta}{2}\right); \quad \ldots \quad (15.)$$

and the complete solution of the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\varphi}\right)^2+n(n+1)(\sin\theta)^2\right\}u_n=0 \text{ is}$$

$$u_n=(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left\{f\left(e^{\varphi\sqrt{-1}}\tan\frac{\theta}{2}\right)+F\left(e^{-\varphi\sqrt{-1}}\tan\frac{\theta}{2}\right)\right\}. \quad \ldots \quad (16.)$$

It may be observed that (15.) is deducible from (16.) by taking $f(x)=C_1+\frac{1}{2}C_2\log x$, $F(x)=\frac{1}{2}C_2\log x$.

5. The expression (16.) may be compared with the following result of Professor BOOLE's first solution*:—

$$u_n = \mathrm{F}\left(\mu, \frac{\sqrt{(1-\mu^2)}}{\mu} e^{\phi\sqrt{-1}}\right),$$

where $\mathrm{F}(\mu, e^{\phi\sqrt{-1}})$

$$= \mu^n \left(\frac{d}{d\mu}\frac{1}{\mu}\right)^n \left\{(\mu+\mu^2)^n \psi\left(\frac{\mu e^{\phi\sqrt{-1}}}{1+\mu}\right) + (\mu-\mu^2)^n \chi\left(\frac{\mu e^{\phi\sqrt{-1}}}{1-\mu}\right)\right\}$$

and $\mu = \cos\theta$. I have stated above my impression that this form must be capable of transformation into identity with (16.); but it is probable that this can only be effected by means of some not obvious theorem of differentiation. Examples of such theorems will be seen below (arts. 14 and 16).

SECTION II.

6. I proceed to discuss some particular applications of the formulæ obtained above.

If the expression $(1-2r\cos\theta+r^2)^{-\frac{1}{2}}$ be developed in the form

$$p_0 + p_1 r + p_2 r^2 + \ldots,$$

then p_n satisfies the equation of LAPLACE's functions, and does not contain ϕ. Hence p_n must be given by the expression (15.), art. 4, if the constants be properly assumed; and since it is evident that C_n must be $=0$ in this case, we have

$$p_n = c_n (\sin\theta)^{-n} \left(\sin\theta \frac{d}{d\theta}\sin\theta\right)^n . 1,$$

where c_n is a function of n.

This form of p_n may be independently verified, and c_n determined at the same time, as follows:—

Put $r(\sin\theta)^{-1} = \varrho$, then

$$(1-2r\cos\theta+r^2)^{-\frac{1}{2}} = \frac{1}{\sin\theta}\left(1+(\cot\theta-\varrho)^2\right)^{-\frac{1}{2}},$$

and if we put $\cot\theta = t$, this may be written

$$\sqrt{1+t^2}\, e^{-\varrho\frac{d}{dt}} \frac{1}{\sqrt{1+t^2}},$$

where ϱ is to be treated as constant till after all operations. Now in general, if P, Q be any symbols whatever, quantitative or operative, we know that $\mathrm{P}f(\mathrm{Q})\frac{1}{\mathrm{P}} = f\left(\mathrm{PQ}\frac{1}{\mathrm{P}}\right)$. Hence it is evident that since $\sqrt{1+t^2} = \frac{1}{\sin\theta}$, and $-\frac{d}{dt} = (\sin\theta)^2\frac{d}{d\theta}$, the above expression is equivalent to

$$e^{\varrho\sin\theta\frac{d}{d\theta}\sin\theta} . 1,$$

and therefore this is a symbolical form of $(1-2r\cos\theta+r^2)^{-\frac{1}{2}}$. Restoring the value of ϱ after development, we find that the coefficient of r^n is

* Cambridge and Dublin Journal, vol. i. p. 18.

$$\frac{1}{1.2.3\ldots n}(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^{n}.1.$$

In other words, if we put $v_n=(\sin\theta)^{-n}(\sin\theta\frac{d}{d\theta}\sin\theta)^n.1$, then

$$(1-2r\cos\theta+r^2)^{-\frac{1}{2}}=v_0+v_1\frac{r}{1}+v_2\frac{r^2}{1.2}+\ldots+v_n\frac{r^n}{1.2\ldots n}+\ldots\ldots$$

(From this it is easy to deduce the relation $v_n=(2n-1)\cos\theta.v_{n-1}-(n-1)^2v_{n-2}$ by means of the known relation between the coefficients in the above series and those in the development of $(1-r^2)(1-2r\cos\theta+r^2)^{-\frac{3}{2}}$.)

7. Adopting the notation of the last article, we have

$$e^{\cos\theta\frac{d}{d\theta}\sin\theta}(\sin\theta)^i=\frac{1}{\sin\theta}e^{-\rho\frac{d}{d\theta}}(\sin\theta)^{i+1}$$

$$=\frac{1}{\sin\theta}\left(1+(t-\xi)^2\right)^{-\frac{i+1}{2}}$$

$$=(\sin\theta)^i(1-2r\cos\theta+r^2)^{-\frac{i+1}{2}}.$$

Hence this theorem, which will be useful afterwards. The coefficient of $\frac{r^n}{1.2\ldots.n}$ in the development of $(1-2r\cos\theta+r^2)^{-\frac{i+1}{2}}$ is

$$(\sin\theta)^{-(n+i)}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n(\sin\theta)^i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

8. Let us next consider the development of the expression

$$\{1-2r(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\varphi)+r^2\}^{-\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (18.)$$

Since the coefficient of r^n is necessarily expressible in the form

$$q_0+q_1\cos\varphi+q_2\cos 2\varphi+\ldots+q_n\cos n\varphi,$$

and must also satisfy the equation (2.), art. 1, we find, on substitution in that equation, that q_i must be a solution of the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+n(n+1)(\sin\theta)^2-i^2\right\}u_n=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.)$$

I proceed to consider this equation in a general manner, without reference, in the first instance, to the problem immediately in hand.

Since the equation (19.) only differs from (3.), art. 1, in having $-i^2$ instead of k^2, we may apply the process used in the solution of that equation. We thus get

$$u_n=(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(C_1\left(\tan\frac{\theta}{2}\right)^i+C_2\left(\cot\frac{\theta}{2}\right)^i\right). \quad . \quad . \quad . \quad (20.)$$

Now, although this expression appears to contain two arbitrary constants, it will be shown presently, that when i is an integer and not greater than n, it really contains only *one*. This is seen immediately to be the case when $i=0$, and may be easily verified in other simple cases, such as $i=1$, $n=2$, &c.

9. In fact, the process of art. 2 gives

$$u_n = \varpi_n u_{n-1} = \varpi_n \varpi_{n-1} u_{n-2} = \ldots = \varpi_n \varpi_{n-1} \ldots \varpi_{n-m+1} . u_{n-m},$$

where u_{n-m} satisfies the equation

$$\left(\varpi_{n-m}\varpi_{-(n-m)} + (n-m)^2 - i^2\right) u_{n-m} = 0;$$

and *in general* we can only arrive at an integrable form by taking $m=n$. But when i is an integer not greater than n, we shall get an integrable form by taking m so as to satisfy the condition $(n-m)^2=i^2$. Suppose i positive and less than n, and take $m=n-i$: we then have for the solution of (19.),

$$u_n = \varpi_n \varpi_{n-1} \ldots \varpi_{i+1} . u_i,$$

where u_i satisfies the equation $\varpi_i \varpi_{-i} u_i = 0$, or

$$\left(\sin\theta \frac{d}{d\theta} + i\cos\theta\right)\left(\sin\theta \frac{d}{d\theta} - i\cos\theta\right) u_i = 0:$$

the integration of this is easy, and the result is

$$u_i = C_1 (\sin\theta)^i + C_2 (\sin\theta)^i \int \frac{d\theta}{(\sin\theta)^{2i+1}};$$

so that, in the case considered, the integral of (19.) may be expressed in the form

$$u_n = (\sin\theta)^{-n}\left(\sin\theta \frac{d}{d\theta}\sin\theta\right)^{n-i}(\sin\theta)^{2i}\left(C_1 + C_2 \int d\theta (\sin\theta)^{-2i-1}\right) \quad . \quad . \quad . \quad (21.)$$

$\left(\text{for } \varpi_n \varpi_{n-1} \ldots \varpi_{i+1} \text{ is equivalent to } (\sin\theta)^{-n}\left(\sin\theta \frac{d}{d\theta}\sin\theta\right)^{n-i}(\sin\theta)^i \text{ (see art. 3)}\right)$.

It is evident that the expression (21.) contains two independent arbitrary constants, whatever be the value of i. And i being supposed a positive integer, the second term will always involve a logarithmic function; hence the form (20.), which involves no such function, cannot be the complete solution of the equation (19.), and therefore we shall not really limit its generality by putting $C_2=0$. Moreover, since the coefficient of $r^n \cos i\varphi$ in the development of (18.) cannot contain any logarithmic function, it follows that, for the purpose of that development, we must put $C_2=0$ in the expression (21.).

Comparing the two forms thus obtained for the coefficient of $r^n \cos i\varphi$, we obtain incidentally the following theorem, namely,

$$\left(\sin\theta \frac{d}{d\theta}\sin\theta\right)^i \left(\tan\frac{\theta}{2}\right)^i = c(\sin\theta)^{2i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.)$$

c being a constant, of which the value will appear afterwards.

The preceding investigation shows that the assumption $\varpi_n^{-1} 0 = 0$ (instead of the general value $C(\sin\theta)^{-n}$) is liable in certain cases to limit the generality of the result. I shall return to this point presently, but proceed now to complete the development of the expression (18.).

10. Since the coefficient of $r^n \cos i\varphi$ must contain θ and θ' symmetrically, and satisfy both the equation (19.) and a similar equation in θ', it is evident that it must be of the form $f(\theta)f(\theta')$. Hence, by means of the conclusions of art. 9, we arrive at the following

result: put, for convenience, $\sin\theta\frac{d}{d\theta}\sin\theta=\Theta$, $\sin\theta'\frac{d}{d\theta'}\sin\theta'=\Theta'$; then the coefficient of r^n in the development of $(1-2r(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\varphi)+r^2)^{-\frac{1}{2}}$ may be expressed in either of the two forms,

$$\frac{1}{(\sin\theta\sin\theta')^n}\Theta^n\Theta'^n\left\{a_0+a_1\tan\frac{\theta}{2}\tan\frac{\theta'}{2}\cos\varphi+a_2\left(\tan\frac{\theta}{2}\tan\frac{\theta'}{2}\right)^2\cos 2\varphi\right.$$
$$\left.+\ldots+a_n\left(\tan\frac{\theta}{2}\tan\frac{\theta'}{2}\right)^n\cos n\varphi\right\} \quad . \quad . \quad . \quad . \quad (23.)$$

$$\frac{1}{(\sin\theta\sin\theta')^n}\{c_0\Theta^n\Theta'^n 1+c_1\Theta^{n-1}\Theta'^{n-1}(\sin\theta\sin\theta')^2\cos\varphi$$
$$+\ldots.+c_i\Theta^{n-i}\Theta'^{n-i}(\sin\theta\sin\theta')^{2i}\cos i\varphi+\ldots+c_n(\sin\theta\sin\theta')^{2n}\cos n\varphi\} \quad . \quad (24.)$$

and it only remains to determine the constants.

11. For this purpose we may adopt a process of which the first part is taken from LAPLACE. Let $\cos\theta=\mu$, $\cos\theta'=\mu'$, and suppose the coefficient of $r^n\cos i\varphi$ in the development of $(1-2r(\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}.\cos\varphi)+r^2)^{-\frac{1}{2}}$ to be itself developed in a series of powers and products of μ and μ'. It is only necessary to ascertain the term independent of μ, μ', and the term containing the product $\mu\mu'$; and for this end we may write the above expression thus, $(1-2r(\mu\mu'+\cos\varphi)+r^2)^{-\frac{1}{2}}$, since the terms μ^2, μ'^2 cannot affect the required result; and this again may (for the present purpose) be considered equal to $(1-2r\cos\varphi+r^2)^{-\frac{1}{2}}+\mu\mu'r(1-2r\cos\varphi+r^2)^{-\frac{3}{2}}$; and if we put $2\cos\varphi=x+\frac{1}{x}$, so that $1-2r\cos\varphi+r^2=(1-rx)\left(1+\frac{r}{x}\right)$, and develope in the usual manner, we easily find the following results:—

1st, if $n-i$ be even, the term independent of μ, μ' in the coefficient of $r^n\cos i\varphi$ is

$$2.\frac{1^2.3^2.5^2\ldots(n+i-1)^2.1^2.3^2.5^2\ldots(n-i-1)^2}{1.2.3\ldots(n+i).1.2.3\ldots(n-i)},$$

and the term containing $\mu\mu'$ vanishes.

2nd, if $n-i$ be odd, the term independent of μ, μ' vanishes, and the term containing $\mu\mu'$ is

$$2.\frac{1^2.3^2\ldots(n+i)^2.1^2.3^2\ldots.(n-i)^2}{1.2.3\ldots(n+i).1.2.3\ldots(n-i)}\mu\mu',$$

the factor 2 to be omitted in each case when $i=0$.

12. On the other hand, the coefficient of $r^n\cos i\varphi$ is (art. 10)

$$c_i(\sin\theta\sin\theta')^{-i}\Theta^{n-i}\Theta'^{n-i}(\sin\theta)^{2i}(\sin\theta')^{2i}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25.)$$

and we may find the term independent of μ, μ', and the term involving $\mu\mu'$ in the development of this, as follows:—

It is easily found, from the theorem of art. 7, that

$(\sin\theta)^{-i}\Theta^{n-i}(\sin\theta)^{2i}=1.2\ldots(n-i)(\sin\theta)^n\times$(coefficient of r^{n-i} in the development of $(1-2r\cos\theta+r^2)^{-(i+\frac{1}{2})}$); and for our present purpose we may put $\sin\theta=1$, and

$$(1-2r\cos\theta+r^2)^{-(i+\frac{1}{2})}=(1+r^2)^{-(i+\frac{1}{2})}+(2i+1)\mu r(1+r^2)^{-(i+\frac{3}{2})}.$$

Hence it is easily found that when $n-i$ is even, the term independent of μ, μ' in (25.) is

$$c_i.1^2.3^2.5^2\ldots(n-i-1)^2.(2i+1)^2(2i+3)^2\ldots(n+i-1)^2;$$

and when $n-i$ is odd, the term involving $\mu\mu'$ is

$$c_i.\mu\mu'.1^2.3^2.5^2\ldots(n-i)^2.(2i+1)^2(2i+3)^2\ldots(n+i)^2.$$

If these expressions be equated to those obtained in art. 11, we find that whether $n-i$ be even or odd the value of c_i is

$$2.\frac{1^2.3^2.5^2\ldots(2i-1)^2}{1.2.3\ldots(n-i).1.2.3\ldots(n+i)},$$

the factor 2 being omitted when $i=0$.

13. We have then the following result:—

The coefficient of $r^n\cos i\varphi$ in the development of

$$\left(1-2r(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\varphi)+r^2\right)^{-\frac{1}{2}} \text{ is}$$

$$2.\frac{1^2.3^2.5^2\ldots.(2i-1)^2}{1.2.3\ldots(n-i).1.2.3\ldots(n+i)}(\sin\theta\sin\theta')^{-n}\Theta^{n-i}\Theta'^{n-i}(\sin\theta)^{2i}(\sin\theta')^{2i}, \quad . \quad . \quad (26.)$$

where $\Theta=\sin\theta\frac{d}{d\theta}\sin\theta$, $\Theta'=\sin\theta'\frac{d}{d\theta'}\sin\theta'$; and with respect to the numerical coefficient, it is to be observed that when $i=0$ the factor 2 is to be omitted, and the numerator considered to become unity. The extreme values of i are of course 0 and n, and the term $1.2.3\ldots(n-i)$ is to be taken $=1$ when $i=n$.

14. It is now easy to ascertain the values of the coefficients a_0, a_1, &c. in the form (23.), art. 10. But first it is necessary to prove that

$$\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(\tan\frac{\theta}{2}\right)^n=1.3.5\ldots(2n-1)(\sin\theta)^{2n}. \quad . \quad . \quad . \quad . \quad (27.)$$

We have proved this already (art. 9, equation (22.)), except in so far as the numerical coefficient was left undetermined, so that we have only to establish its value. Now the expression on the left of the above equation may be written

$$(\sin\theta)^{-1}\left((\sin\theta)^2\frac{d}{d\theta}\right)^n\sin\theta\left(\tan\frac{\theta}{2}\right)^n;$$

and if we put $\cot\theta=t$, so that $\tan\frac{\theta}{2}=\sqrt{1+t^2}-t$, and $(\sin\theta)^2\frac{d}{d\theta}=-\frac{d}{dt}$, the equation (27.) becomes

$$(-)^n\left(\frac{d}{dt}\right)^n\frac{(\sqrt{1+t^2}-t)^n}{\sqrt{1+t^2}}=\frac{1.3.5\ldots(2n-1)}{(1+t^2)^{\frac{2n+1}{2}}};$$

and since it is only the value of the coefficient that is in question, it will be enough to prove this when $t=0$, *i. e.* to show that the coefficient of t^n in the development of

$$\frac{(t-\sqrt{1+t^2})^n}{\sqrt{1+t^2}}$$

is $\frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n}$. Now, expanding the numerator by the binomial theorem, the above

expression becomes

$$\frac{t^n}{\sqrt{1+t^2}}-nt^{n-1}+\frac{n(n-1)}{1.2}t^{n-2}\sqrt{1+t^2}-\frac{n(n-1)(n-2)}{1.2.3}t^{n-3}(1+t^2)+\ldots+(-)^n(1+t^2)^{\frac{n-1}{2}};$$

and if we develope each term and collect the coefficient of t^n, we find that it is

$$1+\frac{n(n-1)}{2^2}+\frac{n(n-1)(n-2)(n-3)}{2^2.4^2}+\ldots+\frac{n(n-1)\ldots(n-2i+1)}{2^2.4^2.6^2\ldots(2i)^2}+\ldots,$$

a series of which the sum * is $\frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n}$. The truth of (27.) is therefore established.

15. It follows that

$$\Theta^n\left(\tan\tfrac{\theta}{2}\right)^i=\Theta^{n-i}\Theta^i\left(\tan\tfrac{\theta}{2}\right)^i=1.3\ldots(2i-1)\Theta^{n-i}(\sin\theta)^{2i},$$

and consequently the coefficient of $r^n\cos i\varphi$ may also be expressed in the form

$$\frac{2}{1.2.3\ldots(n-i).1.2.3\ldots(n+i)}(\sin\theta\sin\theta')^{-n}\Theta^n\Theta'^n\left(\tan\tfrac{\theta}{2}\right)^i\left(\tan\tfrac{\theta'}{2}\right)^i, \quad . \quad . \quad (28.)$$

the factor 2 being omitted as before when $i=0$.

The law of the numerical coefficients in this case is remarkable. For if the expression $(1+\cos\varphi)^n$ be developed in the form $A_0+A_1\cos\varphi+A_2\cos 2\varphi+\ldots+A_n\cos n\varphi$, it will be found that

$A_i=1.2.3\ldots n.1.3.5\ldots(2n-1).\frac{2}{1.2.3\ldots(n-i).1.2.3\ldots(n+i)}$ (omitting the 2 when $i=0$),

so that we may express the law of the development of

$$\left(1-2r(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\varphi)+r^2\right)^{-\frac{1}{2}}$$

as follows:—the coefficient of r^n is

$$\frac{1}{1.2.3\ldots n.1.3.5\ldots(2n-1)}(\sin\theta)^{-n}(\sin\theta')^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(\sin\theta'\frac{d}{d\theta'}\sin\theta'\right)^n U_n,$$

where

$$U_n=A_0+A_1\cos\varphi\tan\tfrac{\theta}{2}\tan\tfrac{\theta'}{2}+A_2\cos 2\varphi\left(\tan\tfrac{\theta}{2}\right)^2\left(\tan\tfrac{\theta'}{2}\right)^2+\ldots+A_n\cos n\varphi\left(\tan\tfrac{\theta}{2}\right)^n\left(\tan\tfrac{\theta'}{2}\right)^n,$$

and A_i is the coefficient of $\cos i\varphi$ in the development of $(1+\cos\varphi)^n$.

Although this form of the development follows a more simple and remarkable law than that of art. 13, it is evident that for actual calculation it would be much less convenient, since the number of operations is much greater, and the differentiations more complex. But as the complete explicit form of the development is known independently, this consideration is not of much practical importance.

* $\frac{1.3\ldots(2n-1)}{1.2\ldots n}$ is the coefficient of x^n in the development of $(1-2x)^{-\frac{1}{2}}$; now

$$(1-2x)^{-\frac{1}{2}}=\left((1-x)^2-x^2\right)^{-\frac{1}{2}}=(1-x)^{-1}+\tfrac{1}{2}(1-x)^{-3}x^2+\tfrac{1.3}{2.4}(1-x)^{-5}x^4+\ldots;$$

and if each term be developed and the coefficient of x^n collected, the result is the series in the text.

The equation (27.) probably admits of some simpler demonstration, which I have failed to perceive.

16. Professor BOOLE, in the paper above referred to, obtains the following expression for the coefficient of $r^n \cos i\varphi$; namely,

$$\frac{2f(\mu)f(\mu')}{1.2....(n-i).1.2...(n+i)},$$

where
$$f(\mu)=(1-\mu^2)^{\frac{i}{2}}\mu^{n-i}\left(\frac{d}{d\mu}\frac{1}{\mu}\right)^n\mu^{n+i}(1+\mu)^{n-i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29.)$$

Here the numerical coefficients are the same as in the form just given. It follows, therefore, that the expression on the right of (29.) must be equivalent to

$$(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(\tan\frac{\theta}{2}\right)^i.$$

This equivalence, and that expressed by equation (27.), art. 14, are instances of theorems by no means obvious or easy to verify directly. (See art. 5.)

For actual calculation, Professor BOOLE'S form would be preferable to either of mine; for though the number of operations is much greater than in the form (26.), their result may be assigned with much greater facility. But considered merely as an analytical expression, the form (26.) is the simplest of the three.

SECTION III.

17. It was shown above (art. 9), that the expression

$$u_n=(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(C_1\left(\tan\frac{\theta}{2}\right)^i+C_2\left(\tan\frac{\theta}{2}\right)^{-i}\right)$$

fails in certain cases to represent the complete solution of the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+n(n+1)(\sin\theta)^2-i^2\right\}u_n=0,$$

namely, when i is an *integer*, and not greater than n. (Similarly, the expression $u=C_1e^{ix}+C_2e^{-ix}$ fails to give the complete solution of $\left(\left(\frac{d}{dx}\right)^2-i^2\right)u=0$, when $i=0$.)

It is desirable, therefore, to investigate the solution of the equation in a more general manner. Putting, as before, $\sin\theta\frac{d}{d\theta}+n\cos\theta=\varpi_n$, the equation to be integrated is

$$(\varpi_n\varpi_{-n}+n^2-i^2)u_n=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30.)$$

Let $u_n=\varpi_n v$, then
$$\left(\varpi_n\varpi_{-n}\varpi_n+(n^2-i^2)\varpi_n\right)v=0,$$

and *any* value of v which satisfies this, will give a value of u_n satisfying (30.). Now $\varpi_{-n}\varpi_n+n^2$ is identically the same as $\varpi_{n-1}\varpi_{-(n-1)}+(n-1)^2$ (10.), art. 2; hence the equation becomes

$$\varpi_n\left(\varpi_{n-1}\varpi_{-(n-1)}+(n-1)^2-i^2\right)v=0;$$

similarly, putting $v=\varpi_{n-1}\omega$, we get

$$\varpi_n\varpi_{n-1}\left(\varpi_{n-2}\varpi_{-(n-2)}+(n-2)^2-i^2\right)\omega=0,$$

and so on, till we have, finally,

$$u_n = \varpi_n \varpi_{n-1} \varpi_{n-2} \dots \varpi_1 . z, \quad \dots \quad (31.)$$

and

$$\varpi_n \varpi_{n-1} \dots \varpi_1 (\varpi_0^2 - i^2) z = 0. \quad \dots \quad (32.)$$

The point to be observed is, that every value of z which satisfies (32.) will give a value of u_n satisfying (30.); hence, although the complete value of z will contain $n+2$ constants, it is certain that n of them will be destroyed by the n direct operations of the expression (31.). *In general*, the two constants left are those introduced by the inverse operation $(\varpi_0^2 - i^2)^{-1}$; but in the exceptional cases noticed above, one of these disappears, and one of those introduced by the other inverse operations remains instead.

18. The complete value of u_n, in its most general form, is therefore

$$u_n = \varpi_n \varpi_{n-1} \dots \varpi_1 (\varpi_0^2 - i^2)^{-1} \varpi_1^{-1} \varpi_2^{-1} \dots \varpi_n^{-1} . 0. \quad \dots \quad (33.)$$

It was shown before that $\varpi_n \varpi_{n-1} \dots \varpi_1 = (\sin\theta)^{-n} \left(\sin\theta \frac{d}{d\theta} \sin\theta \right)^n$; in like manner we have $\varpi_1^{-1} \varpi_2^{-1} \dots \varpi_n^{-1} = \left(\frac{1}{\sin\theta} \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \right)^n (\sin\theta)^n$, the latter operation being the inverse of the former.

The expression $(\varpi_0^2 - i^2)^{-1}$ is equivalent to $\left(\left(\frac{d}{dt} \right)^2 - i^2 \right)^{-1}$, if $t = \log \tan \frac{\theta}{2}$; and this may be resolved into either of the two following forms; namely,

$$\frac{1}{2} \left\{ \left(\frac{d}{dt} - i \right)^{-1} - \left(\frac{d}{dt} + i \right)^{-1} \right\},$$

or

$$\left(\frac{d}{dt} \pm i \right)^{-1} \left(\frac{d}{dt} \mp i \right)^{-1}.$$

The former, omitting the useless factor $\frac{1}{2}$, gives

$$\left(\tan \frac{\theta}{2} \right)^i \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \left(\cot \frac{\theta}{2} \right)^i - \left(\cot \frac{\theta}{2} \right)^i \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \left(\tan \frac{\theta}{2} \right)^i;$$

the latter, taking the lower signs, gives

$$\left(\tan \frac{\theta}{2} \right)^i \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \left(\cot \frac{\theta}{2} \right)^{2i} \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \left(\tan \frac{\theta}{2} \right)^i.$$

Either of these expressions is to be substituted in (33.), and thus the complete value of u_n is obtained by means of $n+2$ integrations, and n differentiations. *In general* the n integrations implied in the expression $\left(\frac{1}{\sin\theta} \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \right)^n 0$ are useless, so that we may substitute 0 for this expression, as was done in the process first given, without ultimate loss of completeness.

19. If we suppose i an integer, and less than n, we may stop the process of art. 17 at an earlier stage, thus

$$u_n = \varpi_n \varpi_{n-1} \dots \varpi_{i+1} z,$$

and

$$\varpi_n \varpi_{n-1} \dots \varpi_{i+1} \varpi_i \varpi_{-i} z = 0,$$

so that we now have

$$u_n = \varpi_n \varpi_{n-1} \dots \varpi_{i+1} \varpi_{-i}^{-1} \varpi_i^{-1} \varpi_{i+1}^{-1} \dots \varpi_n^{-1} . 0;$$

or, reducing as before, and observing that w_{-i}^{i} is equivalent to

$$(\sin\theta)^{i}\left(\frac{d}{d\theta}\right)^{-1}(\sin\theta)^{-i-1},$$

we have, finally,

$$u_n=(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^{n-i}(\sin\theta)^{m}\left(\frac{d}{d\theta}\right)^{-1}(\sin\theta)^{-m}\left(\frac{1}{\sin\theta}\left(\frac{d}{d\theta}\right)^{-1}\frac{1}{\sin\theta}\right)^{n-i+1}.0. \quad . \quad (34.)$$

If in this expression we put 0 for the result of *all but one* of the operations indicated in the last term, we reproduce the form (21.) obtained in art. 9, which appears in fact to be always complete. The form (34.), however, as before remarked, will never contain superfluous constants.

20. It is an interesting question whether the forms thus obtained on the supposition that i is an integer, are not really *general*, inasmuch as they are expressible without reference to that supposition. I believe that they are; but I doubt whether, in the existing state of analysis, it can be proved either that they are or are not (see § IV.).

21. It is easy to obtain other forms analogous to those given in the preceding articles, containing inverse in place of direct operations. But such forms are objectionable, because they necessarily give rise to superfluous constants, the discrimination of which may be a matter of difficulty.

Section IV.

22. It was observed at the beginning of this paper that the equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0,$$

when transformed to polar coordinates, may be written in the form

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\varphi}\right)^2+(\sin\theta)^2r\frac{d}{dr}\left(r\frac{d}{dr}+1\right)\right\}u=0. \quad . \quad . \quad . \quad . \quad (35.)$$

I do not know whether this form has been noticed; but it is remarkable in this respect, that if the symbol $r\frac{d}{dr}$ be replaced by n, it becomes identical with the equation discussed in the first section, and of which the solution was shown to be

$$(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^{n}\left\{f\left(e^{\varphi\sqrt{-1}}\tan\frac{\theta}{2}\right)+\mathrm{F}\left(e^{-\varphi\sqrt{-1}}\tan\frac{\theta}{2}\right)\right\}. \quad . \quad . \quad . \quad . \quad (36.)$$

It is not easy to verify, *by direct substitution*, that this expression satisfies the differential equation of which it is the solution; at least I have not yet succeeded in doing so; and even if this were accomplished, it is most likely that difficulties would remain to be overcome before we could establish the legitimacy of extending this result to the case in which n is a symbol of unlimited signification. Still it seems worth while to try, at least as an experiment, the consequence of assuming that the form (36.) will give the solution of (35.) on putting $r\frac{d}{dr}$ instead of n.

Omitting the second arbitrary function to save space, we should have (observing that r must be considered to enter the arbitrary functions in an arbitrary manner)

$$u=(\sin\theta)^{-r\frac{d}{dx}}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^{r\frac{d}{dx}}f\left(r,\ e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad (37.)$$

23. There is no difficulty in the interpretation of this expression. It is only necessary to recollect that $c^{x\frac{d}{dx}}\varphi(x)=\varphi(cx)$, and we have the following result:—let $\frac{r}{\sin\theta}=\varrho$, then

$$u=f\left(\varrho\sin\theta\frac{d}{d\theta}\sin\theta,\ e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38.)$$

where ϱ is to be put $=\frac{r}{\sin\theta}$ *after* all other operations, and the function must be supposed to be developed as if the two symbols $\varrho\sin\theta\frac{d}{d\theta}\sin\theta$, $e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}$ were commutative, the powers of the former being always prefixed to those of the latter.

It is easy to derive from the form (38.), particular expressions which we already know to be solutions of (35.). For example, supplying the other arbitrary function, it is evident that if we take $f(x, y)=\frac{1}{2}x^n y^i$, and $F(x, y)$ the same, we get

$$u=r^n(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(\tan\frac{\theta}{2}\right)^i\cos i\,\phi,$$

which, we know, satisfies (35.).

Again, we may take $f(x, y)=e^x$, or

$$u=e^{\varrho\sin\theta\frac{d}{d\theta}\sin\theta}.1,$$

which has been already shown (art. 6) to represent

$$(1-2r\cos\theta+r^2)^{-\frac{1}{2}}.$$

These verifications afford, I think, ground to believe that the form (37.) is a true solution of the equation (35.). And here I leave the subject, hoping that it may attract the attention of some mathematician more able than myself to bring it to a satisfactory conclusion.

Note on the last paragraph.

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If the expression $f\left(\varrho\sin\theta\frac{d}{d\theta}\sin\theta,\ e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right)$ be supposed to be developed and arranged according to powers of the term $\varrho\sin\theta\frac{d}{d\theta}\sin\theta$, each term in the result will be of the form

$$r^n(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\psi_n\left(e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right),$$

ψ_n representing a function of which the form may be considered to depend in an arbitrary manner upon the value of n. Now we know that every such term as the above satisfies the differential equation, and it may be asked, what more is required to prove the correctness of the solution? I answer, that the point in question is whether

$$\Sigma . r^n (\sin\theta)^{-n} \left(\sin\theta \frac{d}{d\theta} \sin\theta\right)^n \psi_n \left(e^{\phi\sqrt{-1}} \tan\frac{\theta}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.)$$

is equivalent to the *most general interpretation* of

$$(\sin\theta)^{-r\frac{d}{dx}} \left(\sin\theta \frac{d}{d\theta} \sin\theta\right)^{r\frac{d}{dx}} f\left(r, e^{\phi\sqrt{-1}} \tan\frac{\theta}{2}\right);$$

and, if not, whether the most general interpretation would satisfy the Differential Equation (35.).

The expression (A.) gives, in fact, no new result, being an evident consequence of the conclusions of the former sections.

Postscript (*Added May* 29, 1857).

Mr. Cayley has been kind enough to communicate to me direct verifications of the equation (27.), art. 14, and of the identity referred to in art. 16. Assuming a formula established in Mr. Cayley's paper "On certain Formulæ for Differentiation," &c.*, the former of the two theorems just mentioned is easily obtained, the latter not without a good deal of trouble. The investigations of Section II. may be compared with some of those in the Introduction to Murphy's 'Electricity;' but I have not at present examined the latter with a view to such comparison, and therefore merely mention the subject.

* Cambridge and Dublin Journal, vol. ii. p. 124, equation (2.).

V. *Researches on Organo-metallic Bodies.*—Third Memoir. *On a New Series of Organic Acids containing Nitrogen.* *By* E. FRANKLAND, *Ph.D., F.R.S., Professor of Chemistry in Owens College, Manchester.*

Received June 19,—Read June 19, 1856.

IN the progress made by Organic Chemistry during the past fifteen years, no generalization has perhaps contributed so extensively to the development of this branch of the science, as the doctrine of substitution. The value of this doctrine becomes even still more apparent, when it is remembered that chemists have, until very recently, possessed adequate means for following out its suggestions, in one direction only. The peculiar habits of chlorine render the substitution of an electro-positive constituent by this element, generally a work of little or no difficulty, and even the like substitution of other electro-negative for electro-positive elements in organic bodies, presents no insurmountable obstacles. But the inverse process has hitherto been successfully accomplished only in comparatively few cases, owing to the want of a body capable, like chlorine, of effecting such a replacement with facility. This want is now supplied in zincmethyl and its homologues; bodies which, on account of their intense affinities and peculiar behaviour, possess in an eminent degree the property of removing electro-negative constituents and replacing them by methyl, ethyl, &c. The action of zincmethyl upon water, attended as it is by the substitution of methyl for oxygen,

$$\left.\begin{matrix} C_2H_3Zn \\ HO \end{matrix}\right\} = \left\{\begin{matrix} C_2H_3,H \\ ZnO, \end{matrix}\right.$$

may be regarded as the type of these reactions, which open up a most extensive and perfectly new field of research, from the cultivation of which important discoveries cannot fail to spring. Amongst the reactions of this nature which promise most interesting results, are those with the chlorine and oxygen substitution products derived from the ethers and organic acids, which might lead to the higher members of each homologous series being produced from the lower ones, if not to the building up of some of those series from their inorganic types; a discovery which cannot now remain long in abeyance. Instead of immediately pursuing this line of investigation, however, I determined in the first place to confine my attention to the action of these organo-zinc bodies upon inorganic compounds.

In a former memoir* I endeavoured to give a general view of the rational constitution of all the organo-metallic bodies then known, by showing that they all possessed a molecular isonomy with the inorganic compounds of the respective metals. The only com-

* Philosophical Transactions, 1852, page 438.

pound which at that time did not coincide with this view was Lövig's so-called ethostibylic acid, the formula of which, $SbC_4H_5O_3$, I suggested would probably be found to be erroneous*; and in fact, Lövig has since announced this to be the case: he now assigns to this compound the formula $Sb(C_4H_5)_3O_3$, $2SbO_3$, which harmonizes perfectly with the general view I ventured to propound. The recent researches of Merck† upon the compounds of stibethyl, although they probably prove the existence of certain new compounds of this radical, are by no means conclusive as to the non-existence of the bodies originally described by Lövig. With regard to those stanethyl compounds which have been since discovered, several of them correspond exactly with the known oxides of tin; the remainder are also by no means irreconcileable with my hypothesis, if we consider the polymeric attributes of stannic acid. Nevertheless, I conceive that the formulæ, and even the existence of some of the more complex stanethyl compounds, require confirmation before these bodies can be employed, either for the support or disproof of any general theory of the rational constitution of organo-metallic compounds.

Taking, then, this view of the organo-metallic compounds as my guide, I pointed out in a former memoir‡ that the oxygen compounds of nitrogen might probably be represented by corresponding organic compounds, in which one or more equivalents of oxygen were replaced by an organic radical: thus, to take one example, nitric acid, by the substitution of methyl, should yield the following derivatives:—

I.	II.	III.	IV.	V.
$N\{C_2H_3, O, O, O$	$N\{C_2H_3, C_2H_3, O, O, O$	$N\{C_2H_3, C_2H_3, C_2H_3, O, O$	$N\{C_2H_3, C_2H_3, C_2H_3, C_2H_3, O$	$N\{C_2H_3, C_2H_3, C_2H_3, C_2H_3, C_2H_3$

of which the fourth is already known as oxide of tetramethylammonium. My attempts to produce these derivatives from the oxygen compounds of nitrogen have hitherto been confined to the binoxide, in which I have succeeded in replacing oxygen by ethyl in the manner now to be described.

Action of Zincethyl upon Binoxide of Nitrogen.

If a small quantity of zincethyl, either pure or dissolved in ether, be passed up into dry binoxide of nitrogen confined over mercury, the binoxide is very slowly but completely absorbed in large quantity, without the production of any other gas. The solution may be accelerated by agitation, but even then it is exceedingly slow. At the expiration of from one to four days, rhomboidal crystals begin to be deposited, and increase in number until the liquid finally solidifies. To prepare these crystals in larger quantity, about an ounce of zincethyl, dissolved in an equal bulk of dry ether, was placed in a flat-bottomed flask and supplied with binoxide of nitrogen from a gas-holder,

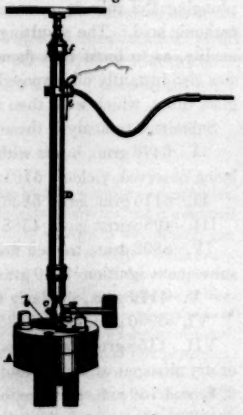
* Philosophical Transactions, 1852, page 442.

† Journal für Prak. Chemie, Bd. lxvi. S. 56.

‡ Philosophical Transactions, vol. cxliii. page 442.

the gas being thoroughly dried, by bubbling through a long series of bulbs filled with concentrated sulphuric acid, which also served to absorb any traces of nitrous gas that might be formed by atmospheric oxygen gaining access to the interior of the apparatus. The gas was conducted into the flask by a tube which terminated just below the cork; whilst a provision was made for its exit by another tube continued to within a short distance of the surface of the liquid, and which terminated outside the cork in a capillary extremity, that could be readily sealed up by the blowpipe and reopened at pleasure. Binoxide of nitrogen, prepared from copper turnings and nitric acid, always contains a considerable per-centage of protoxide, and it was therefore necessary occasionally to allow a stream of the gas to flow through the flask, so as to prevent the absorption being hindered or stopped by the accumulation of protoxide of nitrogen: at other times the exit tube was hermetically sealed and the gas supplied only as it was absorbed. In this way, although the apparatus was in action day and night, six weeks elapsed before the absorption was completed. On another occasion, when the action was accelerated by violent agitation of the liquid for several hours each day, the zincethyl was saturated in about a fortnight. It was evident that such a process was little calculated for the production of considerable quantities of the new compound, and recourse was therefore had to mechanical means in order to expedite and facilitate the operation. Fig. 1 is from a photograph of the apparatus employed for this purpose, and as it will no doubt prove useful in other cases for experiments with sparingly soluble gases, I will describe it somewhat in detail. A is a copper digester similar to the one I have already described for the preparation of zincethyl*; into the aperture *b* is screwed the stopcock *c*, to which can be attached at pleasure the condensing syringe D made of gun-metal, 12 inches long and ·7 inch in diameter. In this syringe a solid steel piston $1\frac{3}{4}$ inch deep works air-tight, and the piston-rod passes through a stuffing-box *f*. The syringe is supplied with gas through the nozzle *e*, to which a flexible tube is attached. When the stopcock *c* is closed, the elevation of the piston produces a vacuum, which is instantly filled with gas, so soon as the piston has passed the nozzle *e*. Between *e* and *f* the interior of the syringe is grooved longitudinally, so as to prevent any compression of gas behind the piston, when it is drawn up to *f*. On forcing down the piston and opening the stopcock *c*, it is obvious that the gas occupying the syringe from *e* to *c* will be forced into A. By repeating this process, it is not difficult for one operator to compress about twenty atmospheres into A; such a degree of compression exerting upon the piston a pressure of about 114 lbs.

Fig. 1.



* Philosophical Transactions, vol. cxlv. p. 261.

In each operation about three ounces of zincethyl in ethereal solution was placed in the copper cylinder A, and the condensing syringe being attached, about twenty atmospheres of dry binoxide of nitrogen were introduced; the syringe was unscrewed and the cylinder A agitated for two or three minutes by rolling upon the floor or otherwise; at the end of which operation, the pressure within A was found to be reduced to three or four atmospheres; the process of condensation and agitation being repeated five or six times, the copper cylinder becomes so much heated as to require immersion in cold water for a few minutes. At this stage of the process it is also desirable to allow the residual gas in A to escape.

This gas consists principally of protoxide of nitrogen and hydride of ethyl; the latter derived from the decomposition of zincethyl by a trace of aqueous vapour introduced with the binoxide of nitrogen. By repeating the above series of operations six or eight times, the zincethyl becomes saturated and the process is completed. If it be desired to obtain the crystalline compound in a state of perfect purity, it is better to place the zincethyl in a wide glass tube open at top and fitting into A; but in this case very moderate agitation only can be used, and consequently the absorption takes place more slowly, and the operation requires two or three days for its completion. It is, however, rarely necessary to have recourse to this modification of the process.

At the conclusion of an operation conducted as above, with the intervention of a glass tube, the contents of the latter consisted of a mass of colourless crystals immersed in an ethereal solution; the latter was poured off and the former were freed from ether, by plunging the tube in a water-bath at 90° C. and passing through it a stream of dry carbonic acid. The resulting crystalline mass attracted oxygen from the air with such avidity, as to burst into flame when any considerable quantity was freely exposed; it was also instantly decomposed by water, and was therefore transferred at once into small glass tubes, which were then immediately sealed hermetically.

Submitted to analysis these crystals yielded the following results:—

I. ·6476 grm., burnt with oxide of copper and free oxygen, the usual precautions being observed, yielded ·6101 grm. carbonic acid and ·3084 grm. water.

II. ·6110 grm. gave ·5836 grm. carbonic acid and ·2909 grm. water.

III. ·4988 grm. gave ·4588 grm. carbonic acid and ·2413 grm. water.

IV. ·8892 grm., treated first with water and then with boiling nitric acid, yielded on subsequent ignition ·3940 grm. oxide of zinc.

V. ·4179 grm., similarly treated, gave ·1837 grm. oxide of zinc.

VI. ·5900 grm. gave ·2585 grm. oxide of zinc.

VII. ·1155 grm., treated according to SIMPSON'S method, gave 52·5 cubic centimetres of dry nitrogen at 9°·2 C. and 210·3 mm. pressure, equal to 14·04 cubic centimetres at 0° C. and 760 mm. pressure, or ·0177 gramme*.

* Considerable difficulty was at first experienced in determining the nitrogen in this and the following compounds. The method of WILL and VARRENTRAPP is quite inapplicable, as it does not in some cases convert more than one-half of the total nitrogen into ammonia. BUNSEN'S method also yields variable results,

VIII. ·3964 grm., similarly treated, gave 88·8 cubic centimetres dry nitrogen at 10°·0 C. and 412·4 mm. pressure, equal to 46·19 cubic centimetres at 0° C. and 760 mm. pressure, or ·05834 gramme.

IX. ·3712 grm. gave 84·0 cubic centimetres dry nitrogen at 411·8 mm. pressure and 15° C., or 43·02 cubic centimetres at 760 mm. pressure and 0° C., or ·05410 gramme.

These numbers correspond with the formula

$$C_4 H_5 Zn N O_3,$$

as is seen from the following comparison:—

	Calculated.		Found.								
			I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.
C_4	24	26·22	25·69	26·05	25·09						
H_5	5	5·46	5·29	5·29	5·38						
N	14	15·30							15·32	14·72	14·63
Zn	32·52	35·53				35·63	35·28	35·16			
O_3	16	17·49									
	91·52	100·00									

The substance used for these analyses was prepared in several different operations. That used for Nos. III. and IV. had undergone considerable oxidation, as it was finely powdered before being weighed, which will account for the deficiency in the per-centage of carbon. These results prove that the new body is formed by the union of an equal number of atoms of zincethyl and binoxide of nitrogen; but from considerations given below, the above formula requires to be doubled, and I shall presently show that the body is a compound of zincethyl with the zinc-salt of a new acid, for which I propose the name *Dinitroethylic acid*. Its formula is therefore

$$N_2 C_4 H_5 O_4 Zn + Zn C_4 H_5.$$

owing to the formation of some carbonate of zinc, which requires a higher heat for complete decomposition. SIMPSON'S absolute method gave very good results without the use of peroxide of mercury, but it was found impossible to prevent the occurrence of a considerable amount of binoxide of nitrogen in the resulting gas. Even when it was made to stream over 12 inches of an intensely ignited mixture of copper turnings and copper reduced from the oxide, the resulting nitrogen still contained 10·9 per cent. of binoxide. It was therefore necessary to estimate the volume of the latter gas by means of solution of protosulphate of iron, and to deduct half its volume from the observed amount of the mixed gases. The presence of protoxide of nitrogen would not interfere with the accuracy of the result, as this gas is unaffected by solution of protosulphate of iron, and contains exactly its own volume of nitrogen. By the observance of this precaution, SIMPSON'S absolute method not only appears to be capable of great accuracy, but it also requires much less expenditure of time, whilst it is easier of execution than any of the other methods, if it be employed in conjunction with the gas analysis apparatus, which I always use in eudiometrical operations (see Journal of the Chemical Society, vol. vi. p. 200); the absorption-tube, containing a layer of concentrated caustic potash solution, being used to receive the products of combustion, at the conclusion of which, the determination of the absolute volumes of nitrogen and binoxide of nitrogen is effected with this apparatus in a few minutes.

This compound is produced from zincethyl and binoxide of nitrogen according to the following equation:—

$$\left.\begin{matrix}2(Zn\,C_4H_5)\\2NO_2\end{matrix}\right\}=N_2C_4H_5O_4Zn+Zn\,C_4H_5.$$

Dinitroethylate of zinc and zincethyl is deposited from its ethereal solution in large colourless and transparent rhomboidal crystals, which instantly become opake on exposure to the air, owing to the formation of an oxidized product. These crystals are tolerably soluble in anhydrous ether without decomposition, but they are instantly decomposed by anhydrous alcohol and by water. Exposed to the gradually increasing heat of an oil-bath, dinitroethylate of zinc and zincethyl fuses at 100° C., froths up and begins slowly to evolve gas. At 180° C. the colour darkens, and a small quantity of a yellowish liquid of a penetrating odour, free from zincethyl and possessing a very powerful alkaline reaction, distils over. This liquid, neutralized with hydrochloric acid and treated with bichloride of platinum, yielded a splendidly crystalline platinum salt, which was obtained, however, in too small quantity to allow of its composition being determined. From 180° to 190°, dinitroethylate of zinc and zincethyl evolved gas very rapidly, and the experiment was then interrupted. The gas consisted of 18·4 per cent. carbonic acid, 23·66 per cent. olefiant gas, and 57·94 per cent. of a mixture of hydride of ethyl, nitrogen and protoxide of nitrogen.

When brought into contact with water, dinitroethylate of zinc and zincethyl is immediately decomposed with lively effervescence. A large quantity of inflammable gas is evolved, and a white flocculent substance formed. At the conclusion of the reaction the latter dissolves almost completely, forming an opalescent solution resembling milk, possessing a powerfully alkaline reaction and a peculiarly bitter taste.

In order to ascertain the exact nature of the gas evolved in this reaction, some crystals of dinitroethylate of zinc and zincethyl were passed up into an inverted receiver filled with mercury, and were then brought into contact with a small quantity of water. The gas thus collected over mercury possessed an ethereal odour, burnt with a slightly luminous flame, and was completely soluble in an equal volume of alcohol. It was perfectly neutral, and underwent no change on being treated successively with caustic potash solution and dilute sulphuric acid.

A specific gravity determination gave the following numbers:—

Temperature of room	7°·7 C.
Height of barometer	761·3 mm.
Height of internal column of mercury	6·5 mm.
Weight of flask and gas	31·2294 grms.
Temperature in balance-case	10°·7 C.
Weight of flask and dry air	31·2199 grms.
Temperature of balance-case	11°·0 C.
Capacity of flask	141·0 cub. cent.

From these data the specific gravity was calculated to be 1·0515.

Submitted to eudiometrical analysis the following data were obtained:—

I.		Temp.
Pressure of gas used (dry)	200·9 mm.	3°·6 C.
Pressure after action of fuming sulphuric acid (dry)	200·9	3°·6
II.		
Pressure of gas used (dry)	20·0 mm.	3°·5
Pressure after admission of oxygen (dry)	267·6	3°·5
Pressure after explosion (dry)	217·9	3°·5
Pressure after absorption of carbonic acid (dry)	177·7	3°·5
Pressure after admission of hydrogen (dry)	845·6	3°·5
Pressure after explosion	312·2	3°·5

These numbers prove that neither nitrogen nor any member of the olefiant gas family is present; they also prove that one volume of the combustible gas consumed almost exactly $3\frac{1}{2}$ volumes of oxygen and generated two volumes of carbonic acid, as seen by the following proportions:—

Vol. of combustible gas.		Vol. of oxygen consumed.		Vol. of CO_2 generated.
20·0	:	69·9	:	40·2
1	:	3·495	:	2·01

These are exactly the results obtained by the combustion of hydride of ethyl, one volume of which consumes $3\frac{1}{2}$ volumes of oxygen and generates two volumes of carbonic acid. This result is also confirmed by the determination of the specific gravity of the gas, for the theoretical specific gravity of hydride of ethyl is 1·03652, which agrees very nearly with the number obtained in the above determination.

On submitting the milky solution, formed by the decomposition of dinitroethylate of zinc and zincethyl in water, to a stream of carbonic acid, a copious precipitate of carbonate of zinc, free from organic matter, was thrown down; the liquid was then heated to boiling and filtered. The filtrate, evaporated almost to dryness in a water-bath, yielded a white radiated crystalline mass, which, after being reduced to powder, pressed between blotting-paper, and dried over sulphuric acid, was submitted to analysis and yielded the following results:—

I. ·9042 grm., exposed to a current of dry air at ordinary temperatures and afterwards at 100° C., at which temperature it fused, lost ·0360 grm. water.

II. ·2843 grm., burnt with oxide of copper with the usual precautions, gave ·1963 grm. carbonic acid and ·1180 grm. water.

III. ·3540 grm. gave ·2410 grm. carbonic acid and ·1518 grm. water.

IV. ·2785 grm., dissolved in water, treated with nitric acid and afterwards ignited, gave ·0910 grm. oxide of zinc.

V. ·3074 grm., similarly treated, gave ·0975 grm. oxide of zinc.

VI. ·2577 grm., similarly treated, gave ·0825 grm. oxide of zinc.

VII. A nitrogen determination according to BUNSEN's method yielded the following results:—

		Temp.
Pressure of mixed gases (dry)	184·2 mm.	13°·7 C.
Pressure after absorption of CO_2 (dry) . . .	65·0	13°·7

The atomic relation of carbonic acid to nitrogen is therefore as follows:—

$$CO_2 : N = 119·2 : 65,$$

or

$$CO_2 : N = 2 : 1·09.$$

VIII. 2·1023 grms., dried at 100° C. until the weight ceased to diminish, lost ·0788 grm. water.

These numbers correspond with the formula

$$2(N_2C_4H_5O_4Zn)+HO,$$

as is seen from the following comparison:—

	Calculated.		Found.					
			II.	III.	IV.	V.	VI.	VII.
C_8 .	48	19·04	18·83	18·57				
H_{11} .	11	4·36	4·61	4·76				
N_2 .	56	22·22						23·78
Zn_2 .	65·04	25·81			26·23	25·46	25·69	
O_9 .	72	28·57						
	252·04	100·00						

These results prove that the action of water upon dinitroethylate of zinc and zincethyl consists in the transformation of the zincethyl into oxide of zinc and hydride of ethyl, thus

$$\left.\begin{matrix} N_2C_4H_5O_4Zn + C_4H_5Zn \\ HO \end{matrix}\right\} = \begin{cases} C_4H_5, H \\ N_2C_4H_5O_4Zn \\ ZnO. \end{cases}$$

The neutral dinitroethylate of zinc, however, thus set at liberty, immediately unites with a second equivalent of oxide of zinc, to form the basic salt $N_2C_4H_5O_4Zn+ZnO$, which is decomposed by carbonic acid into carbonate of zinc and the neutral salt.

Estimations Nos. I. and VIII. prove that dinitroethylate of zinc crystallizes with half an equivalent of water, which is expelled at 100° C., the loss in drying being 3·98 per cent. and 3·75 per cent. respectively, whilst the theoretical number is 3·57.

In order to establish beyond doubt the nature of the transition from dinitroethylate of zinc and zincethyl, to dinitroethylate of zinc, a known quantity of the former compound was treated with water, and the amount of zinc separated by carbonic acid and of that left in solution, separately estimated. The following numbers were obtained:—·5654 grm. dinitroethylate of zinc and zincethyl gave, when treated with carbonic acid, a quantity of carbonate, which yielded on ignition ·1204 grm. oxide of zinc. The filtrate precipitated by carbonate of soda gave ·1148 grm. oxide of zinc. These numbers are equivalent to

21·30 per cent. of oxide of zinc precipitated by carbonic acid, and 20·35 per cent. oxide of zinc left in solution; numbers which nearly approach to equality, and do not differ from the theoretical amount in each case (22·14) more widely than might be expected when it is stated that the dinitroethylate of zinc and zincethyl used, had become partially oxidized before being weighed.

Dinitroethylate of zinc is also produced by the direct oxidation of dinitroethylate of zinc and zincethyl in a stream of dry air, ethylate of zinc being at the same time formed: the completion of the oxidation is known by the product ceasing to effervesce in contact with water. This reaction is expressed by the following equation:—

$$\left.\begin{matrix} N_2C_4H_3O_4Zn + C_4H_5Zn \\ O_2 \end{matrix}\right\} = \left\{\begin{matrix} N_2C_4H_3O_4Zn \\ ZnO,\ C_4H_5O. \end{matrix}\right.$$

When this product is treated with water, alcohol and bibasic dinitroethylate of zinc are produced,

$$\left.\begin{matrix} N_2C_4H_3O_4Zn \\ ZnO,\ C_4H_5O \end{matrix}\right\} = \left\{\begin{matrix} N_2C_4H_3O_4Zn + ZnO \\ C_4H_5O,\ HO. \end{matrix}\right.$$

One of the equivalents of base being removed by carbonic acid, the filtered solution of the neutral salt thus obtained was evaporated to crystallization, and the crystals heated to 100° for some time; at this temperature they fused and afterwards solidified in cooling to a gummy mass, which is the anhydrous dinitroethylate of zinc, as is seen from the following analyses:—

I. ·3964 grm., burnt with oxide of copper and oxygen, gave ·2880 grm. carbonic acid and ·1540 grm. water.

II. ·0835 grm. gave ·0610 grm. carbonic acid and ·0338 grm. water.

III. ·2885 grm., treated with nitric acid and afterwards ignited, gave ·0988 grm. oxide of zinc.

IV. ·3501 grm., similarly treated, gave ·1182 grm. oxide of zinc.

These numbers agree sufficiently with those calculated from the formula of anhydrous dinitroethylate of zinc,

$$N_2C_4H_3O_4Zn.$$

	Calculated.		Found.			
			I.	II.	III.	IV.
C_4 . . .	24	19·75	19·81	19·92		
H_3 . . .	5	4·11	4·32	4·50		
N_2 . . .	28	23·04				
Zn . . .	32·52	26·76			27·49	27·10
O_4 . . .	32	26·34				
	121·52	100·00				

Finally, dinitroethylate of zinc and zincethyl is produced by adding an ethereal solution of zincethyl to anhydrous dinitroethylate of zinc, and corresponding compounds

appear to be formed under similar circumstances, with other salts of dinitroethylic acid. These compounds are evidently of the same nature as that produced by the union of zincethyl with iodide of zinc, which is formed in such large quantity during the preparation of zincethyl.

Dinitroethylate of zinc crystallizes in minute colourless needles containing half an equivalent of water, which they retain when exposed over sulphuric acid *in vacuo*. They fuse below 100° C., and gradually become anhydrous at this temperature. They are very soluble in water and in alcohol. The concentrated aqueous solution solidifies on cooling to a white fibrous crystalline mass. Heated suddenly in air to a temperature of about 300°, this salt does not deflagrate, but it inflames, burning rapidly with a beautiful bluish green flame. When dry dinitroethylate of zinc is treated with concentrated sulphuric acid, and the vessel containing those ingredients is placed in a freezing mixture, dinitroethylic acid is liberated; but it is so unstable, that when the temperature rises a few degrees it begins to effervesce violently, and is rapidly decomposed with the evolution of gases and white vapours. A dilute solution is somewhat more stable; it may be prepared either by decomposing a dilute solution of dinitroethylate of zinc with dilute sulphuric acid and then distilling *in vacuo*, or by adding to a dilute solution of the baryta-salt just sufficient sulphuric acid to precipitate the base. Dilute dinitroethylic acid thus prepared possesses a pungent odour, somewhat resembling that of the nitro-fatty acids, and an acid taste. It reddens litmus-paper strongly, and gradually decomposes even at ordinary temperatures.

The acid procured by distillation *in vacuo* being treated with carbonate of silver, the latter dissolved with evolution of carbonic acid. The filtered solution, evaporated over sulphuric acid, deposited light flocculent crystals of dinitroethylate of silver, which blackened rapidly. They gave, by treatment with nitric acid and subsequent ignition, 55·85 per cent. of metallic silver. The formula $N_2C_4H_8O_4Ag$ requires 54·82 per cent. Another portion of the dilute acid, procured by the decomposition of the baryta-salt as described above, was saturated with magnesia, evaporated to dryness, and the residue treated with strong alcohol. The filtered alcoholic solution, which contained no trace of sulphuric acid, gave, on evaporation, dinitroethylate of magnesia, which, by treatment with nitric acid and ignition, yielded 19·64 per cent. of magnesia. The formula $N_2C_4H_8O_4Mg$ requires 19·80 per cent.

The salts of dinitroethylic acid are all soluble in water and alcohol, and most of them crystallize with more or less difficulty. They are all violently acted upon by concentrated nitric acid, the dinitroethylic acid being entirely decomposed and a nitrate of the constituent base produced. Dilute nitric acid acts in the same manner, but more slowly. They all fuse at a temperature little above 100° C. The potash, soda, lime, and baryta salts deflagrate explosively, like loose gunpowder, at a temperature considerably below redness.

Dinitroethylate of Baryta.

$N_2C_4H_4O_4Ba$.

This salt is produced by adding caustic baryta in excess to a solution of dinitroethylate of zinc, carbonic acid being passed through the solution until the excess of baryta is precipitated. It is then treated with sulphuretted hydrogen to remove a trace of oxide of zinc, which is still held in solution. After being heated to boiling for a few minutes, and then filtered, the solution is concentrated by evaporation, and finally dried down to a gummy mass, which does not crystallize on cooling. This is anhydrous dinitroethylate of baryta. It yielded the following analytical results:—

I. ·8228 grm., burnt with chromate of lead, gave ·4420 grm. carbonic acid and ·2428 grm. water.

II. ·3675 grm., treated with dilute sulphuric acid and ignited, gave ·2737 grm. sulphate of baryta.

These numbers agree with the formula

$N_2C_4H_4O_4Ba$.

		Calculated.		Found.	
				I.	II.
C_4	24	15·23		14·65	
H_4	5	3·17		3·28	
N_2	28	17·77			
Ba	68·59	43·52			43·82
O_4	32	20·31			
	157·59	100·00			

Dinitroethylate of baryta is uncrystallizable, very deliquescent, and very soluble in water. Its solution reacts perfectly neutral.

Dinitroethylic Ether.

Several attempts were made to prepare this compound by the usual methods of etherification, but with only very partial success. When crystallized, dinitroethylate of lime is distilled with sulphovinate of potash, alcohol comes over mixed with an ethereal liquid which dissolves in water, but separates again on the addition of chloride of calcium, in the form of oily drops of a peculiar ethereal odour. I only succeeded, however, in obtaining such minute quantities of this body as to preclude the possibility of fixing its composition.

Dinitroethylate of Lime.

$N_2C_4H_4O_4Ca+3HO$.

This salt is readily prepared by treating solution of dinitroethylate of zinc with excess of hydrate of lime, passing carbonic acid through the solution, and then heating to boiling for a few minutes. The filtered solution deposits, on evaporation, beautiful silky

needles of dinitroethylate of lime, which contain three atoms of water, two of which are expelled at 100° C. An estimation of lime in this salt gave 20·76 per cent.; the above formula requires 20·59 per cent.

Dinitroethylate of Silver

is produced by double decomposition from dinitroethylate of baryta and sulphate of silver. It is very soluble in water, crystallizes in very light scales, and is so speedily decomposed, even with little exposure to light, that no satisfactory analysis could be made.

Double Nitrate and Dinitroethylate of Silver.

$$Ag\,O\,N\,O_3 + Ag\,O\,N_2\,C_4\,H_4\,O_5.$$

This salt is very sparingly soluble in water; it is precipitated in a crystalline granular form when concentrated solutions of dinitroethylate of zinc and nitrate of silver are mixed. The precipitated salt, washed on a filter with distilled water and dried by pressure between bibulous paper, yielded the following analytical results:—

I. ·3850 grm., burnt with oxide of copper, gave ·0941 grm. carbonic acid and ·0500 grm. water.

II. ·1817 grm. gave, on ignition, ·1070 grm. metallic silver.

These numbers agree closely with the above formula.

	Calculated.		Found.	
			I.	II.
C_4	24	6·54	6·67	
H_4	5	1·36	1·36	
N_2	42			
Ag_2	215·94	58·85		58·89
O_{10}	80			
	366·94			

Dinitroethylate of Copper.

$$2(N_2\,C_4\,H_4\,Cu\,O_4) + HO.$$

This salt is prepared by mixing solutions of dinitroethylate of baryta and sulphate of copper. The filtered solution is of a magnificent purple colour; on evaporation *in vacuo* it yields splendid purple needles, which contain half an equivalent of water, and may be obtained several inches in length; they are four-sided prisms. These crystals, separated from the mother-liquor, and dried by exposure over sulphuric acid *in vacuo*, gave the following numbers on analysis:—

I. ·2249 grm., burnt with oxide of copper, gave ·1616 grm. carbonic acid and ·0795 grm. water.

II. ·3038 grm. yielded by Simpson's method 84 cubic centimetres dry nitrogen at

493 mm. pressure and 13° C., or 55·13 cubic centimetres at 760 mm. pressure and 0° C., equal to ·06628 grm.

III. ·1913 grm., treated with dilute nitric acid, evaporated to dryness and ignited, gave ·0593 grm. oxide of copper.

IV. ·1780 grm., dissolved in water and precipitated with caustic potash, gave ·0556 grm. oxide of copper.

The formula above given requires numbers which sufficiently correspond with those obtained:—

	Calculated.		Found.			
			I.	II.	III.	IV.
C_8	48	19·18	19·60			
H_{11}	11	4·39	3·93			
N_4	56	22·37		21·82		
$2CuO$	79·30	31·68			31·00	31·24
O_7	56	22·38				
	250·30	100·00				

Dinitroethylate of Magnesia.

$N_2(C_4H_5)MgO_4$.

Prepared by treating the solution of dinitroethylate of zinc with excess of caustic magnesia, boiling and filtering. The filtered solution, concentrated in a water-bath, yielded granular crystals, which fuse at 100° C. and dry up to a solid amorphous mass. This is the anhydrous salt.

I. ·2504 grm., moistened with nitric acid and cautiously ignited, gave ·0496 grm. magnesia.

II. ·3782 grm., similarly treated, gave ·0758 grm. magnesia.

III. ·0950 grm., treated by SIMPSON'S method, gave 52·5 cubic centimetres nitrogen at 337·0 mm. pressure and 14°·1 C., equal to 22·08 cubic centimetres at 0° C. and 760 mm. pressure, or ·02787 grm.*

The following is a comparison of these results with the theoretical numbers:—

	Calculated.		Found.		
			I.	II.	III.
C_4	24	23·76			
H_5	5	4·95			
N_2	28	27·72			29·37
MgO	20	19·80	19·81	20·04	
O_3	24	23·77			
	101	100·00			

* This determination was made before it was discovered that the nitrogen was always contaminated with binoxide of nitrogen, and hence the number obtained is considerably too high.

Dinitroethylate of Soda.

$N_2C_4H_5O_4Na$.

Prepared by precipitating dinitroethylate of lime with carbonate of soda, and evaporating the filtrate in a water-bath. The residue being treated with strong alcohol, the dinitroethylate of soda dissolves, and is thus separated from the excess of carbonate of soda. The alcoholic solution, evaporated to dryness in a water-bath, yielded minute scaly crystals, which were anhydrous. They gave the following analytical results:—

I. ·1650 grm., treated by SIMPSON'S method, gave 84 cubic centimetres of dry nitrogen at 316·2 mm. pressure and 13° C., or 33·28 cubic centimetres at 760 mm. pressure and 0° C., equal to ·04201 grm.

II. ·2368 grm., moistened with nitro-hydrochloric acid and ignited, gave ·1227 grm. chloride of sodium.

III. ·2337 grm., treated with dilute sulphuric acid and ignited, gave ·1494 grm. sulphate of soda.

The following numbers agree closely with the above formula:—

	Calculated.		Found.		
			I.	II.	III.
C_4	24	21·43			
H_5	5	4·46			
N_2	28	25·00	25·46		
NaO	31	27·68		27·48	27·91
O_4	24	21·43			
	112	100·00			

Products of the Decomposition of Dinitroethylic Acid.

I have stated that when dinitroethylic acid is liberated from its salts, by the addition of concentrated sulphuric acid, it is rapidly decomposed even at 0° C. I have examined the products of this decomposition in the case of the lime-salt, with the following results. A quantity of crystals of dinitroethylate of lime in coarse powder was placed in an apparatus, in which it could be gradually decomposed by concentrated sulphuric acid, the gaseous products collected, and their weight accurately ascertained. The rapidity of decomposition was moderated by the external application of cold water. At the conclusion of the decomposition, it was found that the weight of the gaseous products evolved was equal to 30·6 per cent. of the weight of the lime-salt employed. The weight of gaseous products is therefore almost exactly one-half of the weight of the anhydrous acid contained in the lime-salt (59·6 per cent.). The liquid and solid products of the operation contained sulphate of lime, sulphovinate of lime, and sulphate of ammonia or ethylamine. These gaseous products, after streaming through concentrated sulphuric acid, were collected over mercury and submitted to eudiometrical investigation. The determination

of the specific gravity gave the following numbers:—

Temperature of room	16°·0 C.
Height of barometer	762·8 mm.
Height of inner column of mercury	8·3 mm.
Weight of flask and gas	31·7280 grms.
Temperature in balance-case . .	15°·0 C.
Weight of flask and air	31·6665 grms.
Temperature in balance-case . .	15°·0 C.
Capacity of flask	141·5 cubic centimetres.

From these data the specific gravity was calculated to be 1·3601.

About nine-tenths of the gas was absorbed by an equal volume of recently boiled absolute alcohol. A preliminary examination proved the presence of binoxide of nitrogen, a combustible gas absorbable by fuming sulphuric acid, and an incombustible residue containing combined oxygen.

The estimation of binoxide of nitrogen was effected by adding an excess of oxygen to a known volume of the gas in presence of caustic potash, the excess of oxygen being subsequently removed by pyrogallate of potash. Three determinations gave the following results:—

I.

Pressure of gas used (dry)	178·9 mm.	14°·0 C.
Pressure after absorption of binoxide of nitrogen (dry)	164·3	14°·0

II.

Pressure of gas used (dry)	104·7	13°·0
Pressure after absorption of NO_2 (dry)	95·2	13°·0

III.

Pressure of gas used (dry)	147·8	13°·0
Pressure after absorption of NO_2 (dry)	133·8	13°·0

Estimation of gas absorbed by fuming sulphuric acid, subsequent to the removal of binoxide of nitrogen.

IV.

Pressure of gas used (dry)	243·3 mm.	13°·5 C.
Pressure after action of fuming SO_3 (dry)	177·8	13°·5

V.

Pressure of gas used (dry)	164·3	14°·0
Pressure after action of fuming SO_3 (dry)	121·1	14°·0

The gaseous mixture therefore consisted of—

	I.	II.	III.	IV.	V.	Mean.
Binoxide of nitrogen	8·16	9·07	9·47			8·90
Gas absorbed by fuming SO_3 . .				24·53	23·95	24·24
Gas unabsorbable by fuming SO_3				66·57	67·15	66·86
						100·00

The gas remaining after the action of fuming sulphuric acid yielded the following eudiometrical results:—

VI.

Pressure of gas used (dry)	36·9 mm.	12°·8 C.
Pressure after admission of oxygen (dry)	144·5	12°·8
Pressure after admixture of hydrogen (dry) . . .	191·8	12°·8
Pressure after explosion (dry)	121·8	12°·8
Pressure after action of caustic potash (dry) . . .	121·3	12°·8

VII.

Pressure of gas used (dry)	119·9	14
Pressure after admission of hydrogen (dry)	340·6	14
Pressure after explosion (dry)	231·8	14
Pressure after admission of oxygen (dry)	401·8	14
Pressure after explosion	234·4	14

Analysis No. VI. shows that the gas left unabsorbed by fuming sulphuric acid contains no combustible constituent; for it did not explode with oxygen, and on ignition with hydrogen in presence of excess of oxygen, it neither generated water nor carbonic acid; the diminution of pressure produced on explosion with hydrogen (70·0 mm.) being almost exactly equal to that which ought to be produced by the 47·3 mm. of hydrogen added. Analysis No. VII. proves that on explosion with excess of hydrogen 119·9 volumes of the gas consumed 108·4 volumes of hydrogen, leaving 119·5 volumes of free nitrogen. The gas therefore contained 108·4 volumes of protoxide of nitrogen and 11·5 volumes of nitrogen.

In order to ascertain the composition of the combustible gas removed by fuming sulphuric acid, the original gas, after the removal of binoxide of nitrogen, was exploded with excess of oxygen, so as to determine the amount of oxygen consumed and of carbonic acid generated.

VIII.

Pressure of gas used (dry)	104·3 mm.	14° C.
Pressure after admission of oxygen (dry)	371·5	14
Pressure after explosion (dry)	317·0	14
Pressure after absorption of carbonic acid (dry) . .	260·7	14

IX.

Pressure of gas used (dry)	55·7 mm.	13°·1 C.
Pressure after admission of oxygen (dry)	223·8	13°·1
Pressure after explosion (dry)	193·1	13°·1
Pressure after absorption of carbonic acid (dry) . .	164·5	13°·1
Pressure after admission of hydrogen (dry)	709·8	13°·1
Pressure after explosion (dry)	339·5	13°·1
Pressure after admission of oxygen (dry)	749·1	13°·1
Pressure after explosion (dry)	295·7	13°·1

Analysis No. VIII. shows that 104·3 volumes of gas, containing 27·8 volumes of the body removed by fuming sulphuric acid, consumed 83·0 volumes of oxygen and generated 56·3 volumes of carbonic acid. In analysis No. IX. 55·7 volumes, containing 14·3 volumes of the gas absorbable by fuming sulphuric acid, consumed 44·5 volumes of oxygen and generated 28·6 volumes of carbonic acid. The last three readings in this analysis also prove that the body removed by fuming sulphuric acid is free from nitrogen. The two analyses give the following proportions between the volumes of combustible gas, oxygen consumed, and carbonic acid generated:—

Volume of combustible gas.			Oxygen consumed.			CO_2 generated.	
I.	II.		I.	II.		I.	II.
27·8	14·3	:	83·0	44·5	:	56·3	28·6
or							
1	1	:	2·99	3·007	:	2·02	1·93

The body removed by fuming sulphuric acid is therefore olefiant gas, one volume of which consumes three volumes of oxygen and generates two volumes of carbonic acid.

The whole of these eudiometrical experiments give the mean per-centage composition of the original gas as follows:—

Binoxide of nitrogen	8·90
Olefiant gas	24·24
Protoxide of nitrogen	60·65
Nitrogen	6·21
	100·00

This result is confirmed by the above determination of the specific gravity of the gas, as is seen from the following calculations:—

	Per-centage amount.	Specific gravity.	
Binoxide of nitrogen	8·90	×1·0365 =	9·2249
Protoxide of nitrogen	60·65	×1·5202 =	92·2001
Nitrogen	6·21	×·9674 =	29·4573
Olefiant gas	24·24		
	100·00		130·8823
			100 = 1·3088
Found by experiment			1·3601

I had anticipated that the decomposition of dinitroethylate of lime by sulphuric acid would yield either protoethide of nitrogen ($N C_4 H_5$) or ethoprotoxide of nitrogen ($N C_4 H_5 O$), but the action of the concentrated acid evidently proceeds too far for the production of this result. Further experiments must decide whether or not the employment of a more dilute acid for the decomposition will not lead to the production of one of these compounds.

Action of Zincmethyl upon Binoxide of Nitrogen.

Dinitromethylate of Zinc and Zincmethyl.—Zincmethyl absorbs binoxide of nitrogen much more slowly than zincethyl takes up the same gas; nevertheless at the ordinary atmospheric pressure, the two bodies gradually unite and form colourless crystalline needles, closely resembling, in all their reactions, the dinitroethylate of zinc and zincethyl. I have made no analyses of this body, but considering the homology existing between zincethyl and zincmethyl, together with the product of its decomposition by water, there can scarcely be a doubt that it is dinitromethylate of zinc and zincmethyl, and that its formula is

$$N_2 C_2 H_3 O_4 Zn + C_2 H_3 Zn.$$

It rapidly oxidizes in the air, and takes fire when exposed in considerable quantity. It is instantly decomposed by water, giving light carburetted hydrogen and an opalescent solution of basic dinitromethylate of zinc.

$$\left.\begin{matrix} N_2 C_2 H_3 O_4 Zn + C_2 H_3 Zn \\ HO \end{matrix}\right\} = \left\{\begin{matrix} C_2 H_3, H \\ N_2 C_2 H_3 O_4 Zn + ZnO. \end{matrix}\right.$$

Dinitromethylate of Zinc.

$$N_2 C_2 H_3 O_4 Zn + HO.$$

A quantity of the dinitromethylate of zinc and zincmethyl was prepared by the action of compressed binoxide of nitrogen upon zincmethyl, in the strong copper vessel above described. The resulting crystalline compound was decomposed by water, and the opalescent solution being treated with carbonic acid, boiled, and filtered, yielded, on evaporation, minute crystals of dinitromethylate of zinc. These crystals, dried at 100° C. and submitted to analysis, gave the following results:—

I. 1·4200 grm., burnt with oxide of copper, gave ·5872 grm. carbonic acid * and ·4473 grm. water.

II. ·3585 grm., treated with nitric acid and ignited, gave ·1212 grm. oxide of zinc.

III. ·0982 grm., similarly treated, gave ·0333 grm. oxide of zinc.

* This analysis exhibits a considerable excess of carbon, which is due to the passage of binoxide of nitrogen into the potash bulbs, and the consequent formation of nitrite of potash. It appears to be impossible to decompose the whole of this gas by ignited copper turnings, in conducting the combustion of substances so rich in oxidized nitrogen as the dinitroethylates, and especially the dinitromethylates; hence, in the analyses of the salts of both these acids, the per-centage of carbon is almost invariably too high.

These numbers lead to the formula

$$N_2C_2H_3O_4Zn+HO,$$

which requires the following values:—

	Calculated.		Found.		
			I.	II.	III.
C_2	12	10·30	11·28		
H_4	4	3·43	3·50		
N_2	28	24·03			
ZnO	40·52	34·77		33·81	33·91
O_4	32	27·47			
	116·52	100·00			

Dinitromethylate of Soda.

$$N_2C_2H_3O_4Na+2HO.$$

This salt is formed by treating a solution of dinitromethylate of zinc with carbonate of soda, evaporating to dryness, and treating the residue with strong alcohol. Dinitromethylate of soda dissolves, and the filtered solution, on evaporation, deposits crystals, which, after drying at 100° C., yielded 25·83 per cent. of soda; the above formula requires 26·72 per cent. Dinitromethylate of soda is very soluble both in water and alcohol; it deflagrates violently when heated, and in other respects closely resembles the corresponding salt of dinitroethylic acid.

These determinations, although very imperfect and incomplete, establish the existence of a class of salts containing dinitromethylic acid, homologous with the dinitroethylates; and there can be little doubt that the zinc compounds of the other alcohol radicals will yield corresponding acids, when treated with binoxide of nitrogen.

It is difficult to form any satisfactory hypothesis relative to the rational constitution of this series of acids: they may be regarded as belonging to the type of nitrous acid containing a double atom of nitrogen, and in which one atom of oxygen has been replaced by an alcohol radical, thus—

$$N_2\begin{cases}C_nH_{(n+1)}\\O\\O\\O\end{cases}$$

or they may be viewed as constructed upon the hyponitrous acid type, one equivalent of oxygen being replaced by an alcohol radical, and a second atom by binoxide of nitrogen, thus—

$$N\begin{cases}C_nH_{n+1}\\NO_2\\O\end{cases}$$

Without attaching much value to either hypothesis, I prefer the latter.

By analogous processes, there can be little doubt that many new series of organic acids may be derived from inorganic acids, by the replacement of one or more atoms of oxygen by an alcohol radical; in fact, my pupil Mr. HOBSON is now studying a new series containing sulphur, produced by the action of zincethyl and its homologues upon sulphurous acid: the ethyl acid of this series is formed by the replacement of one equivalent of oxygen, in three equivalents of sulphurous acid, by an alcohol radical.

The following Table exhibits the compounds of the new series of acids, which have been described in the foregoing pages.

	Formulæ.
Dinitroethylic acid	$N_2C_4H_5O_4H$.
Dinitroethylate of silver	$N_2C_4H_5O_4Ag$.
Dinitroethylate of copper	$2(N_2C_4H_5O_4Cu)+HO$.
Dinitroethylate of zinc (crystallized)	$2(N_2C_4H_5O_4Zn)+HO$.
Dinitroethylate of zinc (anhydrous)	$N_2C_4H_5O_4Zn$.
Dinitroethylate of baryta	$N_2C_4H_5O_4Ba$.
Dinitroethylate of lime	$N_2C_4H_5O_4Ca+3HO$.
Dinitroethylate of magnesia	$N_2C_4H_5O_4Mg$.
Dinitroethylate of soda	$N_2C_4H_5O_4Na$.
Double nitrate and dinitroethylate of silver	$NO_6Ag+N_2C_4H_5O_4Ag$.
Dinitroethylate of zinc and zincethyl	$N_2C_4H_5O_4Zn+C_4H_5Zn$.
Dinitromethylic acid	$N_2C_2H_3O_4H$.
Dinitromethylate of zinc	$N_2C_2H_3O_4Zn+HO$.
Dinitromethylate of soda	$N_2C_2H_3O_4Na+2HO$.
Dinitromethylate of zinc and zincmethyl	$N_2C_2H_3O_4Zn+C_2H_3Zn$.

VI. *An account of the Two Methods of Reproduction in* Daphnia, *and of the Structure of the Ephippium.* *By* JOHN LUBBOCK, *Esq., F.G.S. Communicated by* CHARLES DARWIN, *F.R.S.*

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THE genus *Daphnia* is a small group of freshwater Entomostracous Crustacea, which has long been celebrated for possessing the power of reproduction without the intervention of the male, and for laying two different sorts of eggs. In both cases the eggs pass from the ovary into a space or receptacle between the carapace and the back of the animal. The carapace, according to MILNE-EDWARDS, is an excessive development of an anterior, probably the mandibular segment* which encloses the whole of the posterior part of the body, which lies freely in it, as it were in a bivalve shell. This bivalve shell or carapace is widely open below and behind, but is closed along the back, and as it forms there a regular arch, and does not follow the sinuous margin of the back of the animal, there is left between the back and the carapace an open space or receptacle, into which the eggs are laid, and in which they remain until the young *Daphnia* is sufficiently developed to be able to swim about by itself.

This "receptacle" is freely open to the surrounding water, but the eggs are prevented from falling out by a tongue-like projection, developed evidently for that purpose from the back of the animal. When, however, the young are ready for exclusion, the mother has but to depress her abdomen, and they easily slip from the receptacle into the open water. Although both the common or agamic† and the ephippial eggs are thus protected by the carapace or valves until they are hatched, there is this difference, that the agamic eggs are carried about by the animal, and shortly before the next shedding of the skin, the young which have in the meantime been hatched swim away; while for the ephippial eggs, part of the carapace is specialized into a pod-like or saddle-shaped box called the *ephippium*, in which, when the skin is changed, they remain for some time before being hatched.

Several of the earlier naturalists, and more recently Dr. BAIRD, in his excellent 'History of British Entomostraca,' have detailed observations clearly showing that the ordinary eggs are produced, and are fertile, without the intervention of the male, but the mode of production of the ephippial eggs has never been correctly described; nor

* See Ann. des Sc. Nat. 1851, p. 234.

† The agamic eggs certainly do not require impregnation, and I shall give my reasons presently for believing that the male influence is necessary to the hatching of the ephippial eggs; but this is not proved, and it is therefore just possible that both sorts of eggs may be agamic.

has it been determined whether they also are agamic, or whether the stimulus of impregnation is necessary to bring them to maturity.

The agamic reproduction of *Daphnia* has not, indeed, been sufficiently appreciated in the different essays which have recently appeared on this subject, and M. DE QUATREFAGES even says, in an essay on "Les Metamorphoses*," "Des cinq classes composant le sous-embranchement des Annelés supérieurs, ou Annelés à pieds articulés, les insectes seuls paraissent se reproduire par généagénèse: ou n'a du moins encore rien observé de semblable chez les Myriapodes, les Arachnides, les Crustacés, ou les Cirrhipèdes."

The fact, however, rests upon such excellent evidence, that I need not here do more than refer to the observations already mentioned, which are fully confirmed by my own experience, and which may easily be repeated by any naturalist who has still some lingering doubts upon the subject.

I propose now to describe,—

First. The male organs.

Secondly. The formation of the ephippium and the ephippial eggs.

Thirdly. The formation of the common or agamic eggs.

Finally. To collect the scattered notices of parthenogenesis among the Articulata, and to endeavour to show that eggs and buds are not bodies essentially different in their nature, and that between them no distinct line of division can be drawn.

Before, however, entering on my subject, it may perhaps be as well to detail the mode in which I carried on my observations.

My stock of Daphniæ were kept in glass vessels, holding two or three pints each, and they bred there without giving any trouble. Their food consisted of *Scenedesmus obliquus* and other microscopic plants, which covered the sides and bottom of the glasses with a thin green film. *Daphnia Schæfferi* was the species I selected, because it is the largest of the genus, and because it could be at any time easily obtained from a pond in my neighbourhood. This seemed to be important, because, as the species is generally found in horseponds, I feared that my specimens would not live long in confinement. On the contrary, however, they throve well, though in the course of a few generations they diminished considerably in size, suggesting the idea whether they might not after all be merely a gigantic variety of *Daphnia pulex*. This diminution of size in animals kept in vivaria, as compared with those living in their native haunts, is not confined to Daphniæ, nor does it appear, at least in this instance, to be caused by any injurious effect of confinement, because the specimens seemed very healthy and multiplied freely: perhaps, therefore, it may be connected with the known fact that insular specimens have a tendency to become smaller than continental ones of the same species.

When first taken from their native ponds, my *Daphniæ* were generally covered with Vorticellæ, parasitic plants, &c., which materially impaired their transparency; but I was very glad to find that when they had been some time in captivity, they became cleaner and more transparent, and that this was especially the case with those which, for the

* Revue des deux Mondes, 1 Avril, 1855.

purpose of continuous watching, I kept by themselves in tumblers. For the purposes of observation, I used one of SMITH and BECK'S ordinary glass cells, about $\frac{1}{10}$th inch in thickness. This, being glued on to a glass slide, and a piece of thin glass placed over it, formed a cell, which held the *Daphnia*, and prevented it from moving, without doing it any injury. To put them into this cell I used a small metal tube, and in this manner I was able to watch the same specimen at intervals of a few hours, for several days together, and to observe gradually the formation of the eggs in the ovary, without doing any injury to my subject. To identify my specimens without fear of mistake, I placed in each tumbler a small bit of tinfoil, stamped with the number of the page in my note-book.

Until now, "the male organs of generation have never been observed*," and though ZENKER† has described the testes of various species and the generative opening, he has placed and even figured the latter in probably the wrong place. Speaking of *Sida crystallina*, he says, "Sein Ausführungsgang mündet an der den Füssen zugekehrten Seite des Schwanzes auf. Diese Geschlechtsöffnung variert nur sehr wenig ihre Lage durch alle Daphnoiden." Either, however, the situation of the orifice in question *does* differ considerably in different species, or else he is certainly wrong in the place he assigns to it.

If the tails of a male and female *Daphnia* (Pl. VII. figs. 6 and 7) are compared, it will be at once seen that the papillæ (*a*) are much larger in the male than in the female. The vas deferens will almost always be seen leading to this point, and if a mature male be slightly compressed, two clouds of minute bodies will be seen to come out in a stream, one from the papilla of each side of the body, and gradually diffuse themselves in the surrounding waters.

To convince myself that these were really spermatozoa, on the 16th and 27th of June I compressed seven males, and altogether I must have examined about thirty, all with the same result. I then compressed seven females, and neither then nor at any other time could I observe anything similar in this sex. There can, therefore, be no doubt that these are the generative orifices, and that the minute bodies are the spermatozoa.

These latter are rod-like bodies, about ·0004 of an inch in length, and ·000125 of an inch in breadth. They are almost stationary, but if closely watched, a slight wriggling movement can be observed, and several may generally be found curved into an arched form, which, however, may perhaps not be natural, but may be caused by the action of the water. Among these rod-like spermatozoa were several round, probably spermatic, cells, in most of which two bright points could be discerned, which I presume to be the two ends of the spermatozoon.

I have in vain endeavoured to observe the penetration of the spermatozoon into the ovary. Although it is not difficult at certain seasons to find couples attached together, and to place them under the microscope, yet I have always found the male to be merely hanging on to the female by the organs provided for that purpose, and not to be at the

* BAIRD, *loc. cit.* p. 77.

† "Physiologische Bemerkungen über die Daphnoiden," MÜLLER'S Archiv, 1851.

moment engaged in a true act of coition. Moreover, the form of the papilla, which can scarcely be called a penis, justifies the inference that the act of intromission, if any, would be short and easily interrupted. At any rate, I must leave the question as to the mode of introduction of the spermatozoa, undecided for the present. The facts to be hereafter mentioned as to the development of the ephippial egg render it probable that the impregnation takes place at an early period in the development of the ovarian egg, which, unless the act is effected, gradually disappears.

Daphniæ may, however, be found in every stage of ovarian development united to one or two males. I have found them so when the ovaries have been occupied only by nucleated ovarian masses, when the eggs have been almost ready to escape into the receptacle, and when the ephippium has been almost mature. In these cases the coition is probably without any effect; and, indeed, though I have very seldom seen specimens with ephippia where I could find no males, and though the abundance of the males appears to vary with the number of the ephippia, yet I have often confined males and females together in the hope of being thus enabled to trace out the manner of the impregnation of the ephippial egg, but without succeeding in any one instance.

Thus, on the 30th of September, two specimens which were confined together were in coition: this lasted for three-quarters of an hour after I first observed them. The female had then, at the centre of the right ovary, an agamic egg commencing, formed of a somewhat enlarged germinal vesicle, surrounded by some oil-globules and brown granules. On the evening of October 2nd, I again found these two specimens in coition, for at least three-quarters of an hour. At seven o'clock on the morning of the 3rd, I found that eggs had been deposited in the receptacle during the night, and the pair were again in coition. In the ovary were only the usual ovarian masses. On the 4th, 5th and 6th they were again in coition, and on the morning of the 7th a fresh set of eggs had been deposited in the receptacle. Again, on the 19th of September, I placed eleven females in a glass with about as many males, and watched them till the 19th of October, when two females and three males were still alive, and during that time not one ephippium was produced.

These experiments, however, and many others with similar results prove only that the mere stimulus of the presence of males is not sufficient by itself to cause the development of ephippia, but that other concurrent circumstances are necessary.

It has been stated by most writers that the males and ephippia appear only at certain seasons: Zenker, however*, has stated that the males may be found all through the year, and I have myself observed both males and ephippia from the beginning of May till the end of December, with only intervals here and there of a few days. Moreover, the ephippial eggs laid in the spring, summer, and autumn, did not, as is often stated, remain undeveloped till the following spring, but at least in many cases were hatched without any great interval.

I have only once, and then not very satisfactorily, been able to see the actual deposi-

* *Loc. cit.*

tion of the agamic eggs in the receptacle, but I have several times observed the state of the ovary within a few minutes after this took place. The ovary was then as distinct as usual: it consisted of a slightly undulating tube, varying in the middle from ·0033 to ·005 in width, but contracted at both ends, where also the walls became fainter, so that neither the anterior nor the posterior termination could be observed. The anterior contracted part appeared to be empty, or at least a few irregular curved lines only were visible. Nearly opposite to the hind end of the heart was generally an elliptic ovarian mass, with a constriction in the centre of the upper margin, and containing from two to four, or at most five cells, each consisting of a large, perfectly circular nucleus, surrounded by the cells, which formed a sort of narrow border to it. The nucleus can sometimes scarcely be distinguished, but is usually slightly darker than the ovarian mass itself, and this latter than the cells. This ovarian mass occupies the whole width of the ovary, and is followed in a linear series by three or four more, all very similar to the first, all with the lower margin regularly arched, the upper constricted at or near to the centre. The longer axes of these ovarian masses, which is in the direction of the longer axis of the body, is about ·0058 in length, the shorter ·0033: the diameter of the cells varies from ·001 to ·0004. After the first three or four, the masses become smaller, two or more are abreast, and the walls gradually more indistinct: the cells also are usually smaller behind, but not regularly so, often a mass with large cells behind one with small ones. In many cases, however, the order of arrangement is not so regular, nor the masses so distinct; but they appear, as it were, heaped together, and even where the ovary is at first as described above, it gradually changes to this latter appearance. On all sides of the ovary are often scattered circular cells, containing each one or more oil-globules, but they are most numerous at the downward bend of the abdomen. A few hours, or sometimes immediately, after the deposition of the agamic eggs in the receptacle between the back and the shell, it usually happens that at the posterior part of the ovary one of the above-mentioned cells swells a little, and becoming surrounded by brown granules, may be called a germinal vesicle (Plate VI. fig. 2 *g'*). Other cells, sometimes two or three in number, but usually only one, are at first visible in the ovarian mass, but gradually disappear. The contents of this mass continue to grow darker, until it comes to resemble Plate VI. fig. 3 *g'*; after this they generally either fade away, or break up into more or less compact little balls (Plate VI. fig. 8) which gradually disappear.

The ephippial egg in these early stages differs from the corresponding state of agamic eggs in the determinate position and number, being never more than one on each side of the body; and in the constant absence of the large oil-globules, which are constantly present, and show themselves very early in the latter form of eggs.

This embryonic ephippial egg may generally be met with in those specimens which have recently moulted and laid their eggs, and indeed I should say that the cases in which it is not produced are quite exceptional, probably not more than one in five or six.

I have traced the progress of it in thirty-three instances, and in thirty it gradually disappeared again as described above; in the other three the dark granules increased

greatly in number, and formed finally a large, firm-looking, dark homogeneous mass, the ephippial egg. In its later stages it may easily be distinguished from the matter of the agamic eggs, by the darker colour, more regular outline, and absence of large oil-globules.

There are never more than *two* ephippial eggs in *Daphnia*, one being formed in each ovary. Prof. HUXLEY appears to have not very clearly apprehended the relation of these two ova to one another, and to the ephippium, for he compares them to two masses formed in the ephippial egg of *Lacinularia**, whereas they are two distinct eggs, and the ephippium of *Daphnia* is not an egg-shell like the external shell of the winter eggs of Rotifers, but a portion of the carapace, and therefore analogous to the skin of the female Coccus, which acts as a protection to the eggs beneath it. During the last hours before its deposition in the ephippium, the ephippial egg appears to occupy the whole of the ovary, or rather the latter cannot be distinguished; and if at this time the ephippial egg is removed from the body and compressed, there will be seen an immense number of minute cells, some ·00083, but by far the greater number about ·0001 in diameter, and also some of the usual ovarian cells containing the nuclei, which I call germinal vesicles. These latter, however, do not, I believe, belong to the egg itself: the chorion is at this period so delicate, that I have never been able completely to isolate the egg from the other contents of the ovary; but I examined it several times as soon as it was laid, and found it to contain only the above-mentioned minute cells. The mother-cells, or those containing the germinal vesicles, could then be clearly seen in the ovary, and I believe, therefore, that the above-mentioned disappearance of the ovary is only apparent.

The most accurate account given of the ephippium is that by STRAUSS, who says†:—"à certaine époque de l'année, notamment vers les mois de Juillet et d'Août, les valves de la femelle prennent, après la nuce, de l'opacité dans leur partie supérieure, chacune dans une étendue à peu près rectangulaire, s'étendant depuis les environs du premier segment jusqu'au sixième, et descendant jusqu'au dessous de la région des ovaires. Cette partie opaque est d'abord de couleur un peu blanchâtre: mais devenant bientôt plus foncée, elle finit par être d'un gris noirâtre assez obscur. Sur chacune on aperçoit deux ampoules ovulaires, transparentes, placées l'une au devant de l'autre, et formant avec celles du côté opposé deux petites capsules ovales, s'ouvrant comme une coquille bivalve. MÜLLER a nommé ces pièces opaques un *ephippium*, sans dire toutefois ce qu'il en pensait: et comme elles ont en effet, vu leur situation sur le dos de l'animal, quelque ressemblance avec une selle, j'ai cru devoir conserver cette dénomination, pour ne pas introduire de nouveaux termes inutiles. Cet ephippium se partage comme les valves, dont il fait partie, en deux moitiés latérales, réunies par suture le long de leur bord supérieur. Dans son intérieur on en trouve un autre semblable, mais plus petit, à bords libres, et dont les deux moitiés jouent en charnière une sur l'autre."

This account is very accurate: STRAUSS expressly states that the ephippium is a part of the carapace, a fact which has been miscomprehended by subsequent writers. Thus, Dr. BAIRD says (p. 85) that a green matter passes from the ovaries into the receptacle,

* *Loc. cit.* p. 14.

† Ann. de Musée, 1819, vol. v. p. 415.

"and there spreading, forms the ephippium." Professor HUXLEY compares the ephippium with the outer coat of the winter egg of *Lacinularia*, which is a product of the ovary. Prof. MILNE-EDWARDS says*, "On voit alors (vers la fin de l'été) se former dans la cavité ovifère, à la face interne de chaque valve, une lame opaque, qui constitue bientôt un appareil particulier, auquel on a donné le nom d'*ephippium*. Il consiste en deux battans semblables aux valves de la carapace, dont chacun est garni intérieurement d'une ou de deux petites ampoules, transparentes, disposées de façon à former avec leur congénère deux petites capsules bivalves," &c. In this description it is said that the ephippium is formed of two valves, *like* those of the carapace, whereas they are in truth a part of the carapace, and each valve is said to be provided internally with two small transparent "ampoules," which are, however, merely places where the valves are pushed outwards in order to leave the necessary space for the two eggs.

Even Prof. OWEN†, in his very interesting Lectures, has described the ephippium as an opaque layer, "developed on the inner surface of the common incubating cavity," and neither he, nor, so far as I know, any other naturalist, has explained the homologies of the "inner valve," or the limitation of the ephippial eggs to *two*.

I have already described the gradual development of the ephippial egg, until it forms an elongated, dark, homogeneous mass (Plate VI. fig. 10. Plate VII. fig. 1) occupying apparently the whole of the ovary, which at least cannot elsewhere be seen. I have watched the process in three specimens: one unfortunately died, but in the other two I was able to observe the subsequent growth.

At the above stage the receptacle was in both cases occupied by young, nearly ready for exclusion: when this was effected, the only trace of the ephippium was a slight redness in the receptacle. This gradually increased, especially along a bow-shaped line (Plate VII. fig. 1 *p*) very like the caustic which an ordinary teacup throws on the liquid contained in it.

This line and the accompanying redness is formed by a local alteration of the valve, the line being the upper part of the above-mentioned outbowings in which the two eggs will eventually be placed, but the darkening gradually extends until a saddle-shaped piece (Plate VII. fig. 6), of the form described by STRAUSS, is distinguished from the rest of the shell by the darker colour and the smaller size of the cells.

In the meantime the two eggs have left the ovary and deposited themselves in the two ampullæ provided for them. On the 27th of July last, at 5 o'clock in the afternoon, I was so fortunate as to witness this process. I was looking through my compound microscope at a specimen like Plate VII. fig. 1, when suddenly the hind part of the dark ovarian mass turned up along the line (*m*) and elongated considerably. The animal was quite quiet, but the dark matter glided in two streams slowly and steadily into the ephippium, and when there, in a few seconds contracted to less than half its previous length, at the same time becoming oval, and thus forming the two ephippial eggs. The vitelline membrane must, at this period, be very elastic to admit of such alterations of shape.

* Hist. Nat. des Crustacées, vol. iii. p. 377.

† Lectures on Invertebrate Animals, p. 324.

In order to determine, if possible, the condition of the germinal vesicle at the time of deposition, I crushed these two eggs, and found them, as well as many others recently laid, to consist mainly of minute globules, chiefly ·000083 to ·000125 of an inch in diameter, but some as large as ·000250. There were also some large cells from ·0014 to ·0001 of an inch in diameter, and apparently containing a number of the smaller. I could never find in any deposited ephippial egg any of the large oil-globules so characteristic of the agamic eggs, nor any of the "mother-cells" or germinal vesicles so common in the ovary, nor did they appear to possess a double egg-skin. Moreover, if, when the egg has been crushed between two pieces of glass, the upper piece is moved backwards and forwards on the lower piece, the little cells collect into rod-like bundles, remaining, however, distinct, like Plate VII. fig. 11; whereas, if the same power be applied to an agamic egg, the globules appear to run together into a rod, like Plate VII. fig. 12, and are, therefore, probably not true cells.

The ephippium, which at the time of the deposition of the eggs was reddish or light-brown, becomes gradually brown and then black, the protuberances containing the eggs being often darker than the rest. This dark colour seems to be produced by a deposition of granules, which commences gradually after the exuviation of the preceding carapace, and occupies in summer about four days. As the ephippium becomes darker and more matured, so the line dividing it from the rest of the carapace becomes more strongly marked, and the connexion more slight, until at the time that the carapace, of which the ephippium forms a part, is cast, the ephippium is either actually separated, or at any rate can be detached with the greatest ease, and the carapace then looks as if a piece had been cut out from it.

If the outer "saddle"- or "pod"- shaped ephippium is opened, another smaller, but similar case is found inside, enclosing the two eggs. This has been mentioned by all writers on the subject, but no one, so far as I know, has explained the mode of origin, or understood the true nature of it.

It was also some time before I understood this question; but at length, observing one day a specimen with a black ephippium, I remarked that it was already detached along the lines *s*, *s* (Plate VII. fig. 6) from the rest of the carapace, so that I had no difficulty in entirely removing it without injuring the animal; and I then found, to my great surprise, that the "inner valve," instead of being attached to the outer one, along the back hinge, as the descriptions of Strauss*, Milne-Edwards†, and Owen‡ had led me to expect, was actually separated from the outer valve by the new carapace, and was, in fact, in the receptacle between the new carapace and the back.

This artificial removal of the outer valves of the ephippium I repeated several times, and always with the same result. If the operation is effected near the time at which the ephippium would naturally be cast, it may be effected without in any way hurting the Daphnia, which swims about afterwards as if nothing had happened. After a short time, the inner case or valve may be observed slipping out behind from between the

* *L. c.* p. 415. l. 27. † *L. c.* p. 377. l. 36. ‡ *L. c.* p. 324. l. 21.

carapace and the back, the animal leaves off swimming, sinks slowly to the bottom, rests there for a few seconds, and then giving a sudden spring forwards, leaves behind it the inner case, with the rest of the moulted skin.

This experiment renders evident, I think, the true nature of the inner case of the ephippium, which is in fact the inner layer of the carapace, modified in the same manner as the outer. If a thin section of the carapace be made, it will be found to consist of two folds, containing probably between them another, most likely double membrane, the corium.

According to analogy, we may safely conclude that the corium forms round itself the new carapace, which therefore arises between the two layers of the old carapace. Moreover, it is only when fully developed that the lower and side margins of the "inner case" are free; before maturity they may easily be found to be continuous with a very delicate membrane, the inner layer of the carapace.

The diagram (Plate VII. fig. 4) will serve to make this explanation clearer. The lines marked *w* represent the outer and inner layer of the old carapace, in the thickness of which the two layers of the new carapace *x* are formed. The thickness of the carapace is much exaggerated in this diagram, as well as in Plate VII. fig. 5, in order to make the different layers more distinct.

The manner in which the "inner case" with the two eggs is deposited inside the outer one does not need much explanation: it will be seen by the above-mentioned diagram that the lower margins of the carapace are pulled in towards one another, so that the section of each valve, instead of presenting, as at first, a segment of a circle, has a bow shape, the upper or ephippial half being smaller than the other. Plate VII. fig. 4 is a diagram of the section of the carapace along the line Plate VII. fig. 1 *m*, the body of the animal being omitted; and it is therefore evident that the old skin being slipped off, the inner case would naturally come out under and remain enclosed in the outer one.

The receptacle of a specimen which has at the last moult cast an ephippium, will generally be found to be traversed by certain folds: these are caused by the flattening of that part of the carapace which corresponded to the projection in which the eggs were situated.

Plate VII. fig. 3 represents one half of the outer ephippium magnified 250 times, and only partly finished. It consists of small angular cells, about ·000125 in diameter, gradually passing, near the centre of the hollow which is to contain the egg, into small round pit-like cells.

The inner ephippium (Plate VII. fig. 2) is very similar. The lower side margins consist of a plain membrane, in which angular cells gradually become visible: round the central protuberances these cells become elongated, but on the surface of the protuberance regain their first shape. A dotted membrane is also evident here.

I have not succeeded in either obtaining ephippial eggs from isolated specimens, or in clearly proving that these eggs require impregnation, and the facts bearing on this subject may be summed up as follows. In the first place, the common eggs of *D. Schæfferi*,

which I have in this paper called "agamic," certainly do not require to be fertilized in the ordinary manner; and as these two sorts of eggs only, *i. e.* the agamic and the ephippial, are produced when the males are tolerably numerous, and may often be observed *in coitu*, it follows either that impregnation is necessary for the ephippial eggs, or, which in the present state of our knowledge seems less probable, that one or both sorts of eggs may be impregnated, but that neither of them require to be so.

Whenever ephippia were produced, as a general rule, I could also find males; and at the time when there were most ephippia, the males were most numerous; but once when there were a good many ephippia, I could, after careful search, find only one male; and though I continued every morning for several days to look for males, I could not find another specimen, although at the same time ephippia were being developed. In order to determine, if possible, whether ephippia were ever produced without impregnation, I had taken on the 13th of November ten specimens with black ephippia and placed them in a tumbler. These all cast their ephippia; seven of them had then a brood of agamic eggs, three produced ephippia, which, however, might have resulted from impregnation having taken place before I isolated them. On the 29th these two had cast their ephippia, and yet after this three ephippia were developed, which were cast respectively on the 7th, 10th and 17th of December.

Again, on the 7th of December I placed by themselves eight specimens with black ephippia: these all cast their ephippia, and had each a brood of agamic eggs, after which, *i. e.* on the 17th, two more ephippia were visible.

Unless there is a spermatheca in which the spermatozoa can remain in a healthy state, so as to fecundate successive broods, which I do not believe to be the case, these experiments prove that ephippia *can* be produced without male influence. I only, however, met with seven instances*, though I have had at least 400 broods of agamic eggs produced by females kept separate from males.

Moreover, it is not surprising that ephippia should be sometimes so produced, because we know that throughout the animal kingdom eggs are produced *sine concubitu*, although such eggs are usually barren.

Finally, in fifty cases in which I watched at intervals of a few hours the early stages of egg-development, the collection of brown-coloured granules round a germinal vesicle (which is the commencement of the ephippial egg) took place in forty-three; and in seventy in which I watched the subsequent stages it disappeared in all excepting three. I believe, however, that the seven above-mentioned apparently exceptional cases arise only from the animals not having been examined at the right time, and that the early stages of the ephippial egg are always passed through. Something, therefore, must usually be wanting, which is necessary to the further development of it, and I am at a loss to imagine what this want can be, unless it is the absence of impregnation.

We need not long doubt why the *Daphnia* has been provided with this unusual appa-

* Up to the present time (May 26) these ephippia have produced no young, though other ephippial eggs have hatched in the meanwhile.

ratus. The ordinary eggs appear to die if any accident happens to the specimen in whose receptacle they are situated, and are certainly unable to withstand the influence of drought, or of great heat or cold. As the species of this genus are very often found in shallow ponds, which are liable to dry up, or to be entirely frozen, the species would, but for this provision, be often exterminated, in localities usually very favourable to them. An instance of this occurred to me.

The 30th of July was so hot that it killed all the Daphnias which were in glass outside my window, and exposed to the sun. At the same time *Cyclops quadricornis* and *Cypris aurantia* were unhurt. I took some of the ephippia, and in a few days had the satisfaction of finding that some specimens were hatched.

I now pass to the development of the agamic eggs. They are produced in *Daphnia* much more frequently than the ephippial eggs, and I have throughout this paper called them agamic eggs, because, as has long been known, they are remarkable for possessing the power of development without actual impregnation. Almost any amount of evidence might be given on this point. Throughout this summer I have had several females isolated in tumblers, which nevertheless have produced, by the intervention of these eggs, an uninterrupted series of young.

Strauss* says, "Schæffer est le premier qui est reconnu dans ces animaux la faculté qu'ils ont de produire plusieurs générations successives sans accouplement. Il les a observés jusqu'à la quatrième inclusivement. M. de Jurine les a suivis jusqu'à la sixième dans la *D. pulex*, et pense que chez d'autres elle peut aller jusqu'à la quinzième." Dr. Baird also, in his very useful 'History of British Entomostraca' (p. 79), has summed up the evidence of preceding authors, and appended some equally conclusive observations of his own.

The usual appearance of the ovary immediately after the deposition of the eggs has been already described, and is figured in Plate VI. fig. 1. Sometimes, however, the ovarian masses can be only here and there, and indistinctly, seen, and the ovary appears to be chiefly occupied by the cells, some of which afterwards become the germinal vesicles. Gradually, however, the walls of the ovarian masses appear, and generally the brown darkening (*c*), already described as the incipient condition of the ephippial egg, occupies one of the most, if not the most posterior.

In such cases the development of the ordinary eggs begins while the brown darkening is in its most conspicuous condition, and before it has begun to fade away. Where there is no brown darkening, I have usually found the first signs of development to appear from twelve to forty-eight hours after the preceding "laying." In the few cases, however, in which, after this act, the ovary could not be seen, no trace of it reappeared, and the animals died in three or four days.

The place in the ovary at which the agamic eggs arise seems to be quite indeterminate: when there is an ephippial egg begun they may be anywhere in front of it, and in some

* *Loc. cit.* p. 390.

cases where none was present, I have seen agamic eggs beginning quite at the posterior part of the ovary.

The mode of formation is as follows:—we have seen that the brown darkening or first stage of the ephippial egg (*c*) commences round one of the cells, which therefore is doubtless a germinal vesicle. Exactly similar cells and similar ovarian masses occupy also the anterior part of the ovary. See Plate VI. figs. 1 and 2.

One of the cells in one or more of the ovarian masses becomes slightly enlarged, so as to measure about ·002 in diameter, and round it, inside the cell, are deposited several dark granules, and little oil-globules*; the process being the same as in the formation of the ephippial egg, with the addition of the oil-globules. This process may be going on only round one cell or round several at once, as in Plate VI. fig. 3, and the embryonic eggs agree precisely with Dr. MARTIN BARRY'S descriptions and figures of eggs of Vertebrata in a similar condition†. The two sides of the animal do not always present the same appearance, though they seem to have a tendency to do so. At first two or even three other ovarian cells may be seen in the same ovarian masses, but there never appears to be any tendency to development round more than one of them. The others, and the wall of the ovarian mass itself, appear to fade away, leaving the ovule to consist only of the germinal vesicle, granules and oil-globules. These two latter in the meanwhile increase, and the globules become larger and more numerous; the germinal vesicle becomes much less distinct, probably being concealed by the surrounding matter. This is, however, very difficult to determine, because the ovules having now no enveloping membrane, and the component parts not being so firmly kept together by tenacious gelatinous substance, as Dr. NELSON has observed in *Ascaris mystax*, it is almost impossible to extract them from the ovary in anything like a perfect state. In some cases this deposition of granules and globules seemed not to take place round any germinal vesicle, but this I believe is a deception caused by a transverse position of the ovule to the ovary.

As the process of development continues, the ovary-walls begin to be pushed out here and there by the germ-matter, which accumulates in front, and the eggs clothe themselves with an extremely elastic and at first very delicate skin, the vitelline membrane, which does not acquire its full consistency until it has been some time in the receptacle.

The eggs sometimes, after being developed to a certain extent, fade away again and disappear. But these are exceptional cases. I have only seen three or four, although I have watched the process of development so often. These few also appeared to be connected with weakness or disease, for in two of them the animals died soon after, and therefore the phenomenon cannot be compared with the disappearance of the incipient ephippial egg, which is the rule and not the exception, and takes place even when the animal is in excellent health.

* I call them oil-globules merely on account of their appearance.

† See Philosophical Transactions, 1838, Plate V., where there are many cases of embryonic eggs figured as consisting like those in Plate VI. fig. 3, of an outer cell, which afterwards disappears, of a germinal vesicle, oil-globules, and dark granules.

In warm summer weather, about four days after the deposition of the preceding brood of eggs, those whose development we have thus briefly traced are ready to make their entry into the receptacle; the last brood having in the meantime acquired the power of motion, and become fit to swim about freely in the water. Very shortly after the exclusion of the young, the *Daphnia* sheds its skin, and again soon after that the eggs are laid. In what manner this takes place I am unable to say; and that a duct large enough for the passage of the eggs, which, when laid, measure about $\frac{1}{100}$, should be often quite imperceptible, should certainly teach us much caution in describing the anatomy of the lower animals. According to STRAUSS, the eggs pass from the posterior part of the ovary into the receptacle, but in the only specimen in which I observed the process, they *appeared* to emerge close behind the heart.

Several times it happened that at the time of laying, some of the egg-matter remained in the ovary. This was not deposited soon after, as might have been expected, but in one case gradually disappeared; in several it was hidden by the succeeding brood of eggs, and in one it became more diffused, and the creature died.

I have always found these eggs, when crushed shortly after deposition, to consist chiefly of globules of various sizes, which do not appear to possess any distinct membranes, as they gradually run together into large masses. Moreover, if they are placed between two pieces of glass, and after the egg has burst, the upper bit of glass is rolled backwards and forwards upon the lower, the globules run together and form small homogeneous cylinders. If a somewhat older egg is examined, there will be found also a number of minute circular cells, about ·0005 of an inch in diameter, and forming a sort of membrane. These, I believe, constitute the blastodermic layer.

I know not whether the contact of living membrane is necessary for the development of these eggs, but certainly none survived of those which I removed from the receptacle for the purpose of watching.

In many cases the eggs contained a large and distinct globule*, but this is, I believe, merely a large oil-globule; and when the egg is crushed, it appears to have no cell-wall. I have found it much larger and more distinct in the eggs of *D. pulex* than in those of *D. Schæfferi*: in the latter species, indeed, it was not uncommon in the spring, but throughout the summer and autumn, though it was then very conspicuous in *D. pulex*, yet in *D. Schæfferi* the oil-matter was usually in many small globules. That this is in reality a globule, and not a peculiar cell, was proved by pressure often causing it to elongate, and then separate into two smaller balls.

These large oil-globules arise in the ovary, and may be seen there at an early period in the development of the eggs. They appear to be analogous to the "Nahrungsdotter," or nourishment-yolk, described by J. V. CARUS in spider's eggs†.

In the Daphnia eggs they remain unchanged until the young is hatched, and is nearly ready to quit the receptacle, when they break up into other similar but smaller globules.

* See BAIRD, *l. c.* pl. 6. fig. 2.

† Archiv für Wissenschaft. Zool. 1850.

The yolk of the agamic eggs of *D. pulex* at the time of the formation of the blastodermic layer, breaks up into yolk-masses ·0015 in diameter, and containing yolk-globules; in *D. Schæfferi* these masses are smaller and less evident, but still they are present. Zaddach* has noticed the corresponding yolk-masses in *Phryganea grandis*, but asserts that there is nothing similar in *Mystacides*; his observations appear so carefully conducted, that it is presumptuous to suppose him to be in error here, but I cannot help imagining that in *Mystacides*, as in *D. Schæfferi*, the yolk-masses may be present, though small and unobtrusive.

He suggests as a reason for this supposed difference, that the oil-globules are fewer and larger in the *Phryganea* eggs than in those of *Mystacides*; and it is curious that in *D. pulex*, in which the yolk-masses are larger and more evident than in *D. Schæfferi*, the yolk-globules are larger and fewer. In the eggs of *Phryganea*, however, the oil-globules are included in the yolk-masses, while the large oil-globule of *D. pulex* is nearly twice as large as the yolk-masses. In the insects, also, the egg-contents break up into similar masses, as may be readily seen in the wasp or the fly; but this differs from the similar process in *Phryganea*, in preceding not only the appearance of the embryo, but also, if I am not mistaken, the impregnation.

When the eggs are first laid, they are spherical, and from ·009 to ·01 in diameter, but when they have been about two days in the receptacle, they have swollen to ·0125 in diameter, and they then burst their outer shell or vitelline membrane, and appear as broad egg-shaped embryos, consisting of a membranous bag or skin, in which the young animal is contained, and which, not having been present when the egg was laid, must have been formed since.

Mr. Huxley has favoured me with the following notice of a corresponding exuviation in *Mysis*:—"A blastodermic disk, very similar to that described by Rathke in *Astacus*, constitutes the first trace of the embryo in *Mysis*. In the middle of this disk a depression appears, and while the blastodermic disk in front of the depression becomes converted into the bilobed rudiment of the head, posteriorly it is produced into a conical process directed forwards, and applied against the front part of the blastodermic disk, just as is the papillary rudiment of the abdomen and thorax in *Astacus*, with which, in fact, it corresponds.

"Two papillæ, the rudiments of the antennules and antennæ, are developed from each side of the anterior part of the embryo at this stage, and these, as well as the body and the bifurcate or fin-like caudal extremity, become invested by a delicate transparent membrane.

"Up to this point the embryo has been included within the vitelline membrane; but the latter is now burst, and the embryo straightening itself assumes the pear-like form described by Rathke and by Frey and Leuckart, as the first condition of the embryo—these observers having overlooked the earlier condition which has just been described, and

* See his very interesting "Untersuchungen über die Entwickelung und den Bau der Gliederthiere," 1st heft, p. 66.

which is, I believe, of great importance, inasmuch as it seems to be characteristic of the Podophthalmian, as contradistinguished from the Edriophthalmian embryo,—to have the posterior sternal region of the body, or some part of it, at first folded against the anterior.

"The limbs, &c. are now developed in the manner described by RATHKE; but *underneath* the delicate membranous investment of the embryo, which continues to sheath the antennules and the antennæ as they enlarge, but passes smoothly over the rest of the limbs, and eventually extends considerably beyond the end of the abdomen and of the antennules and antennæ, like a glove too long for the fingers.

"This 'larval skin' persists until the embryo has acquired $\frac{1}{14}$th of an inch in length, and has assumed very nearly its adult form. It then disappears, probably becoming cast in the ordinary manner."

So far as I can judge from the figures and descriptions of M. JOLY, a very similar larval skin is developed, and cast by the embryo of *Caridina Desmarestii**.

Mr. SPENCE BATE also, in p. 55 of his very interesting "Report on the British Edriophthalma" to the British Association, 1855, observes,—"It is probable, that about the middle of the period of incubation, the young animal quits the egg, for we have constantly taken them from the pouch, bearing an embryonic character without being closed in their egg-case. The larva at this period is very immature and covered in a general tunic, which, apparently without having any absolute vital connexion with the animal more than the original egg-case had with the embryo, adapts itself in form to the whole creature, and fulfils the duty of a protective tissue."

This remarkable fact has been already observed by the accurate STRAUSS†, who compares it with the gestation of the marsupials. He says, however, "Dans cet état, ils n'offrent qu'une masse arrondie et informe, *sur* laquelle on remarque, quand on l'examine de près, les rudiments obtus du bras;" but here he is in error: the rudimental antennæ are not *on* the spherical membrane enclosing the embryo, but are *inside* it, and are seen through it. Very similar also is the hatching of many Annelids, in which also the young animal quits the egg, as an egg-shaped body without any external organs except cilia, by means of which it swims with activity.

This, however, does not show any disparity in the two cases, for the development of the cilia is probably connected with the free condition of the young animal, and with the tendency to the production of these important organs in the Annelids, and their constant absence in the Crustacea.

It would appear, therefore, that the *Daphnia*, so far from undergoing no metamorphosis, does, in fact, enter the world in a very rudimental condition, and that only after the first change of skin does it assume the well-known characters of the genus. In fact,

* See M. JOLY's paper on that Crustacean in the 'Annales des Sciences Naturelles,' t. xix. See also 'Annals and Magazine of Natural History' (February 1857, p. 163) for an account of a similar phenomenon in *Iulus*.

† *Loc. cit.* p. 417.

the whole process appears strictly comparable to the facts which have been observed in *Monostoma mutabile*.

In the sac above-mentioned is contained the embryo *Daphnia*, in which already, when the vitelline membrane is cast, most of the future limbs may be discerned. The carapace covers as yet only half of the body, and appears to be attached to it by the whole inner surface. The large antennæ also attain about the same length, and are folded down close to the body; their basal portion and two three-jointed branches can be distinguished, and the terminal setæ soon afterwards make their appearance. The small black eye also can now be seen, but the large compound eye, which at first consists of two dark spots which gradually coalesce, does not arise till later. The large upper lip is situated between the two large antennæ, and immediately behind it are the mandibles, which, seen from the side, appear like pears attached by their small ends, but from below appear as hemispherical bosses. Behind the mandibles are the first pair of maxillæ, and behind these again I believe myself to have seen the second pair of maxillæ, as ZADDACH describes them*: but of this I do not feel quite certain. The head is separated from the body by a constriction at the back.

The five pairs of branchial legs form a distinct and separate series, and may about this time be distinguished, arising apparently as folds of the blastodermic membrane. The limbs and outer part of the body at this time consist of a cellular substance, while the centre is occupied by the yolk-masses, and generally a few free oil-globules.

In *Daphnia* therefore, as in Crustacea generally, the appendages make their appearance before the body is divided into segments.

The earlier naturalists who have written about the *Daphniæ* have asserted that all the young developed from any one brood of agamic eggs, are of one sex. I have never met with an exception to this rule, and have little doubt that it always holds good; in one case, indeed, a very large brood, eighteen in number, were all males, though this sex is usually, and was then, much rarer than the female.

Before quitting the subject of the agamic eggs, it will not be uninteresting to compare the process with that of the formation of the corresponding eggs of Aphides, as given by LEYDIG†, as opposed to the very different accounts of J. V. CARUS‡ and Dr. BURNETT§.

It is evident that in LEYDIG's Plate 5. A, fig. 1, 1 is homologous to the mother-cells, which I have in this paper called ovarian masses (*c*); the cells (*a*, *a*) correspond to (*g*), and the enlarged one (*b*) to those marked (*g'*) in my figures.

Indeed, if we except the non-essential difference, that in *Daphnia* the developing eggs do not alter their place in the ovary, the whole process, as now detailed, is so similar to that which is described by LEYDIG in the Aphides, as to confirm in a remarkable manner

* Ueber den Bau und Entwickelung der Gliederthiere.

† SIEBOLD and KÖLLIKER's Archiv, 1850, p. 62.

‡ Zur naheren Kenntniss des Generationswechsels.

§ Dr. BURNETT "On the Development of Viviparous Aphides," Ann. and Mag. of Nat. Hist., Aug. 1854 and in Silliman's American Journal, January 1854.

his observations, and on the other hand, of course, to throw more doubt on the entirely different accounts of J. V. Carus and Dr. Burnett. I must, however, confess, that like the two last-mentioned observers, I have been quite unable to detect any germinal vesicle in the eggs of the viviparous Aphides; it is, however, very difficult to find even in those eggs which are destined to be impregnated in the ordinary manner, or indeed in any insect eggs, while, on the contrary, I have found it to be very large in all the eggs of Crustacea which have come under my notice.

Parthenogenesis in the Articulata.

From the hardness of the outer skin, and from the finality in the mode of growth, parthenogenesis in the Articulata is effected by means of bodies in their origin and structure very similar to true eggs.

The Crustacea are so seldom bred and watched in captivity, that though the Daphniadæ is the only family in the class which is as yet known to reproduce by parthenogenesis, yet the instances may hereafter prove to be much more frequent, as the fact that no males have yet been found of *Polyphemus oculus*, *Limnadia gigas* or *Apus* would seem to suggest.

Similar phenomena have been recorded by Dumeril*, and more recently by Newman†.

In the Insects, excepting the celebrated case of the Aphides and the genera *Cynips* and *Apophyllus*, as stated by Hartig‡, whose assertions have been confirmed by the appearance of *Cynips lignicola* in great numbers in the south-west of England within the last year or two, where in several thousand specimens not a male occurred;—with these exceptions, I say, the phenomenon of parthenogenesis has as yet only been met with in the Lepidoptera, probably because this is the order most frequently bred by collectors.

Pallas has observed this phenomenon in *Euprepia casta*; Bernoulli in *Gastrophaga potatoria* and *Episema cæruleocephala*; Basler in *G. quercifolia*; Suckow in *G. pini*; Treviranus in *Sphinx ligustri*, and Nordmann in *Smerinthus populi*. These instances are mentioned in Burmeister's Manual, p. 312; and Siebold§ has attempted to explain them away, but in the present state of our knowledge the assertions are more probable than the explanation.

Moreover, Siebold himself has observed parthenogenesis in *Solenobia lichenella* and *S. clathrella*‖, in *Psyche helix* and in *Apis*¶.

Lacordaire** mentions the same fact as having occurred in *Liparis dispar* for three successive generations.

* Académie des Sciences, 28th Sept. 1856.

† An Essay on the employment of Physiological Characters in the Classification of Animals.

‡ Germar's Zeitschrift, vol. ii. p. 178.

§ Wahre Parthenogenesis bei Schmetterlinge, &c.

‖ Bemerk. über Psychiden: Jahresbericht der Schlesischen Gesellschaft, &c., 1850; translated also in the Transactions of the Entomological Society of London, N. S. vol. vi.

¶ Wahre Parthenogenesis, &c. ** Introd. to Ent. vol. ii. p. 383.

Mr. P. J. Brown* has observed parthenogenesis in *Sphinx populi* and in *Arctia caja*, and states that M. Wagner has observed the same in *S. ocellata*.

M. H. Lehocq† has met with a similar instance in *Arctia caja*; Dr. Kipp in *Sphinx populi*‡, quoted by Siebold; Mr. Johnston in *Smerinthus ocellatus*§; Mr. Curtis in *Bombyx polyphemus*‖; and in *Bombyx mori* various naturalists have convinced themselves of the same facts.

Mr. Westwood has favoured me with the following extract from his unpublished notes:—"Mr. Davis (the publisher of the Entomological Magazine, and himself a good entomologist) informs me of a singular circumstance connected with the Egger moth, which has been observed by Mr. Tardy, the Irish entomologist, after whom *Mesites Tardii* has been named, this gentleman having reared three generations of this moth from a single impregnation."

The Rev. P. H. Newnham of Guildford, the Rev. Hugh A. Stowell and Mr. E. W. Robinson have kindly informed me by letter that they have observed the same fact respectively in *Smerinthus populi*, *Arctia villica*, and *Arctia caja*.

In *Psyche helix* and *Solenobia* parthenogenesis appears to be the rule, while in the other Lepidoptera it occurs as an exception; taking then these nineteen exceptional cases, it will be observed that they have all occurred in the hawkmoths or moths, and not one in the butterflies; four in *Smerinthus populi* and two in *S. ocellatus*, and three in *Arctia caja*; that, taking the genera, five are in *Smerinthus*, four in *Arctia* and *Gastrophaga*, and two in *Bombyx*. No doubt the Silkworm moth (*B. mori*) has been so much and so carefully observed, that if parthenogenesis did occur even rarely in that species, it would certainly have been noticed by many observers; but I can suggest no explanation of the very unequal distribution of the other cases, except that agamogenesis does occur more frequently in some genera and species than in others, and *à fortiori*, that it takes place in some.

But if Professor Siebold and M. Dzierzon are correct, the Hive-bee (*Apis mellifica*) presents us with the most wonderful instance of parthenogenesis as yet known among the Articulata, since in this instance the males always arise from unimpregnated eggs; and those eggs which, if impregnated, would have produced females, will, if no spermatozoa are present, give birth to males¶.

Several naturalists have convinced themselves that some plants have also the power of producing agamic seeds, and Gærtner has given an abridged account of the experiments on the subject.

The plants in which the most complete experiments have been undertaken, and in which agamic seeds are supposed to have been obtained, are as follows:—

Zea mays, by R. J. Camerer, and again by Henschel.

* Mag. of Nat. Hist. viii. p. 557.
‡ Bienenzeitung, 1853, p. 1752.
‖ Ann. des Sc. Nat. 1851, p. 297.
† Comptes Rendus, Dec. 8, 1856.
§ Zoologist, 1848, p. 2269.
¶ Wahre Parthenogenesis, &c., *passim*.

Cannabis sativa, by Spallanzani, Henschel, Giron de Buzareingues, Bernardi.
Spinacia oleracea, by Spallanzani, Henschel, Giron de Buzareingues.
Cucurbita melopepo, Spallanzani.
Cucurbita citrullus, Spallanzani.
Coyx lacryma, Henschel.
Lychnis dioica, Henschel, Giron de Buzareingues.
Mercurialis annua, Ramisch.

Gærtner, however, is inclined to doubt the truth of the results obtained, and to ascribe the supposed agamic fertility either to pollen brought from a distance in the air, to the unnoticed presence of male flowers, or to "crypto-hermaphroditism." Mr. Smith, Curator of the Royal Kew Gardens, mentions a case of seeds which germinated without any apparent action of pollen as occurring in *Cælebogyne ilicifolia*, a native of Moreton Bay, and belonging to the Euphorbiaceæ, which was introduced into this country by Mr. Cunningham, who sent three plants to the Kew Gardens in 1829.

These plants produced female flowers from which perfect seeds have been obtained for several successive years, although Mr. Smith could never detect "anything like male flowers or pollen-bearing organs:" the ovarium and the seeds presented "the usual structure of Euphorbiaceæ, such as it occurs in *Croton*, *Phyllanthus*, *Cluytia*, &c.*"

Still more recently†, M. Henri Lehocq has observed the same phenomenon. He says, "Mes expériences ont été faites sur le chauvre, l'épinard, le *Mercurialis annua*, le *Trinia vulgaris*, le *Lychnis sylvestris*, et sur une Cucurbitacée dont j'ignorais le nom spécifique. Je n'ai pas besoin de rappeler que j'avais pris toutes les précautions possibles pour isoler mes plantes, et cependant, à l'exception du Cucurbitacée et du *Lychnis*, toutes les autres me donnerent des graines fertiles‡."

Dr. Carpenter§, and after him many authors, as for instance Dr. Burnett‖, Carus¶, and M. de Quatrefages, have sustained with ability the theory that the specimens which produce the so-called internal buds or gemmæ are not females, but entirely without sex. Thus, M. de Quatrefages** says, "Toutes les générations intermédiaires développées entre les termes extrêmes de ce cycle sont agames, c'est à dire, manquent de véritables organes reproducteurs, et se multiplient exclusivement par bouture et par bourgeon interne ou externe."

These and similar assertions, however true with reference to the Zoophytes, require great modifications with regard to the Articulata.

Prof. Siebold, who in his "Bemerkungen über Psychiden," in the 'Jahresbericht der Schlesischen Gesellschaft,' &c., 1850, p. 85, appears to have taken for granted that the self-fertile individuals of *Talæporia* (*Solenobia*) are, "in ihrer organisation von weiblichen Thieren ganz verschiedene geschlechtslose Individuen" (in their organization from the

* See Transactions of the Linnean Society, vol. xviii. p. 509.
† Comptes Rendus de l'Académie des Sciences, 8 Dec. 1856.
‡ See also Journal of Microscopical Science, July 1857, p. 228.
§ *L. c.* p. 443. ‖ *L. c.* p. 96. ¶ *L. c.* ** *L. c.* p. 116.

female animals, entirely different sexless individuals), in his latest and very interesting essay, "Wahre Parthenogenesis bei Schmetterlingen und Bienen," admits that the self-fertile individuals in these insects are true females.

This is also not the less evident in *Daphniæ*; for in them not only may the same specimen produce first agamic eggs, then ephippial, then again agamic, and then again ephippial eggs, so that it would have to be considered first sexless, then female, then sexless and then female again, but actually, and that usually, the embryonic forms of both eggs are present at the same time. Plate VI. fig. 3, for instance, represents a specimen with an ephippial egg, beginning at the posterior end of the ovary, and some agamic eggs in front: and as this is no solitary exception, but in far more than half the specimens I have watched, this contemporaneous development of the early stages of both eggs has occurred; and as in most cases the agamic eggs are developed, and the others disappear, there can be no doubt that the self-fertile specimens are females.

Throughout this paper I have applied the name of *eggs* to the ordinary reproductive bodies of the *Daphniæ*, although aware that it is customary to call all egg-like bodies which are fertile without immediate impregnation "internal buds" or "gemmæ," and to consider them as essentially different from true eggs. Dr. CARPENTER* and Dr. BURNETT† have adopted this theory, and M. DE QUATREFAGES also‡ says, "Seulement chez les dernières [winged female Aphides, in opposition to the so-called sexless specimens] on trouve de véritables œufs, pourvus de toutes leurs parties caractéristiques; chez les premières de petites masses granuleuses, où l'on ne distingue jamais ni vitellus, ni vésicule germinative, ni tâche de WAGNER."

In this statement he probably relies on the descriptions of J. V. CARUS and Dr. BURNETT, but LEYDIG has already shown that there is good reason for supposing them to be in error; at any rate it is by no means applicable to the *Daphniæ*. In these we find in the ovary a number of ovarian masses containing small cells. One of these latter swells, dark granules collect around it, and it becomes a germinal vesicle. This is the foundation of an ephippial egg (see Plate VI. fig. 2). Just in front, another cell swells a little, becomes surrounded in a similar manner by dark granules, and also by oil-globules, and this process continuing, forms the ordinary egg. I say *egg*: for how can we deny this term to a round body developed in the ovary of a female, and round a germinal vesicle?

Truly has M. DE QUATREFAGES observed§, "Ainsi l'on passe de la simple croissance d'un mammifère, au bourgeonnement le mieux caractérisé par des nuances insensibles: et tout nous ramène à cette importante conclusion, que le bourgeonnement, et par conséquent la réproduction agame, ne sont au fond qu'un phénomène d'accroissement." But why stop the series here? Dr. BURNETT says, indeed‖, that "the structure and conditions of all true ova are the same, and there is no passage between them and buds;" but this

* See Dr. CARPENTER 'On the Microscope and its Revelations,' p. 279.
† *Loc. cit.*
‡ *Loc. cit.* p. 121.
§ *Loc. cit.* p. 129.
‖ *Loc. cit.* p. 88.

assertion is disproved by LEYDIG's observations on the Aphides*, and those now related in the *Daphniæ*. Prof. OWEN, on the contrary, has maintained that buds and eggs are not bodies essentially different in their nature, but that we may expect to find every gradation between them. It will probably ere long be convenient to apply some distinguishing name to those egg-like ovarian products which do not require impregnation as a necessary antecedent to development, but for the present, at any rate in the *Daphniæ*, we must call them eggs, which they more nearly resemble than ordinary gemmæ.

Prof. HUXLEY might be supposed to agree with the naturalists as to the essential difference between eggs and buds above-mentioned, since he has proposed, developing the idea of which we owe the germ to Dr. CARPENTER, to extend the use of the word "individual" to the whole product of one impregnation, and to designate as "zooids" the independent forms of the individual†. Prof. HUXLEY, however, proposes this system of nomenclature merely as convenient, and not as expressing any fundamental, structural, or potential difference between eggs and buds.

I believe that the development of the ephippial and agamic eggs of *Daphnia* now described is sufficient evidence of the fundamental identity of eggs and buds, or rather of their forming parts of one and the same series, a point on which I would particularly insist, and the cases of agamogenesis in *Cynips* and many Lepidoptera are equally conclusive.

If, however, SIEBOLD is correct in stating‡ that drones always proceed from unimpregnated eggs, and in fact that it is the presence or absence of impregnation which alone determines the sex of the future bee, a statement not unsupported by careful observations and experiments, I cannot imagine that even the most sceptical naturalists can retain any doubts upon the subject.

Whether we retain the old nomenclature, or dropping the idea of unity in the term "*individual*," adopt the system proposed by Prof. HUXLEY, we shall be met by great difficulties and inconsistencies; but I am not yet convinced of the superiority of the latter system, which would prevent us from knowing whether any given specimen of certain animals, say for instance any *Sphinx* or *Daphnia*, was an individual or not, and according to which, if SIEBOLD's assertions are correct, there are no male individuals of the Honey-bee.

EXPLANATION OF THE PLATES.

PLATE VI.

Fig. 1. Ovary of *Daphnia Schæfferi*, shortly after the deposition of agamic eggs in the receptacle.

Fig. 2. The same, a few hours later.

* *Loc. cit.* † Ann. and Mag. of Nat. Hist. 1852. ‡ Wahre Parthenogenesis, &c.

Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6. } Subsequent changes in the same ovary.

Fig. 7. Early stage of the ephippial egg.

Fig. 8. The posterior part of the ovary, showing the little balls into which the embryonic ephippial egg often breaks up.

Fig. 9. Ephippial egg more developed. It has become so opake that the germinal vesicle cannot be seen.

Fig. 10. Ephippial egg nearly ready to be laid.

PLATE VII.

Fig. 1. Part of the back of a specimen of *D. Schæfferi*, ready to lay an ephippial egg.

Fig. 2. Half of the inner valve of an ephippium, only in part finished.

Fig. 3. Half of the outer valve of an ephippium, only in part finished.

Fig. 4. Diagram of a section of the carapace along the line *m*. The body is omitted.

Fig. 5. Diagram of a section of the carapace along the line *n*. The body is omitted.

Fig. 6. Part of a female *Daphnia Schæfferi*, showing the ephippium in outline.

Fig. 7. Outline of a male *D. Schæfferi*, to show the male generative organs (*q*).

Fig. 8. Part of an ovary of a *D. Schæfferi*, to show agamic eggs just beginning.

Fig. 9. Part of an ovary of a *D. Schæfferi*, to show agamic eggs in the process of formation, each round a distinct germinal vesicle (*g'*).

Fig. 10. Ditto of another specimen.

Fig. 11. Contents of ephippial egg rolled together.

Fig. 12. Contents of agamic egg rolled together.

Fig. 13. An ovarian mass, containing three ovarian cells, any of which may become a germinal vesicle.

The letters refer to the same parts in all the figures:—

b. Line of the back of the body.
c. Ovarian masses.
e. Early stages of the ephippial egg.
g. Ovarian cells, some of which become germinal vesicles, *g'*.
h. Heart.
i. Upper line of intestine.
j. Lower line of intestine.
p. Ephippium.
q. Penis.
r. Receptacle or space between the back and the shell.
w. Outer and inner layers of old carapace.
w'. Outer and inner layers of old carapace modified to form the ephippium.
x. Outer and inner layers of new carapace.

VII. *On the* Scelidothere (Scelidotherium leptocephalum, Owen).

By Professor Owen, *F.R.S. &c.*

Received October 30,—Read December 18, 1856.

The extinct species of large terrestrial Sloth, indicated by the above name, was first made known by portions of its fossil skeleton discovered by Charles Darwin, Esq., F.R.S., at Punta Alta, Northern Patagonia, which were described by me in the chapters of the Appendix to the 'Natural History of the Voyage of H.M.S. Beagle,' treating of the "Fossil Mammalia*" collected during that voyage.

The subsequent acquisition by the British Museum of the collection of Fossil Mammalia brought from the pleistocene beds, Buenos Ayres, by M. Bravard, has given further evidence of the generic distinction of the Scelidothere from the other Gravigrades of the *Bruta phyllophaga*†, and has supplied important characters of the osseous system, and especially of the skull, which the fragments from the hard consolidated gravel of Punta Alta did not afford. The best portion of the cranium from the latter locality wanted the facial part anterior to the orbit, and the greater part of the upper walls; sufficient, however, remained to indicate the peculiar character of its slender proportions, and hence the name *leptocephalum* proposed for the species.

The aptness of the epithet 'slender-headed' proves to be greater than could have been surmised from the original fossil; for the entire skull now in the British Museum (Plates VIII. and IX.) exhibits a prolongation of the upper and lower jaws, and a slenderness of the parts produced anterior to the dental series, unique in the leaf-eating section of the Order *Bruta*, and offering a very interesting approximation to the peculiar proportions of the skull in the Ant-eaters.

The original fossils on which the genus *Scelidotherium* was founded, indicated, by the persistence of the cranial sutures and the condition of the epiphysial extremities of the long bones, that they had belonged to an individual of immature age: the difference of size between them and the corresponding parts in the British Museum, depends on the latter having belonged to full-grown individuals: the slight difference in the shape of the anterior molars seems, in like manner, to be due to such an amount of change as might take place in the progress of growth of a tooth with a constantly renewable pulp. I find, at least, no good grounds for inferring a specific distinction between the mature, if not old, Scelidothere from Buenos Ayres and the younger specimen from Patagonia.

* 4to, 1838–40.

† See 'Conspectus Familiarum et Generum Brutorum frondes carpentium,' in 'Description of the Skeleton of the *Mylodon robustus*,' 4to, 1842.

The first peculiar character which the entire cranium of the Scelidothere presents is the increase in vertical diameter (if the thin pterygoid plates be excluded), from the occiput (3, 4) to the anterior nostril (15, fig. 2, Plate VIII.): this character appears to be due, mainly, to the depth of the upper jaw required for the lodgement of the molar teeth and their persistent pulps.

The occipital condyles (Plates VIII. and IX. 3), though terminal, are less prominent than in the Mylodon or Megathere: they are relatively wider apart: their inner straight sides converge towards their lower end. The foramen magnum (Plate IX. *o*) is wider relatively than in the Megathere: its upper boundary is festooned, a median notch dividing two convex curves of the thin and rather prominent border: just within this border are two pits: on the inner side of each condyle is a depression: the inner sides of the occipital foramen are impressed with the oblique canals leading to the precondyloid foramina, *p*, *p*. The plane of the foramen magnum forms, with that of the basioccipital, an angle of 135°.

The occipital region, which was vertical in the young specimen first described, inclines a little forward, as it ascends, in the adult skulls, but to a less degree than in the Megathere: its height from the upper border of the foramen magnum is half its breadth: it is bisected by a strong median vertical ridge, with a small venous foramen at its lower end: an arched semicircular thicker and rougher ridge forms its upper and lateral boundaries: this ridge bifurcates at its lower part on each side, intercepting thereby a narrow longitudinal fissure, corresponding with the wider lateral fossa of the occipital region in the Megathere, but here rather resembling the digastric fissure in ordinary mammals: the fissure is continued into the rough interspace between the mastoid and paroccipital. In each moderately concave division of the occipital region there is a low subvertical lateral ridge, which is nearer the outer boundary than is the corresponding ridge marked *e* in the skull of the Mylodon, in Plate V. fig. 1, of my "Memoir on the Mylodon," 4to, 1842.

The paroccipital (Plates VIII. and IX. 4) is relatively larger than in the Megathere, the mastoid (6) is relatively smaller: there is no second superoccipital ridge as in the Megathere.

The lateral rough surfaces (Plate IX. *r*) for the attachment of the *recti capitis antici* muscles to the basioccipital, are less prominent than in the Megathere: the under surface of both the basioccipital (1) and basisphenoid (2) is flatter: the lateral basisphenoidal protuberances are low and obtuse, relatively larger than in the Megathere, not angular as in the Mylodon.

The precondyloid canals are relatively smaller than in the Mylodon; their outlets (ib. *p*), as in that animal, are behind and distinct from the foramen jugulare or foramen lacerum (*j*), not blended therewith as in the Megathere.

Corresponding with the indication of the smaller size of the tongue thus given by the canal for its muscular nerve, is the less relative size and depth of the stylohyal depression (*v*). The carotid canal (*c*) seems to have begun at the fore-part of the foramen lacerum. The rough bifid under surface of the petrosal (18) divides it from the auditory cavity.

The loose tympanic bone is wanting. The eustachian groove (*e*) is plainly traceable from the fore-part of the auditory cavity, expanding to behind the base of the pterygoid plate.

The superoccipital extends upon the upper surface of the cranium in an angular form for more than an inch between the parietals. The temporal ridges are separated from each other by an interspace of only 8 lines, at the narrowest part of the intervening space upon the parietals, in the mid-line of which space is a longitudinal furrow.

The sutures between the parietal, the occipital and the temporal bones, remaining in one of the crania of the Scelidothere in the British Museum, show the same small vertical extent of the long squamosal as in the young individual first described*; and the suture dividing the mastoid and squamosal below from the parietal above forms an irregularly undulating, almost straight line to opposite the end of the zygomatic process, where it descends and defines the subangular anterior boundary of the squamosal. At this point the squamosal appears to touch the frontal, an angular process of which extends backwards to the foramen ovale (Plate VIII. fig. 2 *l*), dividing the fore-part of the squamosal from the pterygoid; the three bones appear to concur in forming that foramen.

The zygomatic process, commencing by the ridge continued from the temporal one, as it passes from the parietal to the temporal bone, has an antero-posterior extent of attachment of 3 inches 8 lines; the free projecting end does not exceed 1½ inch in length: it is obtuse, with a slight notch below. The interspace between it and the squamosal plate is 1½ inch. The process (Plate VIII. figs. 1 & 2, *w*) is trihedral, the outer side almost flat, a little concave behind and as much convex in front: the upper surface forms with the narrow squamosal plate a deep and wide channel, which lodged the lower border of the temporal muscle: the under side is the shortest and broadest, and forms the almost flat subtriangular surface (Plate IX. *g*) for the articulation of the mandible. This surface differs from the more concave one of the Megathere, in the absence of any defensive ridge or prominence for the joint either behind or externally.

The pterygoid plates (Plate IX. 24) articulate with the sides of the basi- (5) and pre- (9) sphenoid, leaving a free interspace along the under surface of those bones an inch wide at its back part and gradually narrowing to the rounder styliform prolongation of the presphenoid (9). The posterior border of the pterygoid slopes very obliquely from above downwards and forwards, and then more gradually rises with an irregularly undulating outline to blend with the palatine ridge (20), continued to the back part of the alveolus of the last molar. The combined palatines and pterygoids form a long and wide concavity, in which the oblique ridges, diverging from the back part of the bony palate, define the acutely angular contour of the posterior palatal margin, and at the same time the lower boundary of the posterior nasal aperture (ib. *n*); this aperture, by the production of a ridge from the pointed end of the presphenoid, is shaped like a heart on playing-cards: it gradually expands into the capacious pterygo-palatine fossa (ib. *n*, *n*, 24, 24).

The parietals (Plate VIII. fig. 2, 7) are quadrate bones, 6 inches in fore and aft extent: the posterior boundary of the frontals (ib. 11) is 8 inches from the occipital ridge; the

* Zoology of H.M.S. Beagle, "Fossil Mammalia," p. 73.

antero-median angles of the parietals are produced between the frontals, but to a less extent than the superoccipital in relation to the parietals. The temporal ridges are continued forwards from the parietal on to the frontal bones as far as the low obtuse postorbital process: in this course each temporal ridge makes three curves with two intervening angles, one at the fronto-parietal suture, the second not quite half-way between that suture and the postorbital process. The frontal region of the cranium swells out a little above the zygomatic arch, but this affords no indication of cerebral development; it is anterior to the true cranial cavity, and is due to the development of extensive frontal sinuses. The fore-part of the frontal terminates in an angle, wedged between the nasal and lacrymal bones, above the middle of the orbit: the descending plate forms the back part of the inner wall of the orbit, where it unites with the maxillary, and then joins the palatine and pterygoid, terminating behind at the foramen ovale (ib. *l*). The foramen rotundum (ib. *r*) is 2 inches 3 lines in advance, opening upon the suture between the frontal and pterygoid, and continued forwards by a canal containing the suture between the frontal and palatine.

The palatine bones (Plate IX. $_{20}$) surround the posterior nasal aperture, and support the ridges (*n*, *n*) defining the lower boundary of that aperture: these ridges divide the inner and under surface of the palatines into an intranasal and an infranasal part: both contribute to form the fore-part of the long and wide faucial fossa.

The maxillary bone forms the major part of the narrow, transversely convex, bony palate (Plate IX. $_{21}$), the true palatines terminating anteriorly in a point opposite the penultimate molars (ib. *iv*): the orbital plate of the maxillary (Plate VIII. fig. 2, $_{21'}$) ascends to articulate with the nasal ($_{13}$), frontal ($_{11}$), and lacrymal ($_{73}$), and circumscribes the antorbital foramen (*a*), by an outstanding malar process, which affords a sutural surface behind for articulation with the malar bone (ib. fig. 1, $_{26}$): the maxillary then expands into a broad subquadrate facial plate (ib. fig. 2 *a*, $_{21}$), smooth and convex externally; uniting above with the long nasal ($_{13}$), behind with the anterior half of the lacrymal ($_{73}$) and with the malar ($_{26}$), and below and in front with the premaxillary ($_{22}$). The outer alveolar wall makes a series of prominences corresponding with the three anterior molars: the inner alveolar wall is a thin plate, forming a bony case for the deeply implanted parts of the molars and their matrices; projecting in relief from the inner side of the facial and orbital plates. The length or depth of the first alveolus (Plate IX. $_{1}$) is 3 inches 3 lines; that of the second ($_{2}$) is 3 inches 9 lines; the succeeding ones ($_{3, 4, 5}$) gradually decrease in depth. A canal is continued from that of the dental nerve above the alveoli, forward and downward to the lower and lateral part of the outer nostril.

The alveoli, five on each side of the upper jaw, are close-set, with oblique apertures, the inner boundary descending lower than the outer one: their aggregate antero-posterior extent measures 4 inches 3 lines in one adult skull, 4 inches 6 lines in a second. The teeth (Plate IX. *i*–*v*), when in place, show wider intervals than the sockets; the situation of the thick periosteum and dental capsule being occupied by the infiltrated matrix which, in the fossil under view, binds the tooth in its place: the shape of the socket

nearly corresponds with that of the tooth it contains. The maxillary extends $4\frac{1}{4}$ inches in advance of the first alveolus, in an obtusely pointed form, slightly diverging from its fellow to form the hinder two-thirds of an unusually long prepalatine or incisive fissure (*i*). The side of the fore-part of the maxillary is deeply notched for a corresponding process of the premaxillary ($_{29'}$). The upper surface of the palatine plate is longitudinally grooved, and the inner border of the groove unites with its fellow to form a median ridge.

The premaxillaries ($_{29}$) have partially coalesced anteriorly, where each forms an obtuse, rounded, subcompressed protuberance, expanding as it passes backwards and bifurcating. The inner branch, 3 inches in length, uniting with its fellow, forms a long and slender trihedral process, dividing the prepalatine fissure, at its upper part, into two: the ridged summit of this process is grooved at its fore-part. The outer process of the premaxillary ($_{29'}$), which answers to its ascending branch (*branche montante* of Cuvier) in ordinary mammals, forms the horizontal lower part of the boundary of the external bony nostril, and terminates obtusely a little behind the angle between that part and the vertical part of the boundary, filling up the before-mentioned maxillary notch: the length of this process is $3\frac{1}{2}$ inches. The entire length of the premaxillary is $4\frac{1}{2}$ inches: the obtuse anterior ends of the premaxillaries are partially separated by a notch.

The relative size, shape and connexions of the lacrymal bone are better defined in one of the crania of the Scelidothere in the British Museum than in any other Megatherioid fossil which has hitherto come under my inspection: this bone (Plate VIII. figs. 1 & 2, $_{75}$) forms the anterior two-thirds of the upper boundary of the orbit, where it is wedged between the maxillary and the frontal, articulating by about half an inch of the upper border with the nasal bone, and for about the same extent by its anterior and inferior angle with the malar: it is semioval in shape, irregularly convex externally, and perforated near its anterior border obliquely by the lacrymal canal (*l*), the orifice of which is 6 lines in diameter. The bone forms a slight protuberance behind the orifice, which appears to be outside the upper and fore part of the orbit. The breadth of the lacrymal bone is 2 inches. The size of the lacrymal foramen equals that in the Megathere, and is twice the size of that in the Mylodon: in its extraorbital position it differs from both. The lacrymal canal is continued inward and forward, and then grooves obliquely the inner side of the facial plate of the maxillary, as above described. The secretion of the gland may finally have entered the fore-part of the mouth by the premaxillary fissure.

The malar (Plate VIII. fig. 1, $_{26}$) presents the characteristic form and development of that bone in the Sloth tribe, with the same absence of the postorbital process as in the Mylodon, and a suppression of the zygomatic process (*z*) to a degree which brings it nearly to the condition of the malar in the existing Sloths. The fore part of the malar ($_{26}$) forms the fore part of the orbit, articulating by its upper end with the lacrymal ($_{75}$), and by its fore surface with the outstanding malar process of the maxillary ($_{21'}$). Becoming free, the malar forms a stem with a semi-elliptic transverse section, slightly convex externally, very convex internally, 9 lines in diameter: it then, extending

downward and backward, rapidly expands, decreasing in thickness, and bifurcates. One branch ascends with a slight twist, forming the suprazygomatic process (*s*), and having an obtuse rudiment of the true zygomatic process (*z*) on its posterior border. The suprazygomatic process appears, in the natural position of the bone, to touch or rest upon the end of the zygomatic process of the temporal (*w*). The descending process (*d*) is broader and thinner; is gently concave, is directed obliquely downward and backward, and terminates obtusely.

The cranial cavity is 7 inches in fore and aft extent, including the cavity occupied by the olfactory ganglions, which is $1\frac{1}{2}$ inch long: the greatest transverse diameter of the cavity is 4 inches 9 lines, which corresponds with the back part of the cerebral hemispheres. A low tentorial ridge is continued from the upper edge of the petrosal, in a direction which shows that the cerebellum lay wholly behind the cerebrum; but the major part of the tentorium was membranous, as in the Sloths and true Ant-eaters, not osseous as in the Manis. In the Orycteropus a sharp bony ridge extends from the petrosal into each side of the tentorium. The greatest vertical diameter of the cerebrum of the Scelidothere appears to have been $2\frac{1}{2}$ inches; that of the cerebellum and medulla oblongata 2 inches 3 lines: the transverse diameter of the cerebellum was about 3 inches 9 lines, its fore and aft diameter about 2 inches. The inner surface of the cranial cavity indicates several convex masses of cerebellar convolutions, and also a few simple parallel longitudinal convolutions upon the upper and lateral parts of the cerebral hemispheres.

The small relative size of the brain of the Scelidothere is most clearly demonstrated; it was scarcely one-fourth part the length of the skull: both this proportion and the relative position of its principal masses, succeeding one another lengthwise as in reptiles, closely accord with the low general condition of the cerebral organ in the existing species of the Order *Bruta*.

The upper part of the basisphenoid is impressed with a broad and shallow 'sella turcica,' bounded laterally by two grooves leading forwards and inwards from the carotid foramina. The line of junction between the basisphenoid and basioccipital is indicated by a slight transverse elevation which bounds the 'sella' behind: a low median protuberance forms the anterior boundary: there are neither anterior nor posterior clinoid processes. External to the carotid canal there is a wide groove leading to the foramen ovale. This foramen (Plate VIII. fig. 2 *l*), though absolutely less than in the Mylodon, is relatively larger as compared with the precondyloid foramen, and indicates that the tongue was more sensitive and less muscular or bulky in the Scelidothere; but, in reasoning from the size of the foramen ovale, it must be kept in mind that certain branches both of the second and third divisions of the fifth pair of nerves were associated with the persistence of large dental pulps of which they regulated the formative force. Anterior to the foramen ovale, and at the termination of the large common groove lodging the trunk of the fifth pair of nerves, is the inner orifice of the canal representing the foramen rotundum (ib. *r*); the diameter of this canal is 5 lines, being somewhat less than that of the foramen ovale.

The length of the lower jaw (Plate VIII. figs. 4 & 5) is 1 foot 6 inches 9 lines. The horizontal rami, each 3 inches 8 lines in depth at the base of the coronoid process (*b*), decrease in that diameter, at first gradually, then more rapidly at the beginning of the symphysis (*d*), to its end (*d'*): they are confluent at the symphysis for an extent of 6 inches 9 lines (ib. fig. 5 *d*, *d'*). The ascending ramus of the jaw, or posterior apophysiary part, presents the broad, backwardly produced, slightly inflected angular process (*c*) common to all the Megatherioids; in shape and other minor characters it most resembles that in the Mylodon. The condyle (*a*) is transversely oblong, with the inner and smaller end bent a little back: the articular surface is nearly flat. The posterior border of the ramus is continued from the reflected inner angle of the condyle, the coronoid process (*b*) from the fore-part of the condyle near the thick outer end: the extent of the base of the coronoid process is 4 inches, the deep notch dividing its reflected apex (*b*) from the condyle (*a*) is but 7 lines wide: the fore margin of the process describes a regular convexity: the outer side of the process is flat, the inner side convex near the anterior border, slightly concave elsewhere: the low ridge bounding anteriorly that concavity is continued into another, which runs along the middle of the inner side of the horizontal ramus for 2 or 3 inches. The angular process (*c*) is concave internally: its upper border is notched, and its outer convex side is grooved for muscular attachments. The coronoid and angular processes are obviously the two great handles of the jaw, which, grasped by the opposing masses of muscles, enabled them to move with effect the lower molars horizontally upon the upper ones; the flat condyle and broad flat surface for its articulation offering no obstacle to those grinding movements. The dental canal commences by an oblique orifice, 6 lines in diameter, situated 2 inches behind the last alveolus (*i v*), and from 1 to $1\frac{1}{2}$ inch below the level of its outlet. The canal soon bifurcates, the outer branch leading to the large orifice (ib. fig. 4 *g*) outside and below the anterior part of the base of the coronoid process; the inner division supplies the teeth, and then subdivides into two small canals which have their outlets on the outside of the symphysis, from 2 to 4 inches distance from its extremity. These 'mental foramina' (*h*, *h*) are small in comparison with those in the Mylodon. The horizontal ramus is moderately convex below and tumid externally at the molar region.

The series of four alveoli in each ramus measures 4 inches 3 lines. The outer alveolar wall describes a slight curve, convex outwards; it is higher than the inner one, and forms a kind of 'bead' or rounded prominent border externally: the whole alveolar part of the ramus inclines inwards. The inner alveolar wall is thin, straight, nearly parallel with that of the opposite side; but a little divergent anteriorly, where it blends with the outer wall to form the obtuse upper and outer border of the long symphysial part of the jaw. This part, near 7 inches in length, deeply excavated behind, becomes gradually shallower to the end, which is cut square off, and is slightly notched in the middle. There is a slight median prominence on the under part of the symphysis near the anterior end.

The teeth of the Scelidothere, as in the other known gravigrade Sloths, are eighteen

in number, all of one kind, arranged according to the Megatherioid formula: viz. $m, \frac{1}{1}=18$. They are simple, long, fangless, and of the same thickness from the implanted to the exposed end; the former is excavated by a deep conical cavity, the latter worn into a shallow depression, and somewhat obliquely, reaching to the level of the inner alveolar wall in the upper jaw, and to the outer one in the lower jaw.

In the upper jaw the shape of the anterior tooth (Plate IX. *i*) is slightly modified in two of the skulls of the Scelidothere from Buenos Ayres in the British Museum. In one, its transverse section more resembles that of the same tooth figured in the fragment described in my original memoir on the genus*, the straightest and broadest side being the inner one: in the other skull the outer side shows those proportions; the teeth being naturally implanted in both skulls: the general form of this section is an irregular oval, 11 lines in long diameter; the length of the tooth is $2\frac{1}{2}$ inches; it is very slightly curved, with the convexity directed forwards and inwards.

The other four teeth gradually diminish in size, and are more or less three-sided with the angles rounded off, in transverse section; the longest side being turned outwards and backwards, and being slightly concave: the opposite angle is more developed in one than in the other skull, in the second and third molars: these are more nearly equal in size than the last two molars. The longest diameter of the grinding surface of the penultimate molar is 9 lines, that of the last molar $7\frac{1}{2}$ lines. The longitudinal extent of the dental series in both adult skulls is 4 inches 4 lines. The length of the last tooth is $2\frac{3}{4}$ inches; that of each of the three middle teeth is 3 inches.

In the lower jaw (Plate IX. fig. 2) the first molar (*i*) is broader than the second and third: its transverse section is triangular with the angles rounded off, the longest side turned inwards. The second (*ii*) and third (*iii*) teeth are of equal size and similar shape, the transverse section being triangular with the apex thick and obtuse; the base is much shorter than the sides, is slightly indented, and turned outwards and a little forwards; the rounded apex in the opposite direction; the anterior side is more concave than the opposite side. The last molar (*iv*) is the largest and most complex, its grinding surface consisting of two lobes produced by two opposite unequal-sided channels traversing the tooth longitudinally: the lobes are subequal, and placed obliquely one before the other.

The teeth consist of a central axis of vaso-dentine, a wall between 1 line and 2 lines thick of hard dentine, and a very thin outer coat of cement.

Comparison of the Skull and Dentition.

In the comparative simplicity of the malar bone and of the grinding surface of the teeth, in their close and regular arrangement and in the narrowness of the palate, in the incomplete orbit and the uninterrupted or more simple outline of the under jaw, the Scelidothere manifests its nearer affinity to the Mylodon than to the Megathere; but in many respects it is an intermediate form, as is shown by the long premaxillaries and the corresponding length and slenderness of the symphysis of the lower jaw. The occi-

* *Op. cit.* Pl. XXIII. fig. 3, 1.

pital plane is more vertical than in the Megathere, and still more so than in the Mylodon or in the existing Sloths. In the minor development of the mastoid as compared with the paroccipital, the Scelidothere resembles the Mylodon more than it does the Megathere: it differs from both in the much smaller surface of articulation for the stylohyal; and, apparently, by the want of junction of the malar with the squamosal. The lower jaw, whilst it resembles that of the Mylodon in the form of the ascending ramus and of the dentigerous part, differs in the unusual prolongation of the long and slender symphysis, and in the smaller size of the mental foramina: the intermediate character between Mylodon and Megathere is well illustrated in the mandible of the Scelidothere. The dentition of the Scelidothere differs from that of the Megathere, and resembles that of the Mylodon in the simple concavity of the grinding surface; but it differs from the dentition of the Mylodon in the first upper molar not being divided by a disproportionate interspace from the rest, and by the more unequal sides of the triangular grinding surface. The second molar, in the Mylodon, presents an elliptical transverse section instead of the reniform or subtriangular shape. The third and fourth molars of the Scelidothere are more compressed than in the Mylodon, and their long axis is from before backwards, instead of being transverse. The fifth molar is relatively smaller than in the Mylodon, approaching to that in the Megathere. In the lower jaw of the Scelidothere the differences in the form of the teeth are equally manifest, especially in the prismatic form of the first molar: it is elliptical in the Mylodon. The last molar of the *Mylodon robustus* has the second lobe larger than the first: in the *Mylodon Darwinii* the proportions of the lobes more resemble those in the Scelidothere. The two middle teeth differ more markedly from the corresponding ones in any of the species of *Mylodon*: the second molar in *M. robustus* has a triangular transverse section with the base turned inwards and indented: the third molar is subquadrate.

The retention of the mylodontal character of dentition in three forms of the genus, with modifications not surpassing those of specific value, renders it probable that other species of *Scelidotherium* may have existed besides the one under description. The generic distinction of *Scelidotherium* from *Mylodon* is strengthened by the additional characters which the complete crania in the British Museum have brought to light.

In conclusion I may remark, that, as our knowledge of the great Megatherioid animals increases, the definition of their distinctive characters demands more extended comparison of particulars: hence in each successive attempt at a restoration of these truly remarkable extinct South American quadrupeds, there results a description of details which might else seem prolix and uncalled-for.

I have only to add, that the allotment of the Government Grant for drawings of rare or nondescript fossils has enabled me to bring before the Society without delay the present account, with adequate illustrations of the entire skull of the *Scelidotherium leptocephalum*.

Comparative Table of Dimensions of the Skull of the SCELIDOTHERE, MYLODON, and MEGATHERE.

	Scelidotherium leptocephalum.	*Mylodon robustus.*	*Megatherium americanum.*
CRANIUM.	ft. in. lin.	ft. in. lin.	ft. in. lin.
Length from the occipital condyles to the fore-end of the upper jaw	1 8 4	1 6 6	2 7 0
Length from the occipital condyles to the fore-part of the malar bone	0 11 6	1 2 0	1 8 0
Length from the fore-part of the malar bone to the fore-end of the upper jaw	0 8 8	0 5 0	0 10 6
Breadth across the widest part of the zygomatic arches	0 7 0	0 10 9	
Least breadth at the interspace of those arches	0 3 7	0 5 4	0 6 0
Breadth of the fore-part of the nasal bones	0 2 8	0 3 3	0 4 6
MANDIBLE.			
Length	1 6 6	1 3 6	2 3 0
Breadth between the hinder ends (angles) of the rami	0 4 7	0 6 3	0 7 0
Breadth between the condyles	0 1 8	0 4 2	0 4 0
Breadth between the posterior sockets of the teeth	0 1 6	0 2 10	0 2 3
Breadth between the anterior sockets of the teeth	0 1 8	0 4 0	0 1 9
Breadth across the fore-part of the symphysis	0 2 5	0 5 4	0 4 5
Depth of ascending ramus from the upper part of the condyle	0 5 8	0 5 7	1 5 0
Depth of ascending ramus at the fore-part of the base of the coronoid process	0 3 7	0 3 7	0 9 0
Depth of horizontal ramus at the fore-part of the first socket	0 2 7	0 3 0	0 8 0
Length of the symphysis following the outer curve	0 7 3	0 4 3	1 1 0
Fore and aft extent of base of coronoid process	0 3 9	0 3 8	0 9 0
From the back part of the condyle to the end of the angular process	0 3 6	0 3 2	0 6 0
From the end of the angular process to the last socket	0 6 5	0 6 6	0 9 7
From the first socket to the anterior margin of the jaw	0 7 10	0 3 6	0 9 0
Extent of the alveolar series	0 4 4	0 5 4	0 9 0
Breadth of the condyle	0 1 10	0 2 6	0 3 0

DESCRIPTION OF THE PLATES.

PLATE VIII.

One-fourth nat. size.

Fig. 1. Side view of the cranium, with the malar bone, in outline.

Fig. 2. Side view of a cranium, wanting the malar bone.

Fig. 3. Side view of upper molars. Nat. size.

Fig. 4. Side view of the mandible.

Fig. 5. Upper view of the mandible.

PLATE IX.

Nat. size.

Fig. 1. Base view of skull.

Fig. 2. Grinding surface of the upper molars.

Fig. 3. Grinding surface of the lower molars.

VIII. *On the Structure and Development of the* Cysticercus cellulosæ, *as found in the Muscles of the Pig.* *By* GEORGE RAINEY, *M.R.C.S., Lecturer on Anatomy and Demonstrator of Practical and Microscopical Anatomy at St. Thomas's Hospital. Communicated by* R. D. THOMSON, *M.D., F.R.S.*

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I AM aware that this parasite has been found in various other animals besides the Pig, also that it occasionally infests the human subject, but as I have not had the opportunity of examining it in the latter, and as my opportunities of investigating its structure and development in the former have been very abundant, I shall confine my observations to this entozoon as it occurs in the Pig.

The *Cysticercus* in its developed state is found chiefly in the ordinary muscles lodged in the cellular intervals between the fibres, also on the surface of the muscles immediately beneath their fascial investment. These parasites infest the heart in as great abundance as other muscles composed of striped fibres, where they are situated, some in the substance of the walls of the auricles and ventricles, others directly beneath the pericardium between it and the muscular fibres, and others just under the endocardium. They occur also in the muscular coat of the œsophagus, almost as low down as the stomach. But as in the Pig the striped fibres extend nearly to the inferior extremity of this tube, the presence of *Cysticerci* there forms no exception to their usual position in muscles composed of striped fibres. I found but very few in the lower part of this tube, and as yet I have not met with them in the muscular coat of the intestines, in that of the large vessels close to the heart, in any of the glandular organs, nor in any of the structures of the lungs.

However, in one pig, very much infested with them, I found one *Cysticercus* under the mucous membrane of the larynx, also one behind the peritoneum near to the kidney. As these appear to be probably only accidental cases, I am inclined to regard the striped muscles as the true seat of this species of *Cysticercus*.

This *Cysticercus*, when sufficiently developed to be apparent without the aid of the microscope, is seen to consist of an oval cyst, about half an inch in length, having a white globular body in its centre; but if the muscle in which these animalcules are contained has been exposed for a few hours to the air, the white bodies will be the only part very distinctly visible; the cysts in this case having become collapsed from the evaporation of their fluid. If, however, the muscle be put into water for a short time, the cysts will become redistended, and rendered as apparent as before. As often as

* The observations described in this paper were originally communicated to the Royal Society on the 27th of June, 1855. See Proceedings of the Royal Society for December 13, 1855.

these cysts lose their fluid by exposure, they can be redistended by a fresh application of moisture, even after they have been deprived of life, showing that this property of imbibition of fluid is obviously due to physical causes. These entozoa vary very much in size, according to their various stages of development; but the relative dimensions of the white opaque portion and the transparent sac differ most in those individuals which may be considered to have attained the adult state. Wherever they occur they are contained in sacs formed by the condensation of the surrounding tissues, whose strength and completeness vary in proportion to the closeness of the muscular fibres between which the cysts are found; hence in the substance of the muscular walls of the ventricles of the heart, these adventitious sacs are strong and well-defined, whilst they are thin and sometimes indistinct where the muscular fibres are less closely connected together, or when the *Cysticerci* are situated between a muscle and its fascia; however, in all situations they can be easily distinguished by the microscope from the true cyst of the animalcule, their structure being very dissimilar.

If the adult *Cysticercus* be subjected to slight pressure, the opaque globular-looking portion, which was so coiled up within the oval cyst as to form a spiral of about a turn and a half, will be made to protrude, and then the animal will present a form which will admit of an anatomical division into two parts; namely, a ventral portion and a neck, which is the usual mode of dividing this entozoon. See Plate X. fig. 1.

The ventral part is in the form of an oval cyst, about half an inch in its long diameter. It is composed of an extremely thin membrane, rendered uneven on its external surface by minute rounded projections, and its internal presents a granular appearance, as if covered with coarsely-pounded glass. It is furnished with no visible external opening, and is of the same structure everywhere, excepting where it is connected with the neck. Its cavity contains fluid in which there is amorphous granular matter, and some highly refractive particles having an oily aspect, also occasionally minute earthy particles which effervesce in acid.

No fibres of any kind can be distinguished in this membrane, either in its natural state or when acted upon by acetic acid. As this part of the *Cysticercus* is formed first, and exists as a perfect membranous sac some time prior to the addition and development of the neck, there can be but little doubt, this is the part in the perfect animalcule in which the function of nutrition is chiefly performed, and therefore that it is entitled, as well in a physiological as in an anatomical sense, to the appellation of ventral portion.

The neck is a tubular projection from the middle of the ventral portion; its form is somewhat pyramidal, and at its free extremity it presents a quadrilateral enlargement, whose terminal surface is occupied by four circular discs and a ring of hooklets. See Plate X. fig. 1 *b* and fig. 2.

Its length varies very much in different *Cysticerci* according to their age, this part continuing to increase in size after the other parts have acquired certain fixed dimensions. The membranous parietes of the neck are of considerable strength and thickness, and present two orders of fibres, namely, a longitudinal or superficial set, and a transverse or

deep one. These are connected by a transparent material which composes the chief thickness of the membrane, which is deeply wrinkled transversely, and thus formed internally into a number of transverse sacculi. The cavity of the neck does not communicate directly by any visible opening with that of the ventral portion. It is almost completely filled with oval laminated particles of earthy matter resembling minute calculi (see Plate X. fig. 3), which effervesce briskly when put into diluted muriatic acid, but which leave a residue of animal matter after all the earthy salt has been dissolved. These bodies begin to be formed as soon as the neck appears, and continue to increase in number as it increases in size.

The probable office of bodies so purely mechanical in their form and simple in their composition, is most likely one which is purely mechanical also. Perhaps by giving a degree of solidity to the neck, they enable the circular and longitudinal fibres entering into its composition, to effect its protrusion from the ventral cavity, and thus serve as an example of a very low type of an internal skeleton.

The terminal surface of the neck is of a quadrangular form, each angle being occupied by a circular disc or sucker, and its centre contains an apparatus of hooklets (see Plate X. fig. 2), and thus the four suckers and hooklets are all situated nearly upon the same plane, a position which would be advantageous for the employment of these organs when the animals are transferred from their deeply-seated and confined position between the muscular fibres to the free surface of a mucous membrane. The suckers are of a circular figure, with a diameter of about $\frac{1}{710}$th of an inch. See Plate X. fig. 4. Each consists of two or three membranous folds, placed at different depths from the surface, of about three-fourths of a circle in extent, so that the passage through a sucker from the external surface to the interior of the neck is rendered rather tortuous, and inclining to a spiral. The external opening of this passage varies very much in different *Cysticerci*, both in size and shape; but it is always smaller than the internal one. It is difficult, if not impossible, to demonstrate by the microscope alone, the existence of a passage through a sucker; however, this fact admits of proof from the circumstance of the carbonic acid, which is evolved in the cavity of the neck during the decomposition of the calcareous bodies, when these animalcules are in acetic acid, being seen under the microscope to pass freely through the openings in the suckers. In the membranous folds of the suckers, both circular and radiating fibres are distinctly visible, resembling in their general characters those in the neck, already mentioned. See Plate X. fig. 4.

The hooklets, as before observed, are situated in a circle around the centre of the square space before described. See Plate X. figs. 2 and 5. They are generally twenty-six in number, thirteen long and as many short, arranged alternately, a long and a short one. Occasionally, but very rarely, there are more: I have seen as many as twenty-nine. The longer hooklets are about $\frac{1}{150}$th of an inch in length, the shorter ones are a little less. Each consists of a curved part resembling a bird's claw, and a straight part or handle; and at the junction of these two parts there are rounded processes or tubercles, two in the short hooklets, and only one in the long ones, an anatomical mark by which

the two kinds can be easily distinguished. See Plate X. figs. 6 *b* and 6 *c*. They are composed of a very highly refractive, dense, and perfectly homogeneous substance possessed of a slight degree of elasticity, and not acted upon by acetic acid. In the curved portion there is a cavity filled with a fibrous material, the other parts are perfectly solid. They are placed (see Plate X. fig. 5), like the radii of a circle, with the extremities of their handles turned towards its centre, which corresponds to the centre of the quadrangular area on which they are situated, and their handles not meeting at this point circumscribe a small circular space, which, if perforated, would correspond to the mouth of the animalcule; but at this part there is no opening. The membrane is simply depressed, so as to present a conical hollow. By pressure upon the neck, this part can be made to protrude in the form of a tongue-like process. The handles of the hooklets are connected with this process, so that when it moves they must move with it. See Plate X. fig. 7 *a*.

Besides the radiating fibres which connect the handles of the hooklets to the membrane just described, there are other fibres which occupy the spaces between their curved portions. These have an arched form with their concavity turned towards the hooklets, into which they seem to be inserted. See Plate X. fig. 5 *e*. There are also two zones of circular fibres; one is placed over the curved portion of the hooklets, and the other over their handles. Between these zones there is a circular interval which contains the tubercles of the hooklets. See Plate X. figs. 5 *b* and 5 *c*. These fibres lie close to the hooklets, but in what manner they are attached to them, or whether they are inserted into them or not, I am unable to determine. Although these fibres are sufficiently distinct in the well-developed *Cysticercus* to leave no doubt of their presence, yet their morphological characters are not sufficiently defined to indicate their precise nature. However, their connexion with organs which are obviously intended to perform a mechanical office, and which would be useless unless they admitted of a certain amount of motion, furnishes a high degree of physiological evidence, that if all these fibres are not muscular, at least a part of them is.

The Development of the Cysticercus cellulosæ.

The earliest indication of this species of *Cysticercus*, which admits of certain recognition as a form of cystic entozoon, is the presence of a collection of reniform corpuscles of about $\frac{1}{1000}$th of an inch in length and $\frac{1}{4000}$th in breadth, mixed with very minute, highly refractive molecules of different sizes in the substance of a primary fasciculus of a muscular fibre, or between its sarcolemma and the sarcous elements. See Plate X. fig. 11.

Though such a collection of corpuscles has a moderately definite shape, being somewhat fusiform, yet it has not a complete investment. It soon, however, acquires a very distinct membranous covering, which is first apparent about its middle, and afterwards at its extremities. Its dimensions in this stage of its formation may be about $\frac{1}{150}$th of an inch in length and $\frac{1}{1500}$th in breadth, but these are by no means regular. The external investment at first appears only as a bright line of homogeneous substance, best

defined on the side next the sarcous matter. It soon, however, increases in thickness, and afterwards becomes converted into short fibres, which increase in size and distinctness as the animalcule grows larger. These fibres are peculiar; there is nothing that I am acquainted with analogous to them. They have not the sharp and well-defined outline of true cilia, nor are they pointed like setæ, or curled like cirri. They have somewhat the nature of white fibrous tissue, their distinctness being impaired by acetic acid. They are of different lengths in the same entozoon, and generally longer, though not thicker, in the large than in the small ones. Their length averages about $\frac{1}{1100}$th of an inch.

The most remarkable circumstance connected with them is the great uniformity of their arrangement in different *Cysticerci*. See Plate X. figs. 12 and 13. They cover the whole of the outer surface of the investing membrane, and on opposite sides of the same entozoon, their form, size, and direction are similar, so that the two halves taken longitudinally are in this respect symmetrical. If the direction of these fibres be examined about midway between the two extremities of one of these animalcules, they will be seen to project from the surface at right angles with the axis of its body; but if traced each way from this point they will be observed gradually to incline to this axis at an angle which keeps diminishing as they approach the two extremities, so that the fibres nearest to the two ends almost coincide in their direction with that of the axis, and thus correspond in their situation to the barbs situated on each side of the extremity of an ordinary feather. See Plate X. figs. 12 and 13.

As the first position of these animals is in the very substance of a primary muscular fasciculus (see Plate X. fig. 14), it is obvious that the mechanical action of this apparatus will be to aid their longitudinal development whilst new cells are in progress of formation in their interior. For it is scarcely possible that the muscular fibrillæ by which they are surrounded, can, when in action, fail by their friction to urge the two extremities onwards in opposite directions, whilst at the same time the fibres by which these entozoa are covered are in consequence of their direction preventing the separated ends from regaining their former position, and thus the two ends being always carried in opposite directions without the possibility of a counter movement, a general elongation must ensue. This apparatus also, by splitting up the primary fasciculi, will serve a locomotive purpose, and thus enable these animals to reach the cellular intervals between the muscular fibres, where their further development will be completed. That such is the effect of the fibres in question is evident on a careful inspection of some of the fasciculi in which these animalcules are contained, in which a separation of the fibrillæ can be seen to have been produced by the pointed ends of the entozoon; these fibrillæ having been obviously turned out of their original course, and some directed to one side and some to the other. This explanation receives confirmation from the fact of those *Cysticerci* which are developed in the muscular parietes of the heart being of a different shape from those formed elsewhere, although their structure in all other respects is precisely the same. These *Cysticerci*, in the first or vermicular stage of their develop-

ment, are very short and thick, and of an oval shape. Their locomotive fibres, though perfectly demonstrable, are very short, and in many instances imperfect. See Plate X. fig. 15.

This difference might have been anticipated, considering the close texture of the muscular fibres of the heart, the absence of any well-formed sarcolemma, and the shortness of the fibres occasioned by their frequent interlacement, all which circumstances would materially tend to diminish the effect of, as well as the necessity for, the fibres which cover the surface of the entozoa, and therefore be unfavourable to their longitudinal development. After these *Cysticerci* have reached the spaces between the muscular fibres, their subsequent development is the same as in other situations, and the perfect animals formed in the heart cannot be distinguished from those formed in other muscles. I may also add, that, while in the vermicular stage, the *Cysticerci* developed in the short muscular fibres of the tongue, are of a shape resembling very much those of the heart.

The investing membrane which has just been described as covered with cilia, is entirely filled with corpuscles, all of one kind, remarkably characteristic, and differing only according to their states of development. The perfect cells are best seen in the middle of an entozoon, but their mode of formation, and the subsequent changes which they undergo, must be examined in those parts which are increasing most rapidly, as in the growing ends of an animalcule.

The first appearance indicative of an increase in the length of an animalcule is a thinning of the investing membrane, and a separation or partial detachment of the cilia-like fibres at the growing end. Next, a clear space, of the form of the part which is about to be added, is perceptible a little in advance of this extremity, apparently the result of a very fine membranous protrusion. This contains numerous dark molecules of different forms and sizes mixed with granules more or less perfectly spherical: the most perfect of these globular bodies are those which are nearest to the perfectly formed part of the animalcule. These corpuscles, when completely formed, have a bright oily-looking aspect, and a diameter of about $\frac{1}{3000}$th of an inch. See Plate X. fig. 8.

These corpuscles have the appearance of being formed by the coalescence of molecules which had existed in the clear space before any corpuscles were apparent, by which they are afterwards replaced. After a growing end has become thus filled with these globular bodies, the terminal membrane becomes more and more distinct, and the cilia-like fibres are afterwards added, which are generally neither so regularly disposed, nor so distinct as on other parts of an entozoon. Next, these corpuscles lose their spherical form and become flattened, and lastly, they assume their characteristic elliptical or reniform figure before mentioned (see Plate X. fig. 10), which they retain as long as the entozoon remains in its primary muscular fasciculus. This shape, however, is not essential to these corpuscles, but merely results from the rounded form of the masses into which they are grouped together, each corpuscle, by its convexity, forming a segment of the circular outline of its respective group. These corpuscles contain very fine dark granules, so variously disposed in different ones, as to present a variety of appearances, such as

circular or oval spaces, which might be taken for nuclei or nucleoli. These collections of corpuscles make up nearly the whole of an animalcule, and they frequently give to it a lobulated and sometimes an obscurely annulose appearance. See Plate XI. fig. 2.

The entozoa, as long as they remain in the primary fasciculi, retain all those characters which have so far been described, but these characters gradually disappear after they have broken away from the cavity of the sarcolemma, and gained access to the spaces between the muscular fibres.

In this new situation they gradually lose their former membranous clothing studded with cilia-like fibres, which can occasionally be seen partially deprived of its corpuscular contents, though sufficiently perfect to admit of demonstration. See Plate X. fig. 16. The reniform corpuscles before aggregated together in circular groups now gradually lose their distinctness of outline, and imperfectly coalesce into confused ill-defined masses, having an oily aspect, so that, if in this state one of these vermicules be crushed under the microscope, amorphous oily and granular matter will be seen to have escaped from it, similar to that contained in the ventral part of the adult animal. Here, too, the restraint to the lateral growth of these entozoa being very much diminished, their breadth increases rapidly, and they present globular projections extending out very irregularly from their sides, giving them an irregular figure. See Plate XI. fig. 2. These projections gradually take on the form of those which were described on the ventral part of the perfect entozoon. The largest of the entozoa which I have seen in this stage is about $\frac{1}{15}$th of an inch in length, and $\frac{1}{40}$th in breadth.

The next facts requiring especial notice, are those connected with that stage of development which takes place after the animalcule has become surrounded by an adventitious cyst.

The first indication of the formation of such a cyst is, the turgescency of the capillaries, or some of the smaller vessels in the vicinity of one or more entozoa. Granular bodies, exudation corpuscles, and fibres of different shapes next make their appearance. These at first only partially obscure the entozoon, but afterwards completely conceal it. When the cyst is first formed, the animalcule can, by a good light and careful examination, be obscurely seen within it, and by dissection under the microscope it can be dislodged. See Plate XI. fig. 1, which is an accurate representation of one of these animals folded up in its cyst. Thus folded up, it measures $\frac{1}{18}$th of an inch in length.

Plate XI. fig. 7 shows a *Cysticercus* removed from its adventitious cyst. It has still somewhat of the folded character of the preceding specimen. This is $\frac{1}{3}$th of an inch in length, and $\frac{1}{32}$th in breadth. The interior of a cyst being smaller than the animalcule contained therein, it naturally follows that during its growth one portion must be folded over another. By this means it is adapted to the confined locality in which it is lodged during the period of its development. Hence the ventral portions of all *Cysticerci* are, when first taken from their cysts, very much plicated; but these plicæ disappear after the ventral sac has become distended with the fluid brought into contact with its surface.

Up to this point of the development of the *Cysticercus*, it is a simple cyst growing by

the assimilation of fluid imbibed equally by every part of its surface, no one part differing sensibly in its structure from another. No portion of this surface presents any indication of incipient hooklets or suckers. There is nothing either on its surface or in its interior analogous to the structure of an ovum. Nor is there any other anatomical character which would raise its organization above that of a simple acephalocyst. However, this so exactly resembles in its structure that of the ventral portion of a perfect *Cysticercus*, that it is impossible to doubt their identity of character. Its size, too, is not much beneath that stage where the suckers and hooklets first begin to present obscure indications of the part they are about to occupy. I may observe, moreover, that this form, and all the preceding ones, together with perfect *Cysticerci* provided with the regular number of hooklets and suckers, can, in many specimens of diseased muscle, all be found intermixed together in a space not exceeding a square line.

The dimensions of that form of *Cysticercus* in which there are obscure, though sufficiently certain, indications of the site about to be occupied by the neck and suckers, are about $\frac{1}{10}$th of an inch in length, and $\frac{1}{15}$th in breadth. See Plate XI. fig. 8.

The first indication of the addition of the neck with the suckers and hooklets to the ventral part of a *Cysticercus*, is the appearance about its centre of a slightly raised body, depressed in the middle, with longitudinal folds proceeding from each side of it towards the poles of the ventral cyst, appearing as if at this part the parietes of the latter had been drawn inwards. See Plate XI. fig. 8. On two sides of this hollow there are dark transverse lines, rather more distinct on one side than on the other, indicating the commencement of the transverse rugæ of the neck, mentioned in the description of this part in the perfect animal, in which the laminated earthy bodies are contained. About the central part of the cervical projection there is an ill-defined oval space, having a granular appearance, and containing some minute spherical particles of a dark colour, consisting apparently of a highly refractive material. See Plate XI. fig. 8. In this condition of the entozoon there is nothing in this space which has the slightest resemblance to the parts which are there about to be developed, namely, the hooklets, suckers, and earthy concretions; and it is only by the comparison of these obscure appearances with the other specimens in which the development of the hooklets is a little more advanced, that their true signification can be learned. See Plate XI. figs. 10 and 12.

Of these parts I will first describe the development of the hooklets, these being the most remarkable and characteristic organs of a *Cysticercus*.

By comparing the preceding specimens with one in which some of the hooklets are in progress of development, and sufficiently advanced in their formation to leave no doubt of their real nature, it will be seen that the highly refractive globular particles seen in Plate XI. fig. 8 are portions of the material of which the hooklets are composed. The entire number of these organs is not in progress of formation at the same time, but only about six are being developed at once, and when these are nearly completed, others make their appearance. The primary condition of one of these organs is very remarkable, consisting merely of a confused and irregular group of very bright particles of a pale straw-colour,

which are of various shapes, but still all have a contour more or less curvilinear, and the smaller ones are of a spherical figure. Their size varies from about that of the third part of a handle of a perfect hooklet, to a particle so minute as scarcely to be appreciable by the highest powers of the microscope. There are all the intermediate sizes between these extremes. Notwithstanding, however, these extremes of size and form, all these particles possess the same optical and physical properties, so as to be perfectly recognizable, both when apart and when joined together in the perfect hooklet. See Plate XI. figs. 10 and 12, in which they present no appreciable change in their general appearance.

At a period of the development a little more advanced, some parts of a hooklet can be recognized among these various forms, especially the curved portion, as being most characteristic; also some parts of a handle can be distinguished; lastly, the larger pieces, formed obviously by the coalescence of smaller ones, can be seen fused, as it were, together, more or less completely in a newly-formed hooklet, in which frequently the joining is so incomplete as to amount to little more than mere apposition of the coalescing particles. See Plate XI. figs. 12 and 14.

The part first formed is the hook: this has at first its internal cavity rather larger than that in the older ones, and there is an irregularity of its outline, indicating a want of complete union of its component particles. The handle is formed next: this is more remarkable for its want of symmetry than the curved part, some pieces of it appearing to be merely applied to each other. See Plate XI. figs. 10 and 14. Lastly, the tubercles are added. Neither the whole nor the parts of a hooklet undergo any increase in size after being once formed, but, on the contrary, rather suffer a slight diminution. The hooklets of the old *Cysticercus* are frequently smaller than those of the young ones. But these organs vary a little in different animalcules, being rather smaller in some than in others, but those belonging to the same individual are remarkably regular in this respect.

From the facts that have just been mentioned, the hooklets of the animalcule in question do not appear to be formed by cell-development. For by the most careful examination of these organs, both recent, and after the application of acids, I have not been able to distinguish anything which can be looked upon as a cell or cell-nucleus, calculated to give the idea of their being developed from previously existing cells, or in dependence of cells; but, on the contrary, all the various forms and characters which they present during the process of their formation simply indicate the coalescence of very minute spherules of an homogeneous material, exceeding the number of a complete set of hooklets, into small globular masses, and these again into larger pieces, and so on successively, until recognizable portions of hooklets come into view, which, coalescing, build up, as it were, an entire organ.

It is worthy of remark, that if these structures had been produced directly from the metamorphosis of previously existing cells, the circumstances connected with their formation would have been the most favourable for observing both the original cells and the changes which they passed through; indeed, so much so, that it is almost impossible

that they could have escaped notice. First, because these parts are of such a size and degree of transparency as to admit of examination with the highest powers of the microscope without the necessity of disarranging them, or disturbing their position by manipulation. Secondly, because the material of which they are composed is so dissimilar in appearance to that forming the adjacent tissue, and so characteristic, that it cannot be confounded with the structures in their immediate vicinity. Thirdly, because at one view, in a favourable specimen, hooklets can be seen in every stage of their formation, from the first grouping together of the masses of formative particles to the blending of them into perfect organs; and lastly, because it is not as if a mere thread of tissue were formed amongst other threads, slightly differing in appearance, as fibres of elastic tissue, for instance, in a mass of connective tissue, but the objects referred to are perfect organs, which possess an arrangement of parts connected together with order and remarkable regularity. So that, under such circumstances, if these organs had been preceded by nucleated cells, and the cells had been transformed into hooklets, neither these cells in their primitive state, nor in their several stages of transformation, could have escaped detection.

The parts next to be noticed are the suckers. Indications of these are visible as soon as the hooklets. They appear as four circular spaces, presenting a granular aspect, about the size of perfectly-formed suckers. The two sets of fibres next make their appearance, the radiating and circular, which have not at first the sharp outline which they afterwards acquire, but still appear obscurely granular. As the tissue of these organs possesses nothing characteristic like that of the parts just described, the progressive changes which they undergo during the different periods of their formation can be but imperfectly distinguished; and hence no further description of them will be necessary.

It has been observed in respect to the two sets of organs above described, that their size does not increase materially after once formed; exactly the reverse is the case in reference to the part called the neck, and the quantity, though not the size, of the laminated bodies, which increase in number as the cavity of the latter increases in size. These bodies appear as soon as the hooklets and suckers, and they are as large when first formed as afterwards, but there are indications of the transverse wrinkles of the neck before either hooklets or suckers can be distinguished. See Plate XI. fig. 8. The neck afterwards continues to grow, so that its relative length in respect to the ventral portion is some indication of the age of a *Cysticercus*.

It is probable that this part does not arrive at its full size until after it has been protruded, which I have never seen to be the case in any animalcules occurring in or between the muscular fibres, and which perhaps is not effected until the entozoa quit their confined locality between the muscular fibres, and gain access to the free surface of a mucous membrane, there, as physiologists generally believe, to be further developed into a higher form of entozoon.

In the preceding observations on the development of the *Cysticercus cellulosæ*, I have confined myself entirely to such facts as are so obvious and easy of verification, as to

leave no doubt of their general accuracy, also to such explanations of their meaning as appear to be almost self-evident; and here I might have concluded this paper had there not remained some considerations entirely of a theoretical nature too important to be omitted.

It will be asked, how the *Cysticercus*, in its earliest condition, such as I have described it, finds access to the interior of a primary fasciculus. Before attempting to answer this question, I must observe that my description commences from a condition of this entozoon so complete, that no one, on examining it with the microscope, and comparing it with those forms in which the development is more advanced, would doubt their identity. But this form, though so low, is very far from being the earliest indication of an abnormal state of the muscular fibres in all those animals which are infested with immature *Cysticerci*. These conditions of the muscular fibres can at once be seen to be abnormal, but still they are altogether of an ambiguous character, and therefore I have deferred till now the consideration of them only from a wish to keep that which is certain and distinct from that which is only probable.

Besides the cells and molecules already described as accumulated in sufficient quantity to present the undoubted form of a *Cysticercus* (see Plate X. fig. 11), numerous minute irregularly-shapen particles are also found in the same specimens of diseased muscle.

These occur either singly or grouped together in clusters of different shapes and sizes. Some lie immediately beneath the sarcolemma, others are deeply seated in the substance of the primary fasciculi. See Plate XI. figs. 3 and 4.

They exist both in fasciculi containing imperfect entozoa, and in those of the same specimen in which entozoa are not yet formed.

These particles do not resemble in the least the scattered nuclei normally existing in muscular fibres, nor are they like the spherules of oil into which muscular fibres are sometimes converted. Occasionally, however, there is a state of fatty degeneration of the muscular fibres, coexisting with this disease, which is known by the regularly spherical form of the particles of oil, and their general arrangement in lines; also by their more equal refraction of light. The particles peculiar to the cystic disease very often occur where there is no fatty degeneration whatever; in no instance do they look like the globules of oil seen in fatty muscle and formed at the expense of the sarcous matter, but they exist independently of oil-globules, and appear to be lodged between the smallest fibrillæ, tending to separate them longitudinally.

Now, as these particles always exist in conjunction with the earliest forms of *Cysticerci*, and in such cases only, and as it is almost impossible that such a collection of corpuscles and molecules as the one I have described to be the earliest *unmistakeable* form of this entozoon could have at once come into existence in so characteristic a form as that delineated in Plate X. fig. 11, I believe that these particles are antecedent ambiguous forms of *Cysticerci*, and that this is the first stage of the disease as it exists in muscle. And it is further worthy of remark, that in some specimens of muscle very much infested with *Cysticerci*, I have found some of the capillaries and small blood-vessels containing mole-

cules of an organic character; some were so completely filled with such particles as to be entirely deprived of blood-corpuscles, and to be recognizable as blood-vessels only by their mode of ramification. Their tunics had become so attenuated as not to be visible under the microscope, and the mass of molecules within them presented the appearance of casts of their interior. In some of these vessels the coat appeared to have been destroyed, and the molecular matter to have become extravasated among the muscular fibres. See Plate XI. fig. 5, which is the representation of a blood-vessel $\frac{1}{333}$rd of an inch in diameter, filled with organic molecules, from the heart of a pig very much infested with *Cysticerci*. Although, in the instance above mentioned, the quantity of organic molecules in the blood-vessels was so abundant as to be easily detected, yet I may observe that in most cases of *Cysticerci* I have not been able to find these molecules in the capillaries, especially when they contained blood-corpuscles, so that I am not enabled to state that the presence of molecules in these vessels in sufficient quantity to admit of detection by the microscope is invariably demonstrable. However, in all cases the smaller blood-vessels and capillaries are in an abnormal condition, but whether this is wholly attributable to the irritation of the incipient entozoa, or to some other cause, is a point which I have not yet been able to ascertain.

Their calibre is very irregular. Some are much distended with blood-corpuscles, and others have their coats so attenuated as not to be distinctly visible. This condition of the vessels is best seen in muscles so small as to allow of microscopic examination without the necessity of mechanically separating their fibres, as in the muscles moving the eye-ball, especially near the part where the muscular fibres are connected with the tendon. Although the presence of these abnormal particles of an ambiguous character in the interior of the primary fasciculi, and that of minute molecules in the blood-vessels and their capillaries, with the attenuation of their coats, may not fully answer the question proposed as to the precise manner in which the first forms of *Cysticerci* find their way into the muscular fibres, still I think these facts render it highly probable that such forms of the entozoa as are represented in Plate X. fig. 11, existed at first as those minute particles which I have described in the muscular fibres, and that these had found their way to the muscular fibres through the medium of the blood.

Without urging any opinion as to the source whence these entozoa are derived, I may here state, that from certain data which I have collected, and which I will now briefly mention, I believe it possible to institute a course of experiments which might throw some light upon this question.

Through the kindness of Mr. Fisher, Inspector of Newgate Market, and some other persons employed there as salesmen, I have been enabled to examine specimens of measly pork taken from at least fifty different pigs. In one class of these specimens I found scarcely any but adult *Cysticerci*, after examining, perhaps, fifty different pieces; I might, however, find one or two of the immature ones, but these were always in the last stage of the vermicular form. In others, a second class, I could find only the earliest stages of the *Cysticerci*, and none of the perfectly-formed ones. And in the third class

of specimens, I found them in every possible stage of development, even in the same field of the microscope. I think the last class of specimens was the largest.

Now the inference I deduce from these facts is, first, that the animals from which the first set of specimens were taken had some time prior to their being slaughtered been placed under circumstances favourable for the production of *Cysticerci* (most probably the food had contained the germs of these animalcules), but had for a certain period been withdrawn from them, so that all the younger animalcules—those constituting the vermicular stage—had been, as it were, used up to produce the adult ones; secondly, that those from which the second set had been taken had only been recently placed under such circumstances, and had remained so circumstanced up to the time that they had been slaughtered; and thirdly, that those from which the third set of specimens had been taken, had, both some time previously, and immediately before they were slaughtered, been placed under circumstances favourable for the production and development of *Cysticerci*. Now, as to the determination and discovery of what these favourable circumstances are, it would be necessary that the history of infected animals, giving similar microscopic indications, should be *most* minutely inquired into, especially in relation to those different periods of the disease, and that the investigation of all these circumstances should be tested by a series of careful experiments upon living animals and by examinations of their food, and all the circumstances under which they are placed, both during the period these animalcules are being developed in the muscles and afterwards; so that negative as well as positive evidence might be obtained on the subject.

In concluding this paper, the author thinks it right to append a brief reference to the views at present entertained as to the nature and relations of the Cystic entozoa, in order that the bearing of his own researches may be more clearly apprehended.

A similarity in the form and armature of the head in the *Cysticercus* and Tape-worm, had led the earlier ento-zoologists to suspect that there was some latent affinity between the Cystic and the Cestoid entozoa, and this surmise has, through recent investigations, grown into a consistent doctrine, according to which it is held that the *Cysticercus* and *Tænia* are but different conditions of the same animal. It is accordingly now generally understood that the ovum, or rather embryo, of a tape-worm, being introduced into the body of some higher animal, penetrates through the tissues, probably by aid of the three pairs of hooklets with which it is armed, and on arriving at a suitable resting-place is developed into a *Cysticercus*,—in some cases undergoing intermediate changes in its progress thither; that the *Cysticercus* is destitute of reproductive organs, but that when the flesh of the animal which has afforded it a temporary abode is devoured by a second animal, the parasite, thus introduced into the alimentary canal of a new host, meets with a fitting nidus for further development, and, losing its caudal vesicle, lengthens out, acquires transverse articulation, and assumes the form of a jointed tape-worm. The term *scolex* is applied to the cystic, and *strobila* to the tænioid form of the entozoon; and, as each articulation of the *Tænia* is furnished with a perfect sexual apparatus, and,

when charged with fertile ova or embryos, may separate in due time from its neighbours, a single segment has been looked on as constituting a distinct being, and is named a *proglottis*.

The evidence of the conversion of cystic into cestoid entozoa is ample. A particular form of *Cysticercus* grows into a particular species of *Tænia*, of which it is the *scolex*; at least this definite specific relation is probable; and it has now been shown, by numerous and varied experiments, that when *Cysticerci* are given to an animal with its food, they gradually pass into *Tæniæ*, provided the *Cysticercus* made use of is the true scolex of a species of *Tænia* capable of being harboured by the animal subjected to experiment. The *Tæniæ* thus produced agree in number with the *Cysticerci* swallowed, at least never exceed them; they may be traced in progress of development, and when given successively at suitable intervals, the resulting *Tæniæ* present corresponding differences of advancement.

The *Cysticercus cellulosæ* is believed to be the scolex of the *Tænia solium*, which infests the human alimentary canal; and in an experiment made on a condemned criminal, these cystic parasites put into the food a short time before execution were afterwards found in the intestine converted into incipient *Tæniæ*, recognizable as the *Tænia solium*.

The *Tænia* being thus developed from the *Cysticercus*, it should naturally follow that the *Cysticercus* is, in its turn, derived from the *Tænia*; and the fact is established by experiments which may be regarded as complementary of the former. Mature segments of tape-worm, full of embryos, have been given to different animals with their food, and the tissues of these animals have become infested with *Cysticerci*. In this way the *Tænia cœnurus*, given to sheep, has been followed by the production of *Cœnurus cerebralis* in the brain of these quadrupeds; *Cysticercus fasciolaris* has been produced in mice from the *Tænia crassicollis* of the Cat; *Cysticercus pisiformis* has been generated in rabbits from embryos of the *Tænia serrata* of the Dog; and, not to mention other cases, pigs have become infested with the *Cysticercus cellulosæ*, after receiving with their food ripe segments of the *Tænia solium* of the human subject. Moreover, Stein found in the alimentary canal of the larvæ of *Tenebrio molitor*, the usual six-hooked embryos of an undetermined species of *Tænia*, and on the outer or peritoneal surface of the stomach of the same larvæ, numerous encysted *Cysticerci*, on nearly all of which the six hooklets shed by the embryo could be perceived adhering to the cyst and thus clearly showing the connexion of the two. In like manner Meissner found cysts containing *Cysticerci* in the body of the Slug; and although he did not discover *Tænia*-embryos, he found what were unmistakeably their six hooklets attached to the body of the *Cysticercus*.

While, however, the proof of the derivation of the *Cysticercus* from a *Tænia*-embryo thus seems complete, the process of conversion of the one into the other has not hitherto been followed completely in its different steps. The observations of Leuckart on this head are perhaps most to our present purpose. He found, that on giving the ova of *Tænia serrata* to rabbits, a numerous brood of *Cysticercus pisiformis* appeared in the liver, which situation he conceives they might have reached by the route of the vena

portæ. At first they were very small white bodies having none of the marks of *Cysticerci*; they afterwards enlarged, assumed an oblong shape, and, piercing through the substance of the liver, escaped into the peritoneal cavity; then, in a thickening which appeared at the fore-part of the very simple vermiform body, was developed the characteristic head of the *Cysticercus* with its suctorial disks and circlet of hooks. No trace of the six *embryo-hooks* could be detected in connexion with the *Cysticerci*. These observations of Leuckart were published in 1855, and were not known to the author when he first communicated his researches to the Royal Society in the summer of that year. Allowing for differences not unlikely to modify the process in different cases, the facts observed by Leuckart are in no way irreconcileable with the conclusions arrived at in this paper; and after due consideration of the present state of knowledge concerning the relations subsisting between the Tape-worm and *Cysticercus*, the author sees no reason to distrust the evidence on which he has represented the *Cysticercus cellulosæ* as immediately developed from a peculiar vermiform animal, whatever be the mode in which the latter may be derived from a *Tænia*-embryo.

Finally, the facts made known in this communication may perhaps serve to throw light on the nature of certain bodies of an ambiguous description observed by Von Hessling* to be occasionally present in the muscular tissue of the heart of the Ox, Sheep, and Roe Deer. These objects appear under the form of oval aggregations of granules of a peculiar kind, enclosed in an outer capsule and parted into several smaller masses somewhat like the segments of a yelk in the process of division. Without hazarding an opinion as to their destination, Von Hessling compares these bodies to the elongated tubular capsules filled with similar spore-like granules, which were discovered in the abdominal muscles of mice by Miescher, and subsequently in rats by Bischoff and Siebold. These singular formations are described by the last-named observers as being lodged within the sarcolemma of the muscular fibre. Siebold is inclined to regard them as vegetable parasites; but they offer many points of resemblance to the early condition of the *Cysticercus* as described in the foregoing pages.

Description of the Plates.

PLATE X.

Fig. 1. A *Cysticercus* with its neck protruded.
- *a*. The ventral part.
- *b*. The neck.

Fig. 2. The free quadrilateral surface at the extremity of the neck, showing the four suctorial disks at its angles, and the ring of hooklets at its centre.

Fig. 3. The two sets of fibres, longitudinal and transverse, visible in the membrane of the neck of the *Cysticercus*, also the laminated calcareous particles.

* Zeitschrift für Wissenschaftliche Zoologie, vol. v. p. 196.

Fig. 4. A suctorial disk sufficiently magnified to show the circular and radiating fibres.

Fig. 5. The entire number of hooklets *in situ*:—(*a*) the circular area circumscribed by the handles of the hooklets; (*b*) the external zone of circular fibres; (*c*) the internal zone; (*e*) the radiating fibres placed between the hooklets, and connecting them together; (*f*) fibres connecting the hooklets to the central membrane.

Fig. 6. Three hooklets:—(*a*) one as seen in profile, showing the curved portion and the handle, and at their union a tubercle; (*b*) one of the larger hooklets with only one tubercle; (*c*) a smaller hooklet with two tubercles.

Fig. 7. The circular depressed part circumscribed by the inner extremities of the handles of the hooklets, as shown in fig. 5 (*a*), has been here made to project: (*a*) the fibre connecting it with a hooklet.

Fig. 8. The growing extremity of an entozoon, showing the extreme end filled with minute molecules and globular corpuscles in various states of development.

Fig. 9. The end of an entozoon covered with cilia-like fibres, and containing molecular matter; also a transverse section of the body or middle part covered with cilia-like fibres placed upon an investing membrane, and filled with reniform corpuscles.

Fig. 10. Reniform corpuscles removed from the interior of an entozoon by pressure.

Fig. 11. The earliest recognizable form of the vermicular state of the *Cysticercus cellulosæ* contained in the interior of a primary muscular fibre.

Fig. 12. A *Cysticercus* of the vermicular form removed from the interior of a muscular fibre, showing the direction and the arrangement of its cilia-like fibres.

Fig. 13. A *Cysticercus* similar to the above, showing the exceedingly thin investing membrane at one of the extremities, devoid of cilia, this part being only of very recent formation.

Fig. 14. A *Cysticercus* in the substance of a primary muscular fibre; the cilia-like fibres are only faintly visible, being pressed by the surrounding muscular substance closely to the animalcule.

Fig. 15. A *Cysticercus* in its vermicular state taken from the muscular parietes of the heart, showing its general form and its cilia-like fibres. The latter are much less distinct than in vermicular *Cysticerci* taken from other striped muscles.

Fig. 16. The ciliated investment of a portion of a *Cysticercus* of the vermicular form, from which the reniform corpuscles have been removed.

PLATE XI.

Fig. 1. Shows a *Cysticercus* of the vermicular form folded upon itself in an adventitious cyst, a portion of which is seen upon one of the ends of the animalcule. This *Cysticercus*, as thus folded up, measures $\frac{1}{16}$th of an inch in length.

Fig. 2. A *Cysticercus* which has left the interior of a muscular fibre, and become situated in a cellular space between the muscular fibres, among the fat-vesicles always occurring in greater or less abundance in such situations. The crenate border and obscurely annulose appearance are here shown.

Fig. 3. A portion of a primary muscular fasciculus, in which irregular particles, of different shapes and sizes, are mixed with the muscular fibrillæ.

Fig. 4. Another portion of a primary fasciculus, in which the same kind of particles as those shown in fig. 3 are grouped together.

Fig. 5. A blood-vessel $\frac{1}{333}$rd of an inch in diameter filled with organic molecules, taken from the heart of a pig very much infested with *Cysticerci*.

Fig. 6. A portion of the same blood-vessel, magnified 400 diameters, showing the microscopic appearance of its molecular contents.

Fig. 7. A *Cysticercus* removed from its adventitious cyst, whose length is $\frac{1}{4}$th of an inch, and breadth $\frac{1}{16}$th. This is the highest stage of the vermicular form, and it combines the characters of that represented in fig. 2 and that in fig. 1 of this Plate, also those of the next figure.

Fig. 8. Shows that stage of the development of the *Cysticercus* in which is seen the earliest indication of the future neck with its hooklets and suckers. The whole of the animalcule is here shown as magnified 15 diameters. (*a*) the part to be occupied by the neck, where traces of its plicated form, also suckers and hooklets, are beginning to appear.

Fig. 9. Shows the part marked (*a*) in the preceding figure, magnified 100 diameters, these parts being faintly indicated.

Fig. 10. Shows a stage in which the hooklets are in progress of development; the neck is but little more advanced than in fig. 9. The entire entozoon is here shown as magnified 10 diameters.

Fig. 11. The hooklets of the preceding specimen, magnified 100 diameters. Six of them are sufficiently advanced to be recognizable: (*a*) particles of the material of which others are about to be formed, grouped together, but not yet coalesced sufficiently to form any recognizable portion of a hooklet.

Fig. 12. A form a little more advanced, showing the entire entozoon, magnified 10 diameters.

Fig. 13. The hooklets of the same specimen, magnified 100 diameters, exhibiting the same appearances, indicative of coalescence, as in fig. 11; here there are, however, portions of hooklets sufficiently formed to be recognizable, mixed with particles of the same material not having yet received a recognizable figure.

Fig. 14. Portions of hooklets from different specimens, magnified 200 diameters.

IX. *Electro-Physiological Researches. Physical and Chemical Phenomena of Muscular Contraction.—Tenth Series.* Part I. *By* CARLO MATTEUCCI, *Professor in the University of Pisa. Communicated by* MICHAEL FARADAY, *Esq., D.C.L., F.R.S. &c.*

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THE present Series of these Researches is divided into two parts: in the first I propose to treat of the development of heat, electricity, and *vis viva*, by the muscle in a state of contraction; in the second, of the chemical changes produced by the muscle in contraction on the air.

PART THE FIRST.

§ 1. That the temperature of an animal is raised by muscular exercise, is a fact generally admitted and proved by common experience. M. BECQUEREL having introduced the point of a thermo-electric pile into a muscle of the human body, found an increase of temperature during contraction which amounted to a degree Centigrade: Mr. NEWPORT also discovered a remarkable increase in the temperature of insects while in movement. M. BECQUEREL attributed this increase of temperature in muscles during contraction to the greater activity of sanguineous circulation which apparently takes place in a muscle in that state. In support of this interpretation, M. BECQUEREL showed, that, by compressing one of the large arteries distributed in a muscle under experiment, an immediate decrease of temperature takes place in that muscle. M. BERNARD discovered an increase of heat in one of the ears of a rabbit or dog after having cut the cervical ganglion connected with the nervous filament of that ear: this remarkable fact is also associated in some unknown manner, depending on the peculiar action of the ganglionic system, with a more rapid capillary circulation of blood in the ear in which the temperature is highest. It remained therefore still to be proved by experiment whether an increase of temperature is produced by the act of contraction in a muscle which is separated from the body, and as far as possible is devoid of blood, in which consequently sanguineous circulation is extinct. The experiment was easily made and the result not doubtful.

I employed for this purpose two thermometers such as those used in M. REGNAULT'S hygrometer, and constructed by M. FASTRÉ. The cylindrical bulb of these thermometers is very small, and by means of a telescope I could read distinctly variations of $\frac{1}{10}$th of a degree Centigrade. I had two cylindrical wide-mouthed glass bottles of about 100 cub. cent., provided with good cork stoppers. One of these corks was traversed by two thick copper wires, pointed at the extremities, and bent round horizontally parallel with each other, so as to leave an interval of 10 to 12 millimetres between them: the other cork was furnished with five hooks. A hole was bored through the middle of these corks, through

which the thermometers might be introduced into the bottle. Five frogs were then got ready in the usual manner, that is, by cutting off the heads, and removing the skin and viscera; when thus prepared and wiped dry, they were fixed on the two copper wires by passing these through the pelvis, and the cork with the frogs thus attached, was applied to the mouth of one of the bottles. In the bottom of the other bottle I put five other frogs similarly prepared, or suspended them on the hooks fixed in the cork; in some experiments instead of frogs I put a little water in the second bottle: the two bottles thus prepared with their thermometers, were placed in a room free from currents of air. I then observed the two thermometers: after the lapse of some minutes the column of mercury inserted in the midst of the frogs remained at the same division, or at least it was only after the lapse of many minutes that a diminution of $\frac{1}{10}$th of a degree was visible. In one of the experiments I found the temperature of the bottle in which there were no frogs to be $+12°{\cdot}60$, while that of the bottle containing the frogs remained fixed at $+13°{\cdot}10$. Then by means of two or three small elements of DANIELL, I set in activity a little electro-magnetic machine constructed by FROMENT, remarkable for the uniform sound which it produces when the circuit is closed. On connecting this machine with the two copper wires which issue from the cork of the bottle containing the frogs, the frogs contracted violently from the action of the interrupted current on the lumbar nerves; almost at the same instant the column of the thermometer began to rise, and after a lapse of five or six minutes stood at $+13°{\cdot}50$. If the experiment is prolonged, this temperature is maintained for many minutes. The copper wires used in this experiment must be thick, in order to prevent their being sensibly heated by the passage of the current: I assured myself of this by direct means, which I shall shortly describe.

In another experiment, in which the temperature of the bottle without frogs was $+12°{\cdot}80$, and that of the five frogs $+13°{\cdot}50$, the temperature of the latter rose to $+13°{\cdot}95$. After an hour's repose I recommenced the experiment on the same frogs; the temperature, which was now $+13°{\cdot}40$, rose only to $+13°{\cdot}60$ after the renewal of the contractions, which were naturally weaker than at first.

Finally, in another experiment in which there was no sensible difference between the temperature of the frogs and that of the air, the temperature of the frogs after four or five minutes' contraction rose from $+13°{\cdot}55$ to 14°, where, as usual, it remained constant for a certain time. This experiment was repeated after an interval of fourteen hours; the passage of the current which had no longer any physiological action produced no sensible increase of temperature.

It is therefore proved *that living muscular tissue developes heat by the sole act of its contraction* *.

* This result may be easily shown by employing, instead of thermometers, two thermo-electric pincers of bismuth and antimony, terminating in points. The point of each pair of pincers is introduced into each of the severed thighs of a prepared frog. The strips of the same metal of each pincer are brought into communication, and the other two strips are united to the end of a delicate short-wired galvanometer. On producing contractions, by any means, in one of the thighs, the needle is deflected, indicating heat in the convulsed thigh.

Perhaps it may not be devoid of interest to note the result of a comparative experiment made on the same number of frogs similarly prepared, and suspended in the same way in two separate bottles, with this difference only, that the frogs in one of the bottles had been firmly bound together with a ribbon wound many times round them: the passage of the interrupted current through the lumbar nerves could not awaken in the latter any general movement of the limbs, but produced a palpitation in the muscles which appeared to diminish rapidly. The frogs thus confined showed an increase of temperature during the passage of the current, but to a much less degree, being in each of the two experiments about one-third less than that of the frogs which were suspended freely.

§ 2. I now proceed to treat of the development of electricity in muscles during contraction.

Before entering more fully on this subject, I think it advisable to describe some new researches on muscular electricity. Two methods have been hitherto followed in order to arrive at a rigorous demonstration of the principal facts and laws of muscular electricity. In one of these methods the muscular elements are united in form of a pile, and the intensity of the muscular current increases with the number of muscular elements of the pile: this fact suffices to exclude any doubt of the existence of electromotive power which might be generated by the platinum plates of the galvanometer, or by the liquid, in which the pieces of muscle forming the outer elements of the pile are plunged. This doubt is also excluded by employing the nerve of the galvanoscopic frog instead of a galvanometer.

The other method, which is that employed by NOBILI and myself in my earlier researches on the electrical current of the frog, has been greatly improved by M. DU BOIS REYMOND, who has rendered it sure and of easy application. The apparatus of M. DU BOIS REYMOND consists, as is well known, of two small glasses, in each of which is laid a thick strip or cushion formed by numerous layers of blotting-paper or flannel; one extremity of this cushion touches the bottom of the glass, while the other rests on its edge so as to form a short horizontal stratum. By means of two little wooden columns, the platinum plates of the galvanometer are held vertically in the liquid, which is a solution of common salt, and remain in constant contact with the paper or flannel cushions above described. At first the two cushions are held in contact until all heterogeneity between the platinum plates has ceased; the two glasses are then drawn asunder, so as to leave a small interval between them. A gastrocnemius or a half-thigh of a frog laid on a narrow strip of gutta-percha held in the hand, is placed in contact with the conducting cushions so that the circuit is closed with the extremity of the muscle. The initial or impulsive deflection of the needle, which is of 40° or 50°, when the galvanometer is very delicate, ceases almost immediately if one element only is employed instead of a pile: if the muscle is removed and the conducting cushions are brought into contact, there is a great deflection in the opposite direction, owing, as is already known, to polarization. I had learnt from many experiments made long since on the electrolytic action of induced currents, that the best way of avoiding polarization is the substitution of care-

fully amalgamated plates of zinc for those of platinum, and the use of a saturated solution of chloride of lime or sulphate of zinc, as liquid. The merit, however, of this great improvement in the apparatus for electro-physiological researches is due to M. J. REGNAULD, as this young physicist was the first who published it, he having employed plates of distilled zinc instead of platinum. Now that the existence of muscular electricity, and that of its principal laws, is proved beyond all doubt, we may freely employ this method, the advantages of which will shortly be seen, without fearing, as formerly we justly did, errors which might be introduced into the experiment by the use of zinc electrodes and highly conducting liquids. I need only cite the following instance to show the great difference which results from this mode of making the experiment. With plates of platinum and diluted solution of common salt which I am in the habit of employing, either a gastrocnemius or a half-thigh of a frog of ordinary vigour produces, with a galvanometer of 24,000 coils, a deflection of 30° or 40°, which in a few seconds decreases to 2°, then 1°, and soon ceases almost entirely. With plates of amalgamated zinc immersed in sulphate of zinc, all heterogeneity between the plates having ceased, a gastrocnemius or half-thigh gives at the first impulse a deflection of the entire quadrant, the needle then rests at 70° or 80°, and descends extremely slowly; when the muscle is taken away and the needle fixes at 0°, no sign of polarization is obtained by bringing the two conducting cushions into contact.

I shall now state the results of my recent researches on the muscular current in some general propositions: the experiments of which I shall have occasion to speak have been so often verified that they may be easily repeated without uncertainty.

Prop. I. *The electromotive power of a cut muscle is independent of the size of its transverse section.*

Several half-thighs of frogs of equal vivacity are prepared of the same length, taking care to leave no portion of muscle beyond the articulation of the leg. These half-thighs are ranged opposite to each other in pairs, in order to ascertain the equality of their electromotive power; any elements which are unequal are rejected. Five or six of these half-thighs, laid closely one upon the other in the same direction, are then piled upon the usual gutta-percha holder; a single half-thigh is opposed to this heap. The circuit is closed, and the differential current is either null or very small, sometimes in the direction of the single thigh, sometimes in that of the heap. I repeated this experiment with two elements only, *i. e.* with two half-thighs taken from frogs of different size and weight; the difference of weight between these elements was as between 1 and 5 or 1 and 6. It is easy to perceive that this method is less exact than the former, since the larger frogs are more robust in the same conditions. The differential current was always extremely weak, but in the greater number of cases the direction was determined by the largest, and consequently most robust element.

Prop. II. *The electromotive force of the muscle increases with its length.*

This proposition may also be established by the differential method. Take two half-thighs of equal length, and which have the same electromotive power. Reduce by a

fresh section one of these half-thighs to a third or fourth of its length, and recompose the double pile. A differential current arises due to the longest half-thigh. The same experiment may be made on two gastrocnemi, which, however, are rarely to be obtained of equal electromotive power. Two gastrocnemi, as equal as possible, are opposed, after having previously made in each a transverse section towards the tendo Achillis: if the two muscles thus prepared are of equal length and taken from the same frog, it is almost certain that there will be no differential current. One of these gastrocnemi is then reduced to one-half or one-third of its length, the double pile is again formed, and the differential current is immediately produced by the longest muscle. I ought to remark here that these experiments should be made with the usual platinum plates in diluted saline solution: in employing amalgamated zinc plates and solution of sulphate of zinc, the smallest differences of the electromotive power produce strong and durable currents, and hence it is difficult to have the two muscular elements sufficiently equal to admit of comparison.

When, by a fresh section, the length of one of the elements is diminished, as a further precaution, an exceedingly thin section should be made in the other element. If a small difference exists between the electromotive power of two elements of equal length, the strongest ought to be shortened.

This experiment is still more striking when made on a strip of the long muscle of the spine of a living rabbit; the surface and transverse section of this muscle may be touched with the plates of the galvanometer so as to have a muscular stratum interposed of either 8 to 10 or 70 to 80 millimetres. The current in the second case is eight or ten times greater than in the former. This experiment recalls that made on the electric organ of the Gymnotus or the Torpedo, by employing strata of various lengths of this organ. I think, that, notwithstanding our ignorance as to the form of the electromotive element of muscles, this proposition leads us to believe that muscular fibre acts as a reunion or series of electromotive elements.

This proposition furnishes a clear explanation of several important facts of muscular electricity. The first of these facts is, that a differential current takes place constantly on closing the circuit with the gastrocnemius of a frog united with its half-thigh, the current being determined by the gastrocnemius. The experiment succeeds equally, if, instead of leaving the tendon attached to the gastrocnemius, the latter is cut transversely: in all cases it is essential that one of the muscles be longer than the other, so that either the gastrocnemius or the half-thigh may predominate. The same result may be obtained by employing a prepared frog reduced to two half-thighs of unequal length attached to a piece of the spine.

This same proposition leads us to explain the fact discovered by M. Du Bois Reymond, namely, that a current is obtained by touching with the plates of the galvanometer the central points and the edge of the same transverse section of a large muscle. In fact, the central points belong to the longest muscular fibres, and therefore play the same part towards the points near the edge of this section belonging to the shortest fibres, as the interior of a muscle towards its surface.

I take this opportunity of calling attention to a fundamental difference existing between a muscular electromotor and a voltaic pile. Let us take an entire muscle, such as the thigh of a frog, and cut it in half: it is known that the two sections thus formed have the same electrical state, and that a pile can be formed by placing in contact the internal section of one of the half-thighs with the superficies of the other. This result cannot be obtained with a voltaic pile: on the other hand, the proposition which we are now considering, leads us to admit a certain analogy between a muscular electromotor, and the organ of electric fish or the voltaic pile.

The relation existing between muscular electricity and the physiological conditions of muscle, forms the subject of the next proposition.

Prop. III. *The electromotive force of muscles, in living or recently killed animals, is greater in mammifers and in birds than in fish and in amphibia: the duration of this force, which in all cases decreases most rapidly in the first moments after death, is greater in fish and amphibia than in the muscles of the higher orders of animals. The nerves have no direct influence on the electromotive force of muscles. In general, all the causes which exert an influence on the physical structure and chemical composition of muscle, to modify in ways unknown its irritability or contractility, act equally on its electromotive power.*

This proposition is founded on experiments which I have already published, and particularly on those described in the Memoirs contained in the Philosophical Transactions. Since their publication I have frequently repeated and varied these experiments, employing the new method above described, and have succeeded in fully confirming them. I shall here only add a very brief description of some of these experiments, which place the existence and nature of muscular electricity in the clearest light. The object of the first of these experiments is to compare the electromotive power of the muscles of pigeons, rabbits and frogs.

For this purpose, I form, with muscles of these animals rapidly prepared, the usual elements which I oppose to each other. I form thus several double piles of muscular elements of frog and pigeon, of frog and rabbit, of pigeon and rabbit, and I measure successively the differential currents of these piles, closing the circuit of the galvanometer with the usual platinum plates. At first, the electromotive power of the muscles of the pigeon and of the rabbit greatly predominate over that of the muscles of the frog. After twenty or thirty minutes the differential current is null, or already inverted, and after the lapse of an hour or more, this inverted current, due to the muscles of the frog, is remarkably increased: this takes place in the pile of pigeon and frog sooner than in that of rabbit and frog.

Another very conclusive experiment is made on muscles exposed to great cold. Several gastrocnemi and half-thighs of frogs are left in a glass tube, surrounded by a mixture of salt and ice: after twenty or thirty minutes it is found that these muscles have entirely lost their electromotive power. By forming the usual piles with frozen and unfrozen muscles opposed, we acquire the certainty that the frozen muscles have lost their electromotive power, but not their conductivity. If we operate on muscles of frogs which have perished in nitrous or hydrosulphuric acid, we obtain the same results as from frozen muscles.

Before quitting this subject, I gladly avail myself of this opportunity of mentioning an important fact discovered by M. Du Bois Reymond. From the period of my earliest researches in electro-physiology, I had vainly endeavoured to obtain electrical phenomena from the nerves of a living animal, or of one recently killed; and even now it is demonstrated, that, notwithstanding the use of the most delicate modern galvanometers, no sign of electrical current is to be obtained from a nerve laid bare in a living animal and left in its natural state. M. Du Bois Reymond has, nevertheless, demonstrated, and the fact is easily verified with a very delicate galvanometer, that a piece of nerve detached from a living animal prepared and disposed in the same way as a half-thigh, in the circuit of the galvanometer, produces constantly an electric current, which, like that of the muscle, is directed from the surface of the nerve to the interior in the galvanometer. I have proved, by experiments made on pieces cut from the lumbar nerves of a frog, that the intensity of the current of the nerves increases with the number of elements united to form a pile, and that a pile of five or six of these elements acts on the nerve of a galvanoscopic frog and excites contractions. I have compared the electromotive power of a muscle with that of a nerve, both belonging to the same frog. It is hardly necessary to say that this was done by opposing to a half-thigh as many nervous elements as were required in order to there being no differential current. In two experiments I found that one half-thigh predominated over seven nervous elements, and that nine of the latter predominated over one half-thigh; in a third experiment eleven or twelve nervous elements were required to predominate over the muscle. I left these piles under a glass bell, the air of which was saturated with moisture; I could not perceive any marked difference in the relative duration of the electromotive power of muscles and nerves.

§ 3. Let us now pass on to the development of electricity in the act of contraction. The first fact of this kind, which, for the sake of brevity, I shall continue to call *induced contraction*, was discovered by myself in 1842. I shall give here a very concise summary of the chief characteristics of induced contraction, although already known, having recently been led to repeat my former experiments, and to determine these characteristics with greater precision.

In order to succeed in these experiments, it is necessary to operate on frogs in a vigorous condition, and prepared very rapidly. The fundamental experiment is made by laying the nerve of a galvanoscopic frog upon any muscle of a living animal. In order to simplify the experiment, I employ a single thigh of a frog, to which remain united the lumbar nerve and a portion of the spine. On provoking contractions in the thigh, either with an intermittent current sent through the nerve, or by stimulating it in any other way, simultaneous contractions take place in the galvanoscopic frog. The disposition of the nerve stretched over the thigh is immaterial, as is also to a certain degree the length of the nerve put in contact with the muscle, as long as the muscle and the galvanoscopic frog are vigorous. I have frequently seen the strongest effects of induced contraction when the nerve of the galvanoscopic frog, isolated from the operator, was in contact with any portion of the surface of the thigh for 2 or 3 millimetres only; it is therefore by no means essen-

tial for induced contraction that the nerve of the galvanoscopic frog should give signs of muscular current at the moment in which it is laid on the thigh. The internal part of a muscle may also be used for obtaining this phenomenon; this is done by immerging the nerve of the galvanoscopic frog in a longitudinal wound made in a muscle. From this we may infer that a phenomenon of the same kind as that of induced contraction takes place on a whole living animal, when an electrical current makes one of the muscular masses of the upper part of a limb contract: although it may be admitted that the nervous filaments concealed in that mass, and which pass through it as they branch towards the inferior muscles, are not traversed by the current on account of their low conducting power, yet at the moment in which the superior muscles contract, the lower ones contract also. This fact, of which I treated in the Eighth Series of my Researches*, appears to be thus traced up to its true cause, namely, that it is a natural and perhaps physiological case of *induced contraction*.

I shall add, finally, that when a stratum, however thin, of a good conducting body, or a like stratum of a solid isolating body, is interposed between the muscles in contraction and the nerve of the galvanoscopic frog, no *induced contraction* is obtained, as may be verified with gold-leaf or an extremely thin lamina of mica; on the contrary, with a piece of wet paper, and also with an excessively thin stratum of liquid turpentine, the phenomenon continues to manifest itself.

The many fruitless efforts which I made, and even the contradictory results at which I arrived, in endeavouring to determine the cause of *induced contraction* by means of the galvanometer, are well known. Although, as far back as 1838, in verifying the observations of Galvani, I had assured myself that when a frog enters into violent contractions, or into a state of tetanus, its muscular current is much weakened, yet I had never succeeded in discovering by the galvanometer in a distinct and certain manner, what took place in this current during contraction. I am now convinced, as I shall shortly show, that my want of success was due to defective instruments and method. In the Ninth Series of these Researches†, having given up the use of the galvanometer, and substituted that of the galvanoscopic frog, I succeeded in establishing the true nature of that phenomenon on which the present researches will throw a new light, proving beyond all doubt *the production of an instantaneous electrical discharge in the muscle during the act of contraction*.

But before proceeding to the exposition of my new researches, it is just that I should say to what point M. Du Bois Reymond has brought this subject. The use which this philosopher had made of a highly sensitive galvanometer, and of an electro-magnetic machine for exciting sustained contractions, accounts for his having succeeded in obtaining from muscles in contraction a distinct electrical effect on the galvanometer. The well-known fundamental experiment of M. Du Bois Reymond is made by closing the circuit of the galvanometer with the gastrocnemius of a frog which remained united to its nerve; when the needle comes to rest, M. Du Bois Reymond provokes contractions in the gastrocnemius by sending interrupted currents through the nerve:—"At the moment in which the

* Philosophical Transactions, 1850.

† Philosophical Transactions, 1850.

muscle begins to be convulsed, the needle is deflected through the zero-point, and is seen to oscillate on the negative side of the zero, until the contracting power of the muscle is exhausted, which always happens before the needle has had time to come to rest*." This fact, which M. Du Bois Reymond has analysed with much care, is interpreted by him in the following manner, which, for the sake of greater exactitude, we give in the words of the above-cited *abstract*, which are in accordance with the opinion repeatedly expressed by the author himself in his original work†:—"At first it might be supposed that this showed a current during the tetanus in a direction contrary to that during rest; but this is not the case. Before the tetanus begins, the secondary polarity is evolved on the platinum plates in the conducting vessel. This polarity tends to produce a current in the opposite direction to the muscular current. As soon, then, as the muscular current, in consequence of the tetanus, diminishes to a certain degree, the current of the secondary polarity becomes the more powerful, and the needle is instantly deflected to the negative side‡." M. Du Bois Reymond evidently admits that the electromotive power of a muscle decreases in the act of contraction, which is equivalent to saying that in all cases of *induced contraction*, considered as owing to a negative variation of the muscular current, as well as in all his experiments of this kind made with the galvanometer, the muscular current ought necessarily to circulate previously either in the nerve of the galvanoscopic frog, or in the wire of the galvanometer.

I can only refer here to the descriptions of experiments contained in the Ninth Series of my Researches on Electro-physiology §, several of which prove very clearly that induced contraction is obtained in conditions in which the previous circulation of the muscular current cannot be verified.

On the other hand, I was thus impelled to redouble my efforts to arrive at a clear idea of the phenomena of *induced contraction*, and I feel satisfied in being now enabled to say that I have at length succeeded in attaining my object. I shall be as concise as possible in describing the three principal experiments on which my last conclusions are founded.

It is not necessary that the galvanometer employed in these experiments should be the most delicate, that is, of 24,000 coils, an ordinary one of 1500 or 2000 coils being sufficient for the purpose. The metallic plates with which the circuit is closed should be of zinc, perfectly well amalgamated, that is, recently immersed in diluted sulphuric acid, then in pure mercury, and afterwards washed repeatedly in water. The two plates thus prepared, plunge in a saturated and neutralized solution of sulphate of zinc contained, as before described, in two small glasses, provided with strips or cushions of paper or flannel which imbibe the liquid in the manner adopted by M. Du Bois Reymond. If

* "On Animal Electricity," Abstract of the Discoveries of Émil Du Bois Reymond; edited by Dr. Bence Jones, p. 132.

† Untersuchungen über thierische Electricität.

‡ "On Animal Electricity," Abstract of the Discoveries of Émil Du Bois Reymond; edited by Dr. Bence Jones, p. 134.

§ Philosophical Transactions, 1850.

these cushions are left for a certain time in contact, the needle rests at 0°, and no sign of current is visible on reopening and closing the circuit again.

Exp. 1. I lay on the quadrant of the galvanometer, exactly in contact with the end of the needle in its state of equilibrium, a solid body, which may be either a piece of marble or glass, provided it is dry and polished: the needle resting against this support is prevented from deflecting in one direction, and is free to move only in the opposite. I then lay on the usual strip of gutta-percha either one gastrocnemius, or five or six of these muscles so as to form a pile; I close the circuit in such a way that the deflection cannot take place in the direction in which the needle is prevented from moving, and I leave it closed for some seconds. I then withdraw the pile rapidly, and close the circuit between the two glasses immediately. The needle remains motionless; there is therefore no sign of secondary polarity; and it was foreseen that, in employing this method, there could be none, although the electromotive power brought into action was much stronger than that with which we were to operate subsequently.

Exp. 2. I prepare a vigorous frog which I reduce to two thighs, one of which only is left with its lumbar nerve intact; this nerve is supported on a piece of gutta-percha destined for the purpose. I close the circuit by touching indifferently any two points of the thigh, which are not necessarily the extreme points; it is enough if there be a distance between them of about 12 or 15 millimetres. With a common galvanometer the deflection is small; with the most delicate one it varies from 50° to 60° and upwards, in the direction of the usual current of the thigh, that is, in the direction of the current of the gastrocnemius. I had left the other thigh, though it ought not to form part of the circuit, in order to avoid the possibility of the interior of the muscle in the upper part of the thigh submitted to experiment becoming part of the circuit. Having allowed the needle to fix, keeping the thigh in its place by pressing slightly on it with a piece of gutta-percha (which occasions little or no variation in the deflection), I begin to excite the nerve with an extremely weak interrupted current; for this purpose I prefer a small pile of zinc and platinum wires which I hold in hand, to an electro-magnetic machine, although I do not believe it probable that the use of that machine can cause any error in the experiment. At the very moment in which the thigh begins to contract, the needle descends towards 0°, and the contraction continuing, the needle passes beyond it and comes to rest on the opposite side. When we cease to excite contraction, or when the contractions are weakened from natural lassitude of the muscle, the needle descends again slowly and deflects in the same direction as at first, but rarely attains the point reached at the beginning. This experiment, repeated with the gastrocnemius, gives the same result, as also with the entire muscles of other animals. I prefer, however, as will be readily understood, the use of a single entire thigh of a frog, in order that the muscular current which circulates at the beginning of the experiment may be as small as possible. Among the numerous experiments which I have made, the muscular current has often been excessively weak, sometimes so weak as to leave the needle at 0°. During the contractions the deflection was in the same direction as that found when the needle

was deflected first in the opposite direction, and that deflection was certainly not inferior to the one in the second case.

Since no secondary polarity can interfere in these experiments (Exp. 1), they prove, that in the act of the contraction of these muscles, an electrical current, which, from its instantaneousness, I shall call electrical discharge, takes place, which circulates in the external conducting arc in a direction contrary to that of the muscular current.

Exp. 3. I make a small incision in the upper part of the thigh of a frog, in order to lay open the inside of the muscle: it is easy to understand, that, on closing the circuit with the two extremities of this thigh, the current, which in this case is differential, is null, or is in an opposite direction to that of the usual current of the thigh. The needle having been allowed to come to rest, and the usual contractions excited, the deflection which is obtained is always in the same direction, that is, opposed in the external arc to the usual current of the thigh when it has not been cut. Hence, if the needle is deflected at the beginning by the predominant current due to the transverse section of the upper part of the thigh, the deflection increases rapidly during the contractions.

Finally, the experiment may be made by putting the two thighs of a frog into the circuit, touching the under extremities of the thighs, but leaving, nevertheless, only one of the lumbar nerves. The current which is obtained on closing the circuit is in no fixed direction, being either null, or now in the direction of one of the thighs, now in that of the other. In all cases during contraction of the thigh provided with its nerve, the usual deflection occurs, that is owing to the current circulating in the external arc of the muscle from the inferior towards the superior extremity of that muscle.

I shall now add a few brief observations, which may throw further light on the interpretation of these experiments. I need not stay to prove that the effects observed by means of the galvanometer, produced by a muscle in contraction, cannot depend either on a portion of the current employed to irritate the nerve, or on changes of contact between the muscle and the extremities of the galvanometer, or finally, on the variable resistance of a muscle in contraction; these doubts have already been completely removed by the experiments of M. Du Bois Reymond. In endeavouring to understand the nature of the electrical phenomena which occur in a muscle during contraction, the fact which merits our chief attention is, that the needle of the galvanometer, which deflects at first in the direction of the muscular current of the gastrocnemius, or of the entire thigh, descends rapidly to zero at the moment of contraction and remains deflected in an opposite direction to that of the muscular current, during the whole time that the contractions are frequent and powerful. M. Pouillet, in his remarkable report*, gives a lucid *résumé* of the conclusions at which M. Du Bois Reymond arrived, in the following terms: —"D'après les principes de M. Du Bois Reymond l'effet d'une contraction soutenue n'est pas de faire naître un courant, mais d'affaiblir et de suspendre par intermittance un courant qui pré-existait; il faut donc un courant pré-existant, ou il en faut deux qui soient égaux et opposés et qui se neutralisent," &c. &c. The slightest reflection on the three

* Comptes Rendus, t. xxxi. p. 28.

experiments to which I have just referred, can leave no doubt as to the necessity of giving another explanation of the effects produced on the galvanometer by a muscle in contraction: when a gastrocnemius or a single entire thigh is in the circuit, M. Du Bois Reymond attributes these effects to the secondary polarity of the plates of platinum; but I have clearly proved that the effect is the same when there is no trace of secondary polarity. In operating on a thigh, the upper part of which has been cut transversely, in which case the current is either null from the neutralization of the two electromotive forces, or in an opposite direction to the usual current of the entire thigh, we have seen the needle deflect, during the contractions, as in all other cases, indicating that a current is directed in the galvanometer from the inferior to the superior extremity of the muscle. According to the principles cited in M. Pouillet's report, we should have to admit that the two currents of this muscle, which are identical and have the same origin, are differently modified in the act of contraction, and this would be contrary to all analogy.

We are therefore led by the experiments referred to, that is, by studying the electrical phenomena of muscles in contraction with the galvanometer and by the method we have described, to the same conclusion as that to which we were led by the researches made with the galvanoscopic frog; namely, that "a conducting homogeneous arc, the extremities of which are applied to any two points whatever, at a certain distance from each other, of a muscle of a frog, is at the moment of contraction traversed by an electric current, or rather discharge, directed from the inferior towards the superior extremity of that muscle." In confirmation of this conclusion, I have only to add the principal results obtained with the galvanoscopic frog.

Whoever has once seen a piece of living muscle, however small, on which is laid the nerve of a galvanoscopic frog, produce the phenomena of *induced contraction* each time that contractions are excited in that muscle by cutting it with a fine pair of scissors, cannot but be struck with the analogy which exists between this phenomenon and that which occurs, under like circumstances, in a piece of the electric organ of the torpedo. In order to study *induced contraction* with the galvanoscopic frog, I proceed as follows:—I prepare several galvanoscopic frogs, which are immediately submitted to the passage of a weak electrical current in the direction of the ramification of the nerve; in a few seconds these frogs cease to contract, except at the closing of the circuit, and are therefore fitted to show the existence of a current or electrical discharge only in case that current is *directed* from the nerve to the leg. An assistant has in the meanwhile prepared from a vigorous frog a thigh, to which is united the lumbar nerve and part of the spine. Nerves of several of the galvanoscopic frogs are now stretched in various directions over this thigh; two strips or rolls of paper, 4 or 5 millimetres long, imbibed with slightly salt water, are put in contact at one end with the extremities of the thigh, and the other ends of these strips of paper are united by two nerves of galvanoscopic frogs, turned in an opposite direction. The thigh is then made to contract: at first several of the galvanoscopic frogs enter into contraction; but if the experiment is continued, it will be seen that contractions take place only in those of the galvanoscopic frogs in contact

with the muscle, or with the two rolls of wet paper, which have their nerves stretched from the lower towards the upper extremity of the thigh. I have repeated this experiment on the muscles of living pigeons and rabbits, and although the difficulties are much greater in these cases, the results were generally the same.

In order to obtain a still clearer insight into the signification of this experiment, I have operated under the same conditions on the organ of a torpedo, in which the direction of the discharge is always clearly indicated by the galvanometer. I suspend a living torpedo by the tail, and rapidly lay bare the brain, and with a pair of scissors cut one of the electric organs in half, parallel to the length of the fish. Having thus obtained a section of the organ, I lay on it (as before on the thigh of a frog) several nerves of galvanoscopic frogs, and also the two extremities of a galvanometer. I then stimulate slightly the electrical lobe of the brain, and at the same instant the needle of the galvanometer deflects, and those frogs only contract whose nerve is extended from the back towards the belly of the torpedo, whether that nerve be in contact with the prisms of the organ, or complete the circuit of the two already mentioned strips of wet paper in contact with the extremities of those prisms.

From the whole of the experiments referred to, it appears to me that the truth of the conclusion which we have deduced is placed beyond all doubt; namely, that "A conducting homogeneous arc laid longitudinally over the surface of a muscle, or applied to that muscle at its extremities only, is, at the moment of contraction, traversed by an electric discharge directed from the inferior to the superior extremity of the muscle, considered in its natural position."

Here the experiments finish, and with them the rigorous results which can be deduced from them. The difficulties encountered in experimentizing on superior animals have deprived these conclusions of the character of generality. Admitting the experiment of M. Du Bois Reymond made on the arms of a man, one of which he contracts voluntarily, the direction of the current obtained at the instant of contraction would be contrary to that of the frog. In the obscurity, not to say entire ignorance, in which we are of the structure of the electromotive element of muscle, we are unfortunately obliged to satisfy ourselves with analogous and equally obscure cases; as for instance that of the direction of the electrical discharge of fishes, without a known relation with the structure of the organ or the form of the fish. Because an exterior arc applied over an entire muscle in repose is traversed by a current in a direction opposite to that which occurs at the moment of contraction, ought we to conclude that the two phenomena have a totally different origin? Or ought we rather to say that, should the form of the electromotor element become known to us, we may probably come to explain the opposite direction of the current as being due to variations of form produced by contraction? Were we not unwilling to diverge into hypothesis or undemonstrated analogies, we might gather many arguments tending to show the greater probability of the latter of these two ideas: according to the analogous case of the organ of the torpedo, we might also suppose that the electrical current is directed in the same sense in muscles in repose as it is in those in action. Since there does not exist, as we know, any analogy between the form of the

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voltaic electromotor and that of the muscular electromotor, it is not impossible to conceive that the change of form which takes place in the muscle in contraction may be momentarily followed by the inversion of the current in the exterior arc. Neither would examples be wanting, taken from certain cases of electro-dynamic induction and also of voltaic circuits, in which this inversion of the current can be obtained by change in the form or in the relative distance of different parts of the circuit. But this is a new field of inquiry which cannot be given up to simply hypothetical views; let us be satisfied with having proved that muscular contraction is accompanied by an electric discharge.

In the second part of these Researches we shall show that an augmentation in the chemical phenomena of muscular respiration takes place in the act of contraction. Notwithstanding our present ignorance of the form of the electromotive element of muscle, we may therefore admit that the development of muscular electricity is most probably due to the chemical actions belonging to living muscles.

§ 4. We shall now speak, in the last place, of the mechanical effects of muscular contraction, limiting ourselves to experiments on the frog, because in this case only it is possible to make the research with a certain degree of exactitude, and because after having determined the mechanical effect for a frog, we shall be able to compare the *effective work* of a muscular contraction with the *work* which we shall call *theoretical*, according to the principles of the dynamic theory of heat.

This research should be made on a single muscle, and not on the whole system of muscles, as in an entire frog, because in the latter case the opposite effects of these muscles neutralize themselves almost completely. I therefore employed the gastrocnemius of a frog to which a small weight was attached, and which was made to contract by the passage of an electric current. The object of the experiment is to determine the height to which this weight is raised in the act of contraction; this I did by means of the dynamometer described in the Fourth Series of my Electro-physiological Researches *.

Among the experiments published by M. Helmholtz in his very important memoir on the velocity of the propagation of nervous power, there are several which might have served our purpose, if the weights attached to the muscles had not been too great; for the comparison we propose requires the determination of the mechanical work of the contraction of a muscle in raising its own weight only. Neither could we deduce from the numbers found by M. Helmholtz, the muscular work in the case of muscles to which a small weight is fixed, because it is known and admitted, according to the experiments of Schwann, that the quantity of work produced by a muscular contraction increases, to a certain limit, with the weight which stretches the muscle and which that muscle raises: this singular property of the muscular engine has necessarily the effect of weakening more rapidly the contractile power of the muscle.

The medium result of a great number of my experiments has been, that the contraction of a gastrocnemius which weighs 0·300 gr., stretched by a weight of 10 grs., raises that weight to the height of 1·412 mm. The mechanical work of a contraction of a gastrocnemius is therefore expressed by 0·00001457 kilogramme metres. This number being

* Philosophical Transactions, Part IV. 1846.

found, it was of some interest to determine the quantity of electricity necessary, in acting on the nerves or muscles of a frog, to produce that contraction; the results obtained in this last research led me to the experiments on muscular respiration which form the subject of the second part of this memoir. After having measured the quantity of zinc oxidized in a given time in a pile, the current of which is employed to excite contractions in a gastrocnemius, we have still to find the duration of the passage of that current necessary for obtaining a contraction. For this purpose a current of a couple, amalgamated zinc and platinum, in water slightly salt, is sent for upwards of twenty-four hours into a neutral solution of nitrate of silver. The silver collected on the negative electrode, exactly weighed, gives the quantity of zinc oxidized in the pile. In order to determine the time of the passage of the current in a gastrocnemius required to produce normal contraction, *i. e.* the work already determined, I employed a method differing little from that imagined by M. POUILLET for measuring very short intervals of time. I fix on the radius of a wheel, *six* or *seven* metres in circumference, a strip of amalgamated copper 1 millimetre wide. The wheel being set in motion with a uniform velocity of about four turns in a second, the time employed by the circumference of the wheel in passing an arc of 1 millimetre is necessarily very short, but can nevertheless be measured accurately. While the wheel is in movement, two amalgamated brass springs are brought in contact with its periphery: these springs form part of the circuit of the pile, in which are comprised a gastrocnemius recently prepared and fixed in the dynamometer and a galvanometer. I have thus obtained *normal contraction* by the passage of a current, which produced no sensible deflection of the needle of the galvanometer, and of which the duration was about $\frac{1}{10,000}$th of a second. If we admit, which we may do without error, that the conducting power of the gastrocnemius does not exceed that of the solution of nitrate of silver, the quantity of zinc oxidated in the pile during that time is measured by the 7 *billionth* of a milligramme of zinc*.

Taking the mechanical equivalent of heat determined by the experiments of JOULE, and reasoning according to the principles of the dynamical theory of heat, and finally supposing that all the zinc oxidated in the above experiments is converted into electrical current, and hence into muscular energy, one arrives at a conclusion which is far from being in conformity with the theory: the quantity of *vis viva* of a muscle corresponding to the above-mentioned quantity of zinc, would be enormously greater than that deduced from the principles of that theory. This discordance will cease, when it shall be proved that nervous irritation excites new chemical actions in the muscle, and that these are associated with its state of contraction: in this way the electrical current acting on a nerve should be considered in its mechanical effects as the spark which ignites gunpowder or a mixture of hydrogen and oxygen.

* These numbers represent only a limit; but if one reflects that the contraction of a frog is excited by the discharge of a Leyden jar of the smallest possible dimensions and which has been previously discharged by a metallic arc, it becomes evident that this limit greatly exceeds the reality.

X. The Bakerian Lecture.—*Experimental Relations of Gold (and other Metals) to Light.* By Michael Faraday, *Esq., D.C.L., F.R.S., Fullerian Prof. Chem. Royal Institution, Foreign Associate of the Acad. Sciences, Paris, Ord. Boruss. pour le Mérite, Eq., Memb. Royal and Imp. Acadd. of Sciences, Petersburgh, Florence, Copenhagen, Berlin, Göttingen, Modena, Stockholm, Munich, Bruxelles, Vienna, Bologna, Commander of the Legion of Honour, &c. &c.*

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That wonderful production of the human mind, the undulatory theory of light, with the phenomena for which it strives to account, seems to me, who am only an experimentalist, to stand midway between what we may conceive to be the coarser mechanical actions of matter, with their explanatory philosophy, and that other branch which includes, or should include, the physical idea of forces acting at a distance; and admitting for the time the existence of the ether, I have often struggled to perceive how far that medium might account for or mingle in with such actions, generally; and to what extent experimental trials might be devised which, with their results and consequences, might contradict, confirm, enlarge, or modify the idea we form of it, always with the hope that the corrected or instructed idea would approach more and more to the truth of nature, and in the fulness of time coincide with it.

The phenomena of light itself are, however, the best and closest tests at present of the undulatory theory; and if that theory is hereafter to extend to and include other actions, the most effectual means of enabling it to do so will be to render its application to its own special phenomena clear and sufficient. At present the most instructed persons are, I suppose, very far from perceiving the full and close coincidence between all the facts of light and the physical account of them which the theory supplies. If perfect, the theory would be able to give a reason for every physical affection of light; whilst it does not do so, the affections are in turn fitted to develope the theory, to extend and enlarge it if true, or if in error to correct it or replace it by a better. Hence my plea for the possible utility of experiments and considerations such as those I am about to advance.

Light has a relation to the matter which it meets with in its course, and is affected by it, being reflected, deflected, transmitted, refracted, absorbed, &c. by particles very minute in their dimensions. The theory supposes the light to consist of undulations, which, though they are in one sense continually progressive, are at the same time, as regards the particles of the ether, to and fro transversely. The number of progressive alternations or waves in an inch is considered as known, being from 37,600 to 59,880, and the number which passes to the eye in a second of time is known also, being from

458 to 727 billions; but the extent of the lateral excursion of the particles of the ether, either separately or conjointly, is not known, though both it and the velocity are probably very small compared to the extent of the wave and the velocity of its propagation. Colour is identified with the number of waves. Whether reflexion, refraction, &c. have any relation to the extent of the lateral vibration, or whether they are dependent in part upon some physical action of the medium unknown to and unsuspected by us, are points which I understand to be as yet undetermined.

Conceiving it very possible that some experimental evidence of value might result from the introduction into a ray of separate particles having great power of action on light, the particles being at the same time very small as compared to the wave-lengths, I sought amongst the metals for such. Gold seemed especially fitted for experiments of this nature, because of its comparative opacity amongst bodies, and yet possession of a real transparency; because of its development of colour both in the reflected and transmitted ray; because of the state of tenuity and division which it permitted with the preservation of its integrity as a metallic body; because of its supposed simplicity of character; and because known phenomena appeared to indicate that a mere variation in the size of its particles gave rise to a variety of resultant colours. Besides, the waves of light are so large compared to the dimensions of the particles of gold which in various conditions can be subjected to a ray, that it seemed probable the particles might come into effective relations to the much smaller vibrations of the ether particles; in which case, if reflexion, refraction, absorption, &c. depended upon such relations, there was reason to expect that these functions would change sensibly by the substitution of different-sized particles of this metal for each other. At one time I hoped that I had altered one coloured ray into another by means of gold, which would have been equivalent to a change in the number of undulations; and though I have not confirmed that result as yet, still those I have obtained seem to me to present a useful experimental entrance into certain physical investigations respecting the nature and action of a ray of light. I do not pretend that they are of great value in their present state, but they are very suggestive, and they may save much trouble to any experimentalists inclined to pursue and extend this line of investigation.

Gold-leaf—effect of heat, pressure, &c.

Beaten gold-leaf is known in films estimated at the $\frac{1}{282000}$th of an inch in thickness; they are translucent, transmitting green light, reflecting yellow, and absorbing a portion. These leaves consist of an alloy in the proportions of 12 silver and 6 copper to 462 of pure gold. 2000 leaves 3$\frac{3}{8}$ths of an inch square are estimated to weigh 384 grains. Such gold-leaf is no doubt full of holes, but having, in conjunction with Mr. W. De la Rue, examined it in the microscope with very high powers (up to 700 linear), we are satisfied that it is truly transparent where the gold is continuous, and that the light transmitted is green. By the use of the balance Mr. De la Rue found that the leaf employed was on the average $\frac{1}{178000}$th of an inch thick. Employing polarized light

and an arrangement of sulphate of lime plates, it was found that other rays than the green could be transmitted by the gold-leaf. The yellow rays appeared to be those which were first stopped or thrown back. Latterly I have obtained some pure gold-leaf beaten by MARSHALL, of which 2000 leaves weighed 408 grains, or 0·2 of a grain per leaf: its reflected colour is orange-yellow, and its transmitted colour a warm green. Gold alloy containing 25 per cent. of silver produces pale gold-leaf, which transmits a blue purple light, and extinguishes much more than the ordinary gold-leaf.

So a leaf of beaten gold occupies in average thickness no more than from $\frac{1}{3}$th to $\frac{1}{4}$th part of a single wave of light. By chemical means, the film may be attenuated to such a degree as to transmit a ray so luminous as to approach to white, and that in parts which have every appearance of being continuous in the microscope, when viewed with a power of 700. For this purpose it may be laid upon a solution of chlorine, or, better still, of the cyanide of potassium*. If a clean plate of glass be breathed upon and then brought carefully upon a leaf of gold, the latter will adhere to it; if distilled water be immediately applied at the edge of the leaf, it will pass between the glass and gold, and the latter will be perfectly stretched; if the water be then drained out, the gold-leaf will be left well extended, smooth, and adhering to the glass. If, after the water is poured off, a weak solution of cyanide be introduced beneath the gold, the latter will gradually become thinner and thinner; but at any moment the process may be stopped, the cyanide washed away by water, and the attenuated gold film left on the glass. If towards the end a washing be made with alcohol, and then with alcohol containing a little varnish, the gold film will be left cemented to the glass†.

In this manner the leaf may be obtained so thin, that I think 50 or even 100 might be included in a single progressive undulation of light. But the character of the effect on light is not changed, the light transmitted is green, as before; and though that green tint is due to a condition of the gold induced by pressure, it as yet remains unchanged through all these varieties of thickness and of proportion to the progressive or the lateral undulation.

Gold-leaf, either fine or common, examined in the microscope, appears as a most irregular thing. It is everywhere closely mottled or striated, according as a part at the middle or the edge of a leaf is selected, minute portions which are close to other parts being four or five times as thick as the latter, if the proportion of light which passes through may be accepted as an indication. Yet this irregular plate does not cause any

* The chlorine leaves a film of chloride of silver behind, the cyanide leaves only metal.

† Air-voltaic circles are formed in these cases, and the gold is dissolved almost exclusively under their influence. When one piece of gold-leaf was placed on the surface of a solution of cyanide of potassium, and another, moistened on both sides, was placed under the surface, both dissolved; but twelve minutes sufficed for the solution of the first, whilst above twelve hours were required for the submerged piece. In weaker solutions, and with silver also, the same results were obtained; from sixty to a hundredfold as much time being required for the disappearance of the submerged metal as for that which, floating, was in contact both with the air and the solvent. An action of this kind has probably much to do with the *formation* of the films to be described hereafter.

sensible distortion of an object seen through it, that object being the line of light reflected from a fine wire in the focus of a moderate microscope. Nor perhaps was any distortion due to consecutive convexities and concavities to be expected; for when the thicker parts of the leaf were examined they seemed to be accumulated plications of the gold, the leaf appearing as a most irregular and crumpled object, with dark veins running across both the thicker and thinner parts, and from one to the other. Yet in the best microscope, and with the highest power, the leaf seemed to be continuous, the occurrence of the smallest sensible hole making that continuity at other parts apparent, and every part possessing its proper green colour. How such a film can act as a plate on polarized light in the manner it does, is one of the queries suggested by the phenomena which requires solution.

When gold-leaf is laid upon glass and its temperature raised considerably without disturbance, either by the blowpipe or an ordinary Argand gas-burner, it seems to disappear, *i. e.* the lustre passes away, the light transmitted is abundant and nearly white, and the place appears of a pale brown colour. One would think that much of the metal was dissipated, but all is there, and if the heat has been very high (which is not necessary for the best results), the microscope shows it in minute globular portions. A comparatively low heat, however, and one unable to cause separation of the particles, is known to alter the molecular condition of gold, and the gold-beater finds important advantage in the annealing effect of a temperature that does not hurt the skins or leaves between which he beats the metal.

It might be supposed that the annealed metal, in contracting from the constrained and attenuated state produced by beating, drew up, leaving spaces through which white light could pass, and becoming itself almost insensible through the smallness of its quantity; and if gold-leaf unattached to glass be heated carefully with oil in a tube, it does shrink up considerably even before it loses its green colour, which finally happens. But if the gold-leaf laid upon glass plates by water only be carefully dried, then introduced into a bath of oil and raised to a temperature as high as the oil can bear for five or six hours, and then suffered to cool, the plates, when taken out and washed, first in camphine and then in alcohol, present specimens of gold which has lost its green colour, transmits far more light than before and reflects less, whilst yet the film remains in form and other conditions apparently quite unchanged. Being now examined in the microscope, it presents exactly the forms and appearance of the original leaf, except in colour; the same irregularities appear, the same continuity, and if the destruction of the green colour has not been complete, it will be seen that it is the thicker folds and parts of the mottled mass that retains the original state longest.

This change does not depend upon the substance in contact with which the gold is heated *. If the leaf be laid upon mica, rock-crystal, silver or platinum, the same result

* The disappearance of gold-leaf as metal, when mingled with lime, alumina and other bodies, and then heated, has been already observed; and referred to oxidation (J. A. Buchner). See Gmelin's 'Chemistry,' vi. p. 206, "Purple oxide of gold."

occurs; the surrounding medium also may change, and be air, oil or carbonic acid, without causing alteration. Nor has the gold disappeared; a piece of leaf, altered in one part and not in another, was divided into four equal parts, and the gold on each converted by chlorine gas into crystallized chloride of gold: the same amount was found in each division.

When the gold-leaf is laid by water on plates of rock-crystal, and then gradually heated in a muffle not higher than is necessary, an excellent result is obtained. The gold is then of a uniform pale brown colour by common observation, but when examined by a lens and an oblique light, all the mottle of the original leaf appears. It adheres but very slightly to the rock-crystal, and yet can bear the application of the pressure now to be described.

When gold rendered colourless by annealing is subjected to pressure, it again becomes of a green colour. I find a convex surface of agate or rock-crystal having a radius of from a quarter to half an inch very good for this purpose, the metal having very little tendency to adhere to this substance. The greening is necessarily very imperfect, and if examined by a lens it will be evident that the thinner parts of the film are rarely reached by the pressure, it being taken off by the thicker corrugations; but when reached they acquire a good green colour, and the effect is abundantly shown in the thicker parts. At the same time that the green colour is thus reproduced, the quantity of light transmitted is diminished and the quantity of light reflected is increased. When the gold-leaf has been heated on glass in a muffle, it generally adheres so well as to bear streaking with the convex rock-crystal, and then the production of the reflecting surface and the green transmission is very striking. In other forms of gold film, to be described hereafter, the greening effect of pressure (which is general to gold) is still more strikingly manifested, and can be produced with the touch of a card or a finger. In these cases, and even with gold-leaf, the green colour reproduced can be again taken away by heat, to appear again by renewed pressure.

As to the essential cause of this change of colour, more investigation is required to decide what that may be. As already mentioned, it might be thought that the gold-leaf had run up into separate particles. If it were so, the change of colour by division is not the less remarkable, and the case would fall into those brought together under the head of gold fluids. On the whole, I incline to this opinion; but the appearance in the microscope, the occurrence of thin films of gold acting altogether like plates and yet not transmitting a green ray until they are pressed, and their action on a polarized ray of light, throw doubts in the way of such a conclusion.

It may be thought that the beating has conferred a uniform strained condition upon the gold, a difference in quality in one direction which annealing takes away; but when the gold is examined by polarized light, there is no evidence as yet of such a condition, for the green and the colourless gold present like results; and there is a little difficulty in admitting that such an irregular corrugated film as gold-leaf appears to be can possess any general compression in one direction only, especially when it is considered

that it is beaten amongst tissues softer than itself, and made up with it into considerable masses. The greening effect of pressure occurs with the deposited particles of electric discharges, and here it appears either amongst the larger particles near the line of the discharge, or amongst the far finer ones at a considerable distance. Such results do not suggest a dependence upon either the size of the particles or their quantity, but rather upon the relative dimensions of the particles in the direction of the ray and transverse to that direction. One may imagine that spherical or other particles, which, being disposed in a plane, transmit ruby rays or violet rays, acquire the power, when they are flattened, of transmitting green rays, and such a thought sends the mind at once from the wave of light to the direction and extent of the vibrations of the ether. For it does not seem likely that pressure can produce its peculiar result by affecting the relation of the dimension of the particle to the length-dimension of a progressive undulation of light, the latter being so very much greater than the former; but the relation to the dimension of the direct or lateral vibration of the particles of the ether may be greatly affected, that being probably very small and much nearer to, if not even less than, the size of the particles of gold.

Silver-leaf, as usually obtained by beating, is so opake, as perfectly to exclude the light of the sun. When this is laid by water on plates of rock-crystal and heated in a muffle, it begins to change, at a temperature lower than that required for gold, and becomes very translucent, losing at the same time its reflective power: it looks very like the film of chloride produced when a leaf of silver is placed in chlorine gas. When examined by a lens or an ordinary microscope, the leaf seems to be as continuous as in its original state; the finest hole, or the finest line drawn by a needle-point, appears only to prove the continuity of the metallic film up to the very edges of these real apertures. When pressure is applied to this translucent film, the compressed metal becomes either opake or of a very dark purple colour, and resumes its high reflective power. If a higher heat than that necessary for this first change be applied, then the leaf, viewed in the microscope, assumes a mottled appearance, as if a retraction into separate parts had occurred. At a still higher temperature this effect is increased; but the heat, whether applied in the muffle or by a blowpipe, which is necessary to fuse the metal and make it run together in globules, is very much higher than that which causes the first change of the silver: the latter is, in fact, below such a red heat as is just visible in the dark. Whatever the degree of heat applied, the metal remains as metallic silver during the whole time. When many silver leaves were laid loosely one upon another, rolled up into a loose coil, introduced into a glass tube, and the whole placed in a muffle and heated carefully for three or four hours to so low a degree that the glass tube had not been softened or deformed, it was found that the silver-leaf had sunk together a little and shaped itself in some degree upon the glass, touching by points here and there, but not adhering to it. But it was changed, so that the light of a candle could be seen through forty thicknesses: it had not run together, though it adhered where one part touched another. It did not look like metal, unless one thought of it as divided dead metal, and it even

appeared too unsubstantial and translucent for that; but when pressed together it clung and adhered like clean silver, and resumed all its metallic characters.

When the silver is much heated, there is no doubt that the leaf runs up into particles more or less separate. But the question still remains as to the first effect of heat, whether it merely causes a retraction of the particles, or really changes the optical and physical nature of the metal from the beaten or pressed state to another from which pressure can return it back again to its more splendid condition. It seems just possible that the leaf may consist of an infinity of parts resulting from replications, foldings and scales, all laid parallel by the beating which has produced them, and that the first action of heat is to cause these to open out from each other; but that supposition leaves many of the facts either imperfectly explained or untouched. The Arts do not seem to furnish any process which can instruct us as to this condition, for all the operations of polishing, burnishing, &c. applied to gold, silver and other metals, are just as much fitted to produce the required state under one view as under the other.

To return to gold: it is clear that that metal, reduced to small dimensions by mere mechanical means, can appear of two colours by transmitted light, whatever the cause of the difference may be. The occurrence of these two states may prepare one's mind for the other differences with respect to colour, and the action of the metallic particles on light, which have yet to be described.

Many leaves of gold, when examined by a lens and transmitted light, present the appearance of red parts; these parts are small, and often in curved lines, as if a fine hair had been there during the beating. At first I thought the gold was absolutely red in these parts, but am inclined to believe that in the greatest number of cases the tint is subjective, being the result of the contrast between the white light transmitted through bruised parts, and the green light of the neighbouring continuous parts. Nevertheless, some of these places, when seen in the microscope, appeared to have a red colour of their own, that is, to transmit a true red light. As I believe that gold in a certain state of division can transmit a ruby light, I am not prepared to say that gold-leaf may not, in some cases, where the effect of pressure in a particular direction has been removed, do the same.

Many of the prepared films of gold were so thin as to have their reflective power considerably reduced, and that in parts which, under the microscope and in other ways, appeared to be quite continuous: this agrees with the transmission of all the rays already mentioned, but it seems to imply that a certain thickness is necessary for full reflexion; therefore, that more than one particle in depth is concerned in the act, and that the division of gold into separate particles by processes to be described, may bring them within or under the degree necessary for ordinary reflexion.

As particles of pure gold will be found hereafter to adhere by contact, so the process of beating may be considered as one which tends to weld gold together in all directions, and especially in that transverse to the blow,—a point favourable to continuity in that direction, both as it tends to preserve and even reproduce it.

If a polarized ray be received on an analyser so that no light passes, and a plate of annealed glass, either thick or thin, be interposed vertically across the ray, no difference is observed on looking through the analyser, the image of the source of light does not appear; but if the plate be inclined until it makes an angle of from 30° to 45°, or thereabouts, with the ray, the light appears, provided the inclination of the glass is not in the plane of polarization or at right angles to it, the effect being a maximum if the inclination be in a plane making an angle of 45° with that of polarization. This effect, which is common to all uncrystallized transparent bodies, is also produced by leaf-gold, and is one of the best proofs of the true transparency of this metal according to the ordinary meaning of the term. In like manner, if a leaf of gold be held obliquely across an ordinary ray of light, it partly polarizes it, as Mr. De la Rue first pointed out to me. Here again the condition of true transparency is established, for it acts like a plate of glass or water or air. But the relations of gold and the metals in different conditions to polarized light shall be given altogether at the close of this paper.

Deflagrations of Gold (and other metals)—heat—pressure, &c.

Gold wire deflagrated by explosions of a Leyden battery produces a divided condition, very different to that presented by gold leaves. Here the metal is separated into particles, and no pressure in any direction, either regular or irregular, has been exerted upon them in the act of division. When the deflagrations have been made near surfaces of glass, rock-crystal, topaz, fluor-spar, card-board, &c., the particles as they are caught are kept separate from each other and in place, and generally those which remain in the line of the discharge have been heated by the passage of the electricity. The deposits consist of particles of various sizes, those at the outer parts of the result being too small to be recognized by the highest powers of the microscope. Beside making these deflagrations over different substances, as described above, I made them in different atmospheres, namely, in oxygen and hydrogen, to compare with air; but the general effects, the colours produced, and the order of the colours, were precisely the same in all the cases. These deposits were insoluble in nitric acid and in hydrochloric acid, but in the mixed acids or in chlorine solution were soluble, exactly in the manner of gold. There is no reason to doubt that they consisted of metallic gold in a state of extreme division.

Now as to the effects on light, *i. e.* as to the coloured rays reflected or transmitted by these deposited particles, and first, of those in the line of the discharge where the wire had been. Here the mica was found abraded much, the glass less, and the rock-crystal and topaz least. Where abraded, the gold adhered; in all the other parts it could be removed with the slightest touch. The gold deposited in this central place was metallic and golden by reflected light, and of a fine ruby colour by transmitted light. On each side of this line the deposit had a dark colour, but when particularly examined gave a strong golden metallic reflexion, and by transmission a fine violet colour, partaking of green and ruby in different parts, and sometimes passing altogether into green. Beyond this, on each side, where the tints became paler and where the particles appeared to be

finer, the transmitted tint became ruby or violet-ruby, and this tint was especially seen when the deposit was caught on a card. As to the reflected light, even at these faintest parts it is golden and metallic. This is easily observed by wiping off a sharp line across the deposit on glass in the very faintest part, and then causing the sun's rays collected in the focus of a small lens to travel to and fro across that edge; the presence of the metallic gold on the unwiped part is at once evident by the high illumination produced there. It is evident that all the colours described are produced by one and the same substance, namely gold, the only apparent difference being the state of division and different degrees of the application of heat. The thickest parts of these deposits are so discontinuous, that they cannot conduct the electricity of a battery of two or three pairs of plates, *i. e.* of a battery unable to produce a spark among the particles.

When any of these deposits of divided gold are heated to dull redness, a remarkable change occurs. The portions which before were violet, blue, or green by transmitted light, now change to a ruby, still preserving their metallic reflecting power, and this ruby is in character quite like that which is presented in the arts by glass tinged by gold. This change is often far better shown in the more distant and thinner parts of the deposit, than in those nearest to the line of discharge, for near the latter place, where the deposit is most abundant, the metal appears to run up into globules, as with gold-leaf, and so disappears as a film. I believe that the ruby character of the deposit *in* the line of discharge, is caused by the same action of heat produced at the moment by the electricity passing there. In the distant parts, the deposit, rubified by after heat, is not imbedded or fused into the glass, rock-crystal, topaz, &c., but is easily removed by a touch of the finger, though in parts of the glass plate which are made very hot, it will adhere.

If the agate pressure before spoken of, in respect of gold-leaf, be applied to ruby parts not too dense, places will easily be found where this pressure increases the reflective power considerably, and where at the same time it converts the transmitted ray from ruby to green; making the gold, as I believe, then accord in condition with beaten gold-leaf. On the other hand, if parts of the *unheated* electric deposit, where they are purple-grey, and so thin as to be scarcely visible without care, be in like manner pressed, they will acquire the reflective power, and then transmit the green ray; and I think I am justified by my experiments in stating, that fine gold particles, so loosely deposited that they will wipe off by a light touch of the finger, and possessing one conjoint structure, can in one state transmit light of a *blue-grey* colour, or can by heat be made to transmit light of a *ruby* colour, or can by pressure from either of the former states be made to transmit light of a *green* colour; all these changes being due to modifications of the gold, as gold, and independent of the presence of the bodies upon which for the time the gold is supported; for I ought to have said, if I have not said so, that these changes happen with all the deposits upon glass, mica, rock-crystal, and topaz, and whatever the atmosphere in which they were formed.

When gold is deflagrated by the voltaic battery near glass (I have employed sovereigns

laid on glass for the terminals), a deposit of metallic gold in fine particles is produced. The densest parts have a dark slate-violet colour passing into violet and ruby-violet in the outer thinner portions; a ruby tint is presented occasionally where the heat of the discharge has acted on the deposit. The deposited gold was easily removed by wiping, except actually at the spot where the discharge had passed. When these deposits were heated to dull redness they changed and acquired a ruby tint, which was very fine at the outer and thinner parts. The portions nearer the place of discharge presented ruby-violet and then violet tints, suggesting that accumulation of that which presented a fine ruby tint would, by stopping more and more light, transmit a ruby-violet or violet ray only. Pressure with the agate surface had a like effect as before, both with the heated and the unheated portions, *i. e.* with the violet and the ruby particles; but the effect was not altogether so good, and the tint of the transmitted ray was rather a green violet than a pure green. Still the difference produced by the pressure was very remarkable. The unheated particles at the surface, away from the glass, presented by reflexion almost a black; being heated, they became much more golden and metallic in appearance.

I prepared an apparatus by which many of the common metals could be deflagrated in hydrogen by the Leyden battery, and being caught upon glass plates could be examined as to reflexion, transmission, colour, &c. whilst in the hydrogen and in the metallic, yet divided state. The following are briefly the results; which should be considered in connexion with those obtained by employing polarized light. *Copper*: a fine deposit presenting by reflexion a purplish red metallic lustre, and by transmission a green colour, dark in the thicker parts, but always green; agate pressure increased the reflexion where it was not bright, and a little diminished the transmission, rendering the green deeper, but not changing its character as in the case of gold. *Tin* gave a beautiful bright white reflexion, and by transmission various shades of light and dark brown; agate pressure diminished the transmission and increased the reflexion in places before dull or dead; the effect appeared to be due simply to the lateral expansion of the separate particles filling up the space. *Iron* presented a fine steel grey, or slate metallic reflexion and a dark brown transmission; agate pressure gave the same effect as with tin, but no change of colour. *Lead*: a bright white reflexion, the transmission a dark smoky brown; agate pressure appeared to change this brown towards blue. *Zinc*: the reflexion bright white and metallic; the transmission a dark smoky colour with portions of blue-grey, brown-grey and pale brown; agate pressure tended to change the blue-grey to brown. *Palladium*: the reflexion fine metallic and dark grey; the transmitted light, where most abundant, sepia-brown; agate pressure converted the tint in the thinner places from brown towards blue-grey. *Platinum*: the reflexion white, bright and metallic; the transmission brown or warm grey with no other colours; agate pressure increased the reflexion and diminished the transmission as with tin. *Aluminium*: the reflexion metallic and white, very beautiful; the transmitted light was dark brown, bluish brown, and occasionally in the thinner parts orange; agate pressure caused but little change.

Films of Gold (and other metals) by Phosphorus, Hydrogen, &c.—effect of heat—pressure.

The reduction of gold from its solution by phosphorus is well known. If fifteen or twenty drops of a strong solution of gold, equal to about 1½ grain of metal, be added to two or three pints of water, contained in a large capsule or dish, if four or five minute particles of phosphorus be scattered over the surface, and the whole be covered and left in quietness for twenty-four or thirty-six hours, then the surface will be found covered with a pellicle of gold, thicker at the parts near the pieces of phosphorus, and possessing there the full metallic golden reflective power of the metal; but passing by gradation into parts, further from the phosphorus, where the film will be scarcely sensible except upon close inspection. If plates of glass be introduced into the fluid under the pellicle, and raised gradually, the pellicle will be raised on them; it may then be deposited on the surface of pure distilled water to wash it; may be raised again on the glass; the water allowed to drain away, and the whole suffered to dry. In this way the pellicle remains attached to the glass, and is in a very convenient condition for preservation and examination.

If phosphorus be dissolved in two or three times its bulk of sulphide of carbon, and a few drops of the fluid be placed on the bottom of a dry basin, vapour of the phosphorus will soon rise up and bring the atmosphere in the basin to a reducing state. If a plate of glass large enough to cover the basin have six or eight drops of a strong neutral solution of chloride of gold placed on it, and this be spread about by a glass stirrer, so as to form a flowing layer on the surface, the glass may then be inverted and placed over the dish. So arranged the gold solution will keep its place, but will have a film of metal reduced on its under surface. The plate being taken off after twenty, thirty, or forty minutes, and turned with the gold solution upwards, may then gradually be depressed in an inclined position into a large basin of pure water, one edge entering first, and the gold film will be left floating. After sufficient washing it may be taken up in portions on smaller plates of glass, dried, and kept for use. Mr. Warren De la Rue taught me how to make and deal with these films: they may by attention be obtained very uniform, of very different degrees of thickness, from almost perfect transparency to complete opacity, and by successive application of the same collecting glass plate may be superposed with great facility.

These films may be examined either on the water or on the glass. When thick, their reflective power is as a gold plate, full and metallic; as they are thinner they lose reflective power, and they may be obtained so thin as to present no metallic appearance, all the coloured rays of light then passing freely through them. As to the transmitted light, the thinner films generally present one kind of colour; it appears as a feeble grey-violet, which increases in character as the film becomes thicker and sometimes approaches a violet; a greenish violet also appears; and the likeness of the grey-violet tint of these films to the stains produced by a solution of gold on the skin or other organic reducing substance, or the stain produced on common pottery, cannot be mistaken. Superposition

of several grey-violet films does not produce a green tint, but only a diminution of light without change of colour. In those specimens made by particles of phosphorus floating on the solution of gold, very fine green tints occur at the thicker and golden parts of the film. The colour of the gold here may depend in some degree on the *manner* in which these films are formed: the thicker parts are not produced altogether by the successive addition of reduced gold from the portion of fluid immediately beneath them. When a particle of phosphorus is placed on pure water, it immediately throws out a film which appears to cover the whole of the surface; in a little while the film thickens around the particle and is easily distinguished by its high reflective power. It is this film which reduces the gold in solution, being itself consumed in the action; the result is a continued extension from the phosphorus outwards, which, after it has covered the solution with a thin film of gold, continues to cause a compression of the parts around the phosphorus and an accumulation there, rendering the gold at a distance of half an inch from the phosphorus so thick, that it is brilliant by reflexion and nearly opake by transmission, whilst near to the phosphorus the forming film is so thin as to be observed only on careful examination, and is still travelling outwards and compressing the surrounding parts more and more. The phosphorus is very slowly consumed; a particle not weighing $\frac{1}{100}$th of a grain will remain for four or five days on the surface of water before it disappears.

Though the particles of these films adhere together strongly, as may be seen by their stiffness on water, still the films cannot be considered as continuous. If they were, those made by vapour of phosphorus could not thicken during their formation, neither could they dry on glass in the short time found sufficient for that purpose. Experimentally also, I find that vapours and gases can pass through them. Very thin films without folds did not sensibly conduct the electricity of a single pair of Grove's plates; thicker films did conduct; yet with these proofs that these films could not be considered as continuous, they acted as thin plates upon light, producing the concentric rings of colours round the phosphorus at their first formation, though their thickness then could scarcely be the $\frac{1}{100}$th, perhaps not the $\frac{1}{100}$th of a wave undulation of light. Platinum, palladium, and rhodium produced films, showing these concentric rings very well.

Many of these films of gold, both thick and thin, which being of a grey colour originally, were laid on a solution of cyanide of potassium to dissolve slowly, changed colour as they dissolved and became green; if change occurred, it was always towards green. On the other hand, when laid on a solution of chlorine, the change during solution was towards an amethyst or ruby tint. The films were not acted upon by pure nitric, or hydrochloric, or sulphuric acids, or solutions of potassa or brine. They dissolved in damp chlorine gas, not changing in colour during the solution. I believe them to consist of pure gold.

When these gold films were heated to dull redness they changed. The reflexion, though not much altered, was a little more metallic and golden than before; more light was transmitted after the heating and the colour had altered from greenish to violet, or

from grey-green to ruby or amethyst; and now two or three films superposed often gave a very ruby colour. This action is like that of heat on the particles separated by electric explosions. If not overheated, the particles were not fused to the glass, but could be easily wiped off. Whenever these heated particles were pressed by the convex agate, they changed in character and transmitted green light. Heat took away this character of the gold, the heat of boiling oil, if continued, being sufficient; but on applying pressure at the same spot, the power of transmitting green light was restored to the particles. In many cases where the gold adhered sufficiently to the glass to bear a light drawing touch from the finger or a card, such touch altered the light transmitted from amethystine to green; so small is the pressure required when the particles are most favourably disposed.

Heating injured the conducting power for electricity of these films, no doubt by retraction of the particles, though there was no such evident appearance in these cases, as in the unattached gold-leaf of the particles running up into globules.

A given film, examined very carefully in the microscope by transmitted lamp-light, with an aperture of 90° and power of 700 linear, presented the following appearances. The unheated part was of a grey colour, and by careful observation was seen to be slightly granular. By very close observation this grey part was often resolvable into a mixture of green and amethystine striæ, it being the compound effect of these which in general produce the grey sensation in the eye. When a part of such a film was heated, the transmitted colour was changed from grey to purple, as before described, and the part thus heated was evidently more granular than before. This difference was confirmed in other cases. That the heated part should thus run up, seems to show that many of the particles must have been touching though they did not form a continuous film; and on the other hand, the difference between the effect here and with unattached gold-leaf, shows that the degree of continuity as a film must be very small. When these heated films were greened by agate pressure, or the drawing pressure of a card, the green parts remained granulated, apparently in the same degree as when purple. The green was not subjective or an effect of interference, but a positive colour belonging to the gold in that condition. Every touch of the agate was beautifully distinct as a written mark. The parts thus greened and the purple parts appeared to transmit about the same amount of light. Though the film appeared granulated, no impression was made upon the mind that the individual particles of which the film consisted were in any degree rendered sensible to the eye.

The unheated gold films when pressed by agate often indicated an improved reflective power, and the light transmitted was also modified; generally it was less, and occasionally tended towards a green tint; but the effect of pressure was by no means so evident as in particles which had been heated.

Films of some other metals were reduced by phosphorus in like manner, the results in all these cases being of course much affected by the strength of the solution and the time of action; they are briefly as follows. *Palladium*: a weak solution of the chloride

gave fine films, apparently very continuous and stiff; the reflexion was strong and metallic, of a dark grey colour; the transmission presented every shade of Indian ink. *Platinum* chloride gave traces of a film excessively thin, and very slow in formation. *Rhodium* chloride in three or four hours gave a beautiful film of metal in concentric rings, varying in reflecting and transmitting power over light and also in colour; those which reflected well, transmitted little light; and those which transmitted, reflected little light; one might have thought there was no metal in some of the rings between other rings that reflected brilliantly, but the metal was there of transmitting thickness; the transmitted colour of rhodium varied from brown to blue. *Silver*: a solution of the nitrate gave films showing the concentric rings; the light transmitted by the thinner parts was of a warm brown, or sepia tint; the film becomes very loose and mossy in the thicker parts and is wanting in adherence; pressure brings out the full metallic lustre in every part, and in the thin places converts the colour from brown to blue, being in that respect like the result with pale gold-leaf, in which the silver present dominates over the colour of the gold. I do not think there is phosphorus combined with this silver; I did not find any, and considering the surface action on metals which float as films between air and water, it seems improbable that it should be there.

Hydrogen was employed to reduce some of the metals, their solutions being placed in an atmosphere of the gas. The action differed considerably from that of phosphorus, as might be expected. *Gold* produced a very thin film, too thin to be washed; it had a faint metallic reflexion, and transmitted a slate-blue colour like the former films. *Platinum* chloride was acted on at once; minute spots appeared here and there on the surface; these enlarged, became rough and corrugated at the middle, though brilliant at the edges, and at last formed an irregular coat over the fluid; at the part where the film was flat and brilliant, it resembled that produced by the electric explosion, and by transmission gave a dark grey colour. *Iridium* required much time, and formed a crust from centres like the platinum. *Palladium* gave an instant action, but most of the reduced metal sunk in a finely divided state; a film may be obtained, but it has very little adhesion. *Rhodium* is reduced, but the film consists of floating particles, having so little adhesion that it cannot be gathered up. *Silver* is reduced, but the film is very thin and has no tenacity.

A copper film of very beautiful character may be obtained as follows in all varieties of thickness. Let a little oxide of copper be dissolved in olive-oil to form a bath, and having immersed some plates of glass, for which purpose microscope plates 3×1 inches are very convenient, let the whole be heated up to the decomposing temperature of the oil; being left to cool, and the plates then drained and washed successively in camphine and alcohol, they will be found covered with a film of copper, having the proper metallic lustre and colour by reflexion; and by transmission, presenting a green colour, which, though generally inclining to olive, is in the thinner films often more beautiful than the green presented by pressed gold.

Diffused particles of gold—production—proportionate size—colour—aggregation and other changes.

Agents competent to reduce gold from its solution are very numerous, and may be applied in many different ways, leaving it either in films, or in an excessively subdivided condition. Phosphorus is a very favourable agent when the latter object is in view. If a piece of this substance be placed under the surface of a moderately strong solution of chloride of gold, the reduced metal adheres to the phosphorus as a granular crystalline crust. If the solution be weak and the phosphorus clean, part of the gold is reduced in exceedingly fine particles, which becoming diffused, produce a beautiful ruby fluid.

This ruby fluid is well obtained by pouring a weak solution of gold over the phosphorus which has been employed to produce films, and allowing it to stand for twenty-four or forty-eight hours; but in that case all floating particles of phosphorus should be removed. If a stronger solution of gold be employed, the ruby fluid is formed, but it soon becomes turbid and tends to produce a deposit. When the gold is in such proportion that it remains in considerable excess, still the ruby formation is not prevented, and being formed it mingles unchanged with the excess of gold in solution. If an exceedingly weak solution of gold be employed, the production of ruby appears to be imperfect and retarded. The nearer the solution is to neutrality at the commencement the better; when a little hydrochloric acid was added the effect was not so good, and the colour of the fluid was more violet than ruby.

If a pint or two of the weak solution of gold before described be put into a *very clean* glass bottle, a drop of the solution of phosphorus in sulphide of carbon added, and the whole well shaken together, it immediately changes in appearance, becomes red, and being left for six or twelve hours, forms the ruby fluid required; too much sulphide and phosphorus should not be added, for the reduced gold then tends to clot about the portions which sink to the bottom.

Though the sulphide of carbon is present in such processes and very useful in giving division to the phosphorus, still it is not essential. A piece of clean phosphorus in a bottle of the gold solution gradually produces the ruby fluid at the bottom, but the action is very slow. If the phosphorus be attached to the side of the bottle, but always beneath the surface of the solution, the streams of ruby fluid may be seen moving both upwards and downwards against the side of the glass, and forming films in the vicinity of the phosphorus perfect in their golden reflexion, and yet transmitting light of ruby, violet, and other tints, thus giving, first a proof that the particles are gold, and then connecting the present condition of the gold with that of the films already described. On the other hand, the phosphorus may be excluded and the sulphide of carbon employed alone; for when it and the solution of gold are shaken together, the gold is reduced and the ruby fluid formed; but it soon changes to purple or violet.

A quick and ready mode of producing the ruby fluid, is to put a quart of the weak solution of gold (containing about 0·6 of a grain of metal) into a clean bottle, to add a little solution of phosphorus in ether, and then to shake it well for a few moments: a

beautiful ruby or amethystine fluid is immediately produced, which will increase in depth of tint by a little time. Generally, however, the preparations made with phosphorus dissolved in sulphide of carbon, are more ruby than those where ether is the phosphorous solvent. The process of reduction appears to consist in a transfer of the chlorine from the gold to the phosphorus, and the formation of phosphoric or phosphorous acids and hydrochloric acid, by the further action of the water.

The fluids produced may easily be tested for any gold yet remaining unreduced, by trial of a portion with solution of protochloride of tin. If any be found, it is easily reduced by the addition of a little more of the phosphorus in solution. After all the gold is separated as solid particles, the fluid may be considered in its perfected state. Occasionally it may smell of phosphorus in excess, even after it has been poured off from the deposited particles of it and the sulphide. In that case it is easy to deprive it of this excess by agitation in a bottle with air. When kept in closed vessels mouldiness often occurs. If this be in groups it is collected with facility at the end of a splinter of wood and removed, or the whole fluid may be poured through a wet plug of cotton in the neck of a funnel, the reduced gold passing freely. All the vessels used in these operations must be very clean; though of glass they should not be supposed in proper condition after wiping, but should be soaked in water, and after that rinsed with distilled water. A glass supposed to be clean, and even a new bottle, is quite able to change the character of a given gold fluid.

Fluids thus prepared may differ much in appearance. Those from the basins, or from the stronger solutions of gold, are often evidently turbid, looking brown or violet in different lights. Those prepared with weaker solutions and in bottles, are frequently more amethystine or ruby in colour and apparently clear. The latter, when in their finest state, often remain unchanged for many months, and have all the appearance of solutions. But they never are such, containing in fact no dissolved, but only diffused gold. The particles are easily rendered evident, by gathering the rays of the sun (or a lamp) into a cone by a lens, and sending the part of the cone near the focus into the fluid; the cone becomes visible, and though the illuminated particles cannot be distinguished because of their minuteness, yet the light they reflect is golden in character, and seen to be abundant in proportion to the quantity of solid gold present. Portions of fluid so dilute as to show no trace of gold, by colour or appearance, can have the presence of the diffused solid particles rendered evident by the sun in this way. When the preparation is deep in tint, then common observation by reflected light shows the suspended particles, for they produce a turbidness and degree of opacity which is sufficiently evident. Such a preparation contained in a pint bottle will seem of a dull pale-brown colour, and nearly opake by reflexion, and yet by transmission appear to be a fine ruby, either clear or only slightly opalescent.

That the ruby and amethystine fluids hold the particles in suspension only, is also shown by the deposit which occurs when they are left at rest. If the gold be comparatively abundant, a part will soon settle, *i. e.* in twenty-four or forty-eight hours; but if the pre-

paration be left for six or eight months, a part will still remain suspended. Even in these portions, however, the diffused state of the gold is evident; for where, as in some cases, the top to the depth of half an inch or more has become clear, it is seen that the ruby portion below is as a cloud sinking from it; and in the part which has apparently been cleared from colour by the settling of the particles, the lens and cone of light still shows the few, or rather the fine diffused particles yet in suspension, though the protochloride of tin can show no gold in solution. The mould or mucus before spoken of, often collects the larger, heavier particles, and becomes of a dark blue colour; it may then be taken out by a splinter of wood, and being shaken in water, disengages the particles, which issue from it in clouds like the sporules from a ripe puff-ball.

A gradual change goes on amongst the particles diffused through these fluids, especially in the cases where the gold is comparatively abundant. It appears to consist of an aggregation. Fluids, at first clear or almost clear to ordinary observation, become turbid; being left to stand for a few days, a deposit falls. If the supernatant fluid be separated and left to stand, another deposit may be obtained. This process may be repeated, and whilst the deposition goes on, the particles in the fluid still seem to aggregate; it is only when the fluid is deprived of much gold that the process appears to stop. Even after the fluid has attained a fine marked ruby tint, if allowed to stand for months in a place of equable temperature, the colouring particles will appear in floating clouds, and probably the aggregation is then still going on. That the particles of gold when they touch each other do in many cases adhere together with facility, is shown in many experiments. In order to test this matter mechanically, I gave much agitation to a dense ruby fluid, but did not find it cause any sensible change in the character. When gold particles of a much larger size were agitated in water, they did cohere together, and the fluid, which required a certain time for settling at the beginning of the experiment, settled in a much shorter time at the termination.

If these fluids be examined generally their appearances differ, not merely under different circumstances, but also under the same circumstances, though they always consist of a colourless liquid and diffused particles of gold. A certain fluid in a bottle or glass, looked at from the front, *i. e.* the illuminated side by general daylight, may appear hazy and amethystine, whilst in bright sunlight it will appear light brown and almost opake. From behind, the same fluid may appear of a pure blue in both lights, whilst from the side it may appear amethystine or ruby. These differences result from the mixture of reflected and transmitted lights, both derived from the particles, the former appearing in greatest abundance from the front or side, and the latter from behind. The former is seen by common observation in a purer state if a black background be placed behind the fluid; when a white background is there, much of the transmitted light from that source comes to the eye, and the appearance is greatly altered. A mode of observing the former by a strong ray of light and a lens has been already described; but even in that case some effects of transmitted light are observed if the focus is thrown deep into the fluid; and it is only the particles near the surface, whether illuminated by the base

or the apex of the cone, which give the nearly pure effect of reflexion. In order to observe the transmitted ray in an unmingled state, a glass tube closed at one end was surrounded with a tube of black paper longer than itself, and with the black surface inwards. When a fluid (or the particles in it) was to be examined, it was put into this tube, and a surface of white paper illuminated by daylight or the sun, regarded through it, other light being excluded from the eye; or the tube was sometimes interposed between the eye and the sky, and sometimes the rays of the sun itself were reflected up to the eye through it. In speaking hereafter of the tints of the light transmitted by the particles (which will of course vary with the proportion of different rays in the original beam of light), a pure white original light is to be understood, but occasionally differently-tinted papers were employed with this tube as sources of different coloured lights.

The very oblique angle at which reflected light comes to the eye from the diffused particles, is well seen when the lens cone, or a direct ray of the sun, is passed into the fluid and observed from different positions; it is only when the eye is behind and nearly in the line of the ray, that the unmixed transmitted ray is observed. In the dark tube I think that no reflected light arrives at the eye: for if half an inch in depth of water be introduced, white light passes; if a drop of the washed deposit, to be hereafter described, be introduced, the light transmitted is either blue or ruby, or of other intermediate tint, according to the character of the deposit; but if water be then added until the column is six inches or more in length, the quantity of light transmitted does not sensibly alter, nor its tint; a fact which I think excludes the idea of any light being reflected from particle to particle, and finally to the eye.

If a given ruby-tinted fluid, containing no gold in solution, be allowed to stand for a few days, a deposit will fall from which the fluid may be removed by a siphon; being now allowed to stand for a week, a second deposit will be produced; if the fluid be again removed and allowed to stand for some months, another deposit will be obtained, and the fluid will probably be of a bright ruby; if it be now allowed to stand for several months, it will still yield a deposit, looking however more like a ruby fluid than a collection of fine particles at the bottom of the fluid, whilst traces of yet finer particles of gold in suspension may be obtained by the lens. All these deposits may be washed with water and will settle again; the coarser are not much affected, but the finer are, and tend to aggregate; nevertheless specimens often occur, especially after boiling, which tend to preserve their fine character after washing, if the water be very clean and pure.

The colour of these particles whilst under, or diffused through water, is by common reflected light brown, paler and richer, sometimes tending to yellow, and sometimes to red. The same difference is shown when illuminated by sunlight. Everything tends to show that the light reflected is very bright considering the size of the particles, and therefore of the reflecting surfaces; yet comparing by the cone of light a ruby fluid when first prepared and before it has become very sensibly turbid, with the same fluid after the evident turbidity is produced, in both of which cases I believe the gold to

be in solid metallic particles, though of different sizes, it would seem that more light is transmitted and absorbed and less reflected by the finer particles than by the coarser set, the same quantity of gold being in the same space. I believe that there may be particles so fine as to reflect very little light indeed, that function being almost gone. Occasionally some of the fluids containing the very finest particles in suspension, when illuminated by the sun's rays and a lens, appeared to give a fine green reflexion, but whether this is a true colour as compared to white light, or only the effect of contrast with the bright ruby in the other parts of the fluid, I am not prepared to say.

When the deposits were examined in the dark tube by transmitted light, being first diffused in more or less water to give them the form of fluid, those first deposited, and therefore presumed to be the heavier and larger, transmitted a pure blue light. The second and the third had the same character, perhaps the fourth, if the subdivision into portions had been numerous; then came some which transmitted an amethystine ray from the white of paper; and others followed progressing to the finest, which transmitted a rich ruby tint. It is probable that many of these deposits were mixtures of particles having different characters, and this is perhaps the reason that in some cases, when the fluids were contained in round-bottomed flasks, the lens-like deposit was ruby at the edges, though deep violet in the middle, the former having settled last; but as a pure blue deposit could be obtained, and also one transmitting a pure ruby ray, and as a comparatively pure intermediate preparation transmitting a ruby violet, or amethystine ray was obtained, it is probable that all gradations from blue to ruby exist; for the production of which I can see no reason to imagine any other variation than the existence of particles of intermediate sizes or proportions.

When light other than white was passed through the fluids, then of course other tints were produced, yet some of these were unexpected. A fluid of a pure blue colour, whilst in the dark tube, would in an open glass and by reflected light appear of a strong ruby-violet tint. Dropping some of the wet deposit into pure water, the striæ which it formed would in one part be ruby in colour and in another violet: these effects were referable to the light reflected from the solid particles back through the fluid to the eye, but it seemed redder than any which light reflected from gold was likely to produce. However, upon regarding the surface of dull gold-leaf, or the thick wet deposit of gold, or the hand, it was found that the red rays easily passed through the blue fluid and formed a ruby-violet tint. Prevost showed in old times, how much the red and warm rays are reflected by gold, in preference to the others contained in white light.

The supernatant fluid in specimens that had stood long and deposited, was always ruby; yet because it showed no dissolved gold, because it showed the illuminated cone by the lens, and because by standing ruby clouds settled in it, there was every reason to believe that the gold was there in separated particles, and that such specimens afforded cases of extreme division, which by long standing would form deposits of the finest kind.

Those fluids which on standing gave abundance of deposits, transmitting blue light, consisted in the first instance of particles transmitting a ruby light, and in these cases

it would seem that the particles at their first separation were always competent to transmit this ruby light; and if the preparation were not too rich in gold the ruby condition appeared to be retained, the division being then most extreme. But purple or amethystine fluids could be procured, which, containing no colouring particles other than suspended gold, still retained them in suspension for many months together, so that they must have been as light or as finely divided as those in the ruby fluids. When the phosphorous ether was employed for the reduction of the gold such fluids occurred; also when the solution of the phosphorus in sulphide of carbon was used, provided the solution of gold had a very little chloride of sodium contained in it. They appear to show that the mere degree of division is not the only circumstance which determines the aptitude to transmit in preference this or that ray of light.

Considering the fluids as owing their properties to diffused particles, it may be observed, that many of them which in small quantities in the dark tube transmit an amethystine light, send forward a ruby light when the quantity is increased; and this appears to be the general progression. I have not found any which by increase in quantity tended to transmit the blue rays in preference to the red.

Elevation of temperature had an effect upon these fluids which is advantageous in their preparation. On boiling an apparently clear ruby fluid for some time, its colour passed a little towards amethystine, and on boiling a like amethystine fluid, its tint passed towards blue. The separation of the gold particles was also facilitated, for now they would settle in three or four days from a fluid which, prior to this operation, would not have deposited them in an equal degree for weeks. In the case of the ruby fluids the colour often became more rosy and luminous, and by reflected light the fluid seemed to have became more turbid, as if the particles had gained in reflective power; in fact the boiling often appeared to confer a sort of permanency on the particles in their new state. When settled, they formed collections looking like little lenses of a deep ruby or violet colour, at the bottom of the flasks containing the fluid; when all was shaken up the original fluid was reproduced, and then, by rest, the gold re-settled. This effect could be obtained repeatedly. The particles could fall together within a certain limit, but many weeks did not bring them nearer or into contact; for they remained free to be diffused by agitation. The space they occupied in this lens-like form must have been a hundredfold, or even a thousandfold, more than that, which they would have filled as solid gold. Whether the particles be considered as mutually repulsive, or else as molecules of gold with associated envelopes of water, they evidently differ in their physical condition, for the time, from those particles which by the application of salt or other substances are rendered mutually adhesive, and so fall and clot together.

In preparing some of these fluids, I made the solution of gold hot and boiling before adding the solution of phosphorus. The phenomena were the same in kind as before: but when the phosphorus was dissolved in sulphide of carbon, the gold soon fell as a dark flocculent deposit; when it was dissolved in ether a more permanent turbid ruby fluid was obtained, which, if it does not go on changing in aggregation, may give a good ruby deposit.

The particles in these fluids are remarkable for a set of physical alterations occasioned by bodies in small quantities, which do not act chemically on the gold, or change its intrinsic nature; for through all of them it seems to remain gold in a fine state of division. They occur most readily where the particles are finest, *i. e.* in the ruby fluids, and so readily that it is difficult to avoid them; they are often occasioned by the contact of vessels which are supposed to be perfectly clean. An idea of their nature may be obtained in the following manner. Place a layer of ruby fluid in a clean white plate, dip the tip of a glass rod in a solution of common salt and touch the ruby fluid; in a few moments the fluid will become blue or violet-blue, and sometimes almost colourless: by mingling up the neighbouring parts of the fluid, it will be seen how large a portion of it can be affected by a small quantity of the salt. By leaving the whole quiet, it will be found that the changed gold tends to deposit far more readily than when in the ruby state. If the experiment be made with a body of fluid in a glass, twelve or twenty-four hours will suffice to separate gold which in the ruby state has remained suspended for six months.

The fluid changed by common salt or otherwise, when most altered, is of a violet-blue, or deep blue. Any tint, however, between this and the ruby may be obtained, and, as it appears to me, in either of two ways; for the intermediate fluid may be a mixture of ruby and violet fluids, or, as is often the case, all the gold in the fluid may be in the state producing the intermediate colour; but as the fluid may in all cases be carried on to the final violet-blue state, I will, for brevity sake, describe that only in a particular manner. The violet or blue fluid, when examined by the sun's rays and a lens, always gives evidence showing that the gold has not been redissolved, but is still in solid separate particles; and this is confirmed by the non-action of protochloride of tin, which, in properly prepared fluids, gives no indication of dissolved gold. When a ruby solution is rendered blue by common salt, the separation of the gold as a precipitate is greatly hastened; thus when a glass jar containing about half a pint of the ruby fluid had a few drops of brine added and stirred into the lower part, the lower half of the fluid became blue whilst the upper remained ruby; in that state the cone of sun's rays was beautifully developed in both parts. On standing for four hours the lower part became paler, a dark deposit of gold fell, and then the cone was feebly luminous there, though as bright as ever in the ruby above. In three days no cone was visible in the lower fluid; a fine cone appeared in the upper. After many days the salt diffused gradually through the whole, first turning the gold it came in contact with blue, and then causing its precipitation.

Such results would seem to show that this blue gold is aggregated gold, *i. e.* gold in larger particles than before, since they precipitate through the fluid in a time which is as nothing to that required by the particles of the ruby fluid from which they are obtained. But that the blue particles are always merely larger particles does not seem admissible for a moment, inasmuch as violet or blue fluids may be obtained in which the particles will remain in suspension as long as in the ruby fluids; there is probably some physical

change in the condition of the particles, caused by the presence of the salt and such affecting media, which is not a change of the gold as gold, but rather a change of the relation of the surface of the particles to the surrounding medium.

When salt is added in such quantity as to produce its effect in a short time, it is seen that the gold reflexion of the particles is quickly diminished, so that either as a general turbidness or by the cone of rays it becomes less visible; at last the metal contracts into masses, which are comparatively so few and separate, that when shaken up in the fluid, they confer little or no colour or character, either by reflected or transmitted light. In these cases no re-solution of the metal is effected, for neither the salt nor hydrochloric acid, when used in like manner, have any power to redissolve the gold. The same aggregating effect is shown with all the fluids whatever their colour, and also with the deposits that settle down from them. When salt is added to the solution of gold *before* the phosphorus, and therefore before the reduction of the gold, the fluid first produced is always ruby, but it becomes violet, purple, or blue, with a facility in proportion to the quantity of salt present. If that be but small, the ruby will remain for many days unchanged in colour, and the violet-ruby for many weeks, before the gold will be deposited, the degree of dilution or concentration always having its own particular effect, as before described; the more finely divided preparations, *i. e.* the ruby and amethystine, appear to be more permanent than when the salt is added after the separation of the gold.

Many other bodies besides salt have like action on the particles of gold. A ruby fluid is changed to or towards blue by solutions of chlorides of calcium, strontium, manganese; sulphates of magnesia, manganese, lime; nitrates of potassa, soda, baryta, magnesia, manganese; acetates of potassa, soda, and lime; these effect the change freely: the sulphate of soda, phosphates of soda and potassa, chlorate of potassa, and acetate of ammonia acted feebly. Sulphuric and hydrochloric acids produce the change, but show no tendency to dissolve the gold. Nitric acid acts in the same manner, but not so strongly: it often causes re-solution of the gold after some time, because of the hydrochloric acid which remains in the fluid.

Amongst the alkalies, potash produces a similar action in a weak degree. So also does soda. Lime-water produces a change in the same direction, but the gold quickly precipitates associated with the lime.

Ammonia causes the ruby fluid to assume a violet tint; the deposit is slow of formation and often ruby in colour; the alkali apparently retards the action of common salt.

Chlorine or nitromuriatic acid turns the ruby fluid blue or violet-blue before they dissolve the gold.

Solution of sulphuretted hydrogen changes the ruby slowly to purple, and finally to deep blue. Ether, alcohol, camphine, sulphide of carbon, gum, sugar and glycerine cause little or no change in the fluids; but glycerine added to the dense deposits causes serious condensation and alteration of them, so that it could not be employed as a medium for the suspension of particles in the microscope.

All endeavours to convert the violet gold back into ruby were either failures, or very imperfect in their results. A violet fluid will, upon long standing, yield a deposit and a supernatant ruby fluid, but this I believe to be a partial separation of a mixture of violet and ruby gold, by the settlement of the blue or violet gold from ruby gold, which remains longer in suspension. Mucus, which often forms in portions of these fluids that have been exposed to the air, appears sometimes to render a fluid more ruby, but this it does by gathering up the larger violet particles; it often becomes dark blue or even black by the particles of gold adhering to it, many of which may be shaken out by agitation in water; but I never saw it become ruby-coloured as a filter can, and I think that in these cases it is the gathering out of the blue or violet particles which makes the fluid left appear more ruby in tint. I have treated blue or violet fluid with phosphorus in various ways, but saw no appearance of a return in any degree towards ruby. Sometimes the fluids possess a tendency to re-solution of the gold, a condition which may often be given by addition of a very little nitric acid, but in these cases the gold does not become ruby before solution. It would rather appear that the finer ruby particles dissolve first, for the tint of the fluid, if ruby-violet at the commencement, changes towards blue. One effect only seemed to show the possibility of a reversion. Filtering-paper rendered ruby by a ruby fluid was washed and dried; being wetted by solution of caustic potash, it did not change; but being heated in a tube with the alkali, it became of a gray-blue tint; pouring off the alkali, washing the paper, and then adding dilute sulphuric or nitric acid to it, there was no change; but on boiling the paper in the mixed acids there was a return, and when the paper was washed and dried it approached considerably to the original ruby state. Again, potash added to it rendered it blue, which by washing with water, and especially with a little nitric acid, was much restored towards ruby. These changes may be due to an affection of the surface, or that which may be considered the surface of the particles.

The state of division of these particles must be extreme; they have not as yet been seen by any power of the microscope. Whether those that are ruby have their colour dependent upon a particular degree of division, or generally upon their being under a certain size, or whether it is consequent in part upon some other condition of the particles, is doubtful; for judging of their magnitude by the time occupied in their descent through the fluid, it would appear that violet and blue fluids occur giving violet deposits, which still consist of particles so small as to require a time equally long with the ruby particles for their deposition, and indeed in some specimens to remain undeposited in any time which has yet occurred since their formation. These deposits, when they occur, look like clear solutions in the fluid, even under the highest power of the microscope.

I endeavoured to obtain an idea of the quantity of gold in a given ruby fluid, and for this purpose selected a plate of gold ruby glass, of good full colour, to serve as a standard, and compared different fluids with it, varying their depth, until the light from white paper, transmitted through them, was apparently equal to that transmitted by the standard glass. Then known quantities of these ruby fluids were evaporated to dryness,

the gold converted into chloride, and compared by reduction on glass and otherwise with solutions of gold of known strengths. A portion of chloride of gold, containing 0·7 of a grain of metal, was made up to 70 cubic inches by the addition of distilled water and converted into ruby fluid: on the sixth day it was compared with the ruby glass standard, and with a depth of 1·4 inch was found equal to it; there was just one hundredth of a grain of gold diffused through a cubic inch of fluid. In another comparison, some gold leaves were dissolved and converted into ruby fluid, and compared; the result was a fluid, of which 1·5 inch in depth equalled the standard, a leaf of gold being contained in 27 cubic inches of the fluid. Hence looking through a depth of 2·7 inches, the quantity of gold interposed between the light and the eye would equal that contained in the thickness of a leaf of gold. Though the leaf is green and the fluid ruby, yet it is easy to perceive that more light is transmitted by the latter than the former; but inasmuch as it appears that ruby fluids may exist containing particles of very different sizes (or that settle at least with very different degrees of rapidity), so it is probable that the degree of colour, and the quantity of gold present, may not be always in the same proportion. I need hardly say that mere dilution does not alter the tint sensibly, *i. e.* if a deep ruby fluid be put into a cylindrical vessel, and the eye look through it along the axis of the vessel, dilution of the fluid to eight or ten times its volume does not sensibly alter the light transmitted. From these considerations, it would appear that one volume of gold is present in the ruby fluid in about 750,600 volumes of water; and that whatever the state of division to which the gold may be reduced, still the proportion of the solid particles to the amount of space through which they are dispersed must be of that extreme proportion. This accords perfectly with their invisibility in the microscope; with the manner of their separation from the dissolved state; with the length of time during which they can remain diffused; and with their appearance when illuminated by the cone of sun's rays.

The deposits, when not fixed upon glass or paper, are much changed by drying; they cannot be again wetted to the same degree as before, or be again diffused; and the light reflected or refracted is as to colour much altered, as might be expected. Whilst diffused through water, they seem to be physical associations of metallic centres with enveloping films of water, and as they sink together will lie for months at the bottom of the fluid without uniting or coming nearer to each other, or without being taken up by metallic mercury put into the same vessel. This is consistent with what we know of the manner in which gold and platinum can be thoroughly wetted if cleaned in water, and of the difference which occurs when they are dried and become invested with air. I endeavoured to transfer the gold particles unchanged into other media, for the purpose of noting any alteration in the action on light. By decanting the water very closely, and then carefully adding alcohol with agitation, I could diffuse them through that fluid; they still possessed a blue colour when looked through in the dark tube, but seemed much condensed or aggregated, for the fluid was obscure, not clear, and the particles soon subsided. I could not transfer them from alcohol to camphine; they

refused association with the latter fluid, retaining a film of alcohol or water, and adhering by it to the glass of the vessel; but when the camphine was removed, a partial diffusion of them in fresh alcohol could be effected, and gave the colour as before. All these transfers, however, injured the particles as to their condition of division. In one case I obtained a ruby film on a white plate; on pouring off the water and allowing parts to become dry, these became violet, seen by the light going through them to the plate and back again to the eye. I could not wet these places with water, a thin feebly reflecting surface remained between it and them. Using alcohol, the parts already dry remained violet, when wetted by it; but wetting other parts with alcohol before they were dry from water they remained rosy, became bluish when dry from the alcohol, and became rosy again when re-wetted by it.

It will be necessary to speak briefly of the reduction of gold into a divided state by some other chemical agents than those already described*. If a drop of solution of protosulphate of iron be introduced to, and instantly agitated with, a weak neutral solution of chloride of gold in such proportion that the latter shall be in excess, the fluid becomes of a blue-grey colour by transmission and brown by reflexion; and a deposit is formed of a green colour by transmitted light, greatly resembling the colour of beaten or pressed metal. It is not however pure gold, but an association of it and oxide of iron. Hydrochloric or other acids remove the iron and reduce the gold to a dark, dense, insoluble set of particles, in very small quantity apparently, yet containing all that was present in the bulky green deposit. If the solution of gold be made slightly acid beforehand, then the change and precipitation is to appearance much less; the reflexion by the particles is feeble but of a pale brown colour, the general transmitted light is amethystine; in the dark tube the tint is blue; the particles are much condensed and settle quickly, but occasionally leave a good ruby film on the side of the glass, which has all the characters of the ruby films and particles before described. The loose gold particles quickly adhere together. Hence it appears that the green precipitate often obtained by protosulphate of iron is not pure gold in a divided state; and that when care is taken to produce such pure divided gold, it presents the appearances of divided gold obtained by other means, the gold being competent to produce the ruby, amethystine, and blue colours by transmission. Usually the gold rapidly contracts and becomes almost insensible, and yet the test of protochloride of tin will show *that* all has been separated from solution; it then forms a *striking* contrast to the depth of colour presented by the same solution of gold precipitated by phosphorus, and most impressively directs attention to the molecular condition of the metal in the latter state.

A very small quantity of *protochloride of tin*, added to a dilute solution of gold, gave, first the ruby fluid, showing diffused particles by the cone of rays; this gradually became purple, and if the gold were in sufficient quantity a precipitate soon began to fall, being the purple of Cassius. If the chloride of tin were in larger quantity, a more bulky pre-

* See Gmelin's 'Chemistry,' vi. p. 219, "Terchloride of gold," for numerous references in relation to changes of these kinds.

cipitate fell and more quickly. Acid very much reduced this in quantity, dissolving out oxide of tin, and leaving little else than finely-divided gold, which, when diffused and examined in the dark tube, transmitted a blue colour. I believe the purple of Cassius to be essentially finely-divided gold, associated with more or less of oxide of tin.

Tartaric acid being added to a weak solution of gold gradually reduced it. The amethystine tint produced by diffused particles first appeared, and then a blue deposit of larger particles, whilst the side and bottom of the glass became covered by an adhering film of finer particles, presenting the perfect ruby tint of gold.

Ether added to a weak solution of gold gradually reduced it; the fluid was brown by reflected light, fine blue by transmitted light, and gave a good cone by the sun's rays and lens. The blue colour was not deep, though all the gold had been separated from solution; the preparation closely resembled that made with protosulphate of iron and a little acid.

A weak solution of gold, mingled with a little sugar, being heated, yielded a very characteristic decomposition. The gold was reduced into diffused particles, which rendered the fluid of a ruby-amethystine colour, and which, upon standing for twenty-four hours, gave signs of separation by settling as on former occasions. A little glycerine with solution of gold reduces it at common temperatures, producing a fluid, brown by reflexion, blue by transmission, giving a fine cone of rays by its suspended particles. Heat quickens the action, and causes a blue deposit.

Organic tissues often reduce solutions of gold, light if present assisting the action; and they afford valuable evidence in aid of the solution of the question relative to the condition of the metal in the divided state. If the skin be touched with a solution of gold, it soon becomes stained of a dull purple colour. If a piece of the large gut of an ox be soaked first in water, then in a solution of gold, and be afterwards taken out and allowed to dry, either exposed to light or not, the inner membrane will become so stained, that though of a dull purple colour by common observation, a transmitted ray will show it to be generally a very fine ruby, equal to that of ruby-coloured glass, or the gold fluids already described, though perhaps in places of a beautiful violet hue. The character of the particles which are here located and not allowed to diffuse and aggregate, as in the fluids, will be resumed when dealing with the whole question of the metallic nature of the particles of the variously divided gold.

Chloride of gold is reducible by heat alone. If a drop of solution of chloride of gold be evaporated in a watch-glass, or on a plate of rock-crystal, and then heated over a spirit-lamp until the gold is reduced, it will generally be found that the vapour has carried a portion of gold on to the neighbouring part of the glass, and that this part, when placed over a sheet of white paper, has the ruby tint. With the rock-crystal both ruby and blue parts are produced; and when the ruby parts are subjected to rock-crystal pressure, they become beautifully green. In the arts also glass is oftentimes coloured ruby by gold; I think that glass in this state derives its colour from diffused divided gold; and if either the ruby glass or the watch-glass be examined by a lens and

the cone of rays, it will be seen that the colours are not due to any gold dissolved, but to solid and diffused particles. There is nothing in any of the appearances or characters, or in the processes resorted to to obtain the several effects, that point at any physical difference in the nature of the results; and without saying that gold cannot produce a ruby colour whilst in combination or solution, I think that in all these cases the ruby tint is due simply to the presence of diffused finely-divided gold.

Metallic character of the divided gold.

Hitherto it may seem that I have assumed the various preparations of gold, whether ruby, green, violet, or blue in colour, to consist of that substance in a metallic divided state. I will now put together the reasons which cause me to draw that conclusion. With regard to *gold-leaf* no question respecting its metallic nature can arise, but it offers evidence reaching to the other preparations. The green colour conferred by pressure, and the removal of this colour by heat, evidently belong to it as a metal; these effects are very striking and important as regards the action on light, and where they recur with other forms of gold, may be accepted as proof that the gold is in the metallic state. Although I do not attach equal importance to the fact already described, that gold-leaf frequently presents fine parts that appear to be ruby in colour, I am not as yet satisfied that they are not in themselves ruby; and if they should be so, it will be another proof by analogy of the metallic nature of other kinds of preparations eminently ruby.

The *deflagrations* of gold wire by the Leyden discharge can be nothing but divided gold. They are the same whatever the atmosphere surrounding them at the time, or whatever the substance on which they are deposited. They have all the chemical reactions of gold, being, though so finely divided, insoluble in the fluids that refuse to act on the massive metal, and soluble in those that dissolve it, producing the same result. Heat makes these divided particles assume a ruby tint, yet such heat is not likely to take away their metallic character, and when heated they still act with chemical agents as gold. Pressure then confers the green colour, which heat takes away, and pressure reconfers. All these changes occur with particles attached to the substances which support them by the slightest possible mechanical force, just enough indeed to prevent their coalescence and to keep them apart and in place, and yet offering no resistance to any chemical action of test agents, as the acids, &c., not allowing any supposition of chemical action between them and the body supporting them. Still this gold, unexceptionable as to metallic state, presents different colours when viewed by transmitted light. Ruby, green, violet, blue, &c. occur, and the mere degree of division appears to be the determining cause of many of these colours. The deflagrations by the voltaic battery lead to the same conclusion.

The *gold films* produced by phosphorus have every character belonging to the metallic state. When thick, they are in colour, lustre, weight, &c. equal to gold-leaf, but in the unpressed state, their transmitted colour is generally grey, or violet-grey. The progres-

sion of their lustre and colour is gradual from the thickest to the thinnest, and the same is generally true, if thick films are gradually thinned and dissolved whilst floating on solvents; the thick and the thin films must both be accepted as having the same amount of evidence for their metallic nature. When subjected to chemical agents, both the thick and the thin films have the same relations as pure metallic gold. These relations are not changed by the action of heat, yet heat shows the same peculiar effect that it had with preparations of gold obtained by beating, or by electric deflagrations. The remarkable and characteristic effect of pressure is here reproduced, and sometimes with extraordinary results; since from the favourable manner in which the particles are occasionally divided and then held in place on the glass, the mere touch of a finger or card is enough to produce the result. Yet with gold thus proved to be metallic, colours including grey, grey-violet, green, purple, ruby, especially by heat, and green again by pressure, and by thinning of grey films, may be obtained by transmitted light, almost all of them at pleasure.

It may be thought that the *fluid preparations* present more difficulty to the admission, that they are simply cases of pure gold in a divided state; yet I have come to that conclusion, and believe that the differently-coloured fluids and particles are quite analogous to those that occur in the deflagrations and the films. In the first place they are produced as the films are, except that the particles are separated under the surface and out of the contact of the air; still, when produced in sufficient quantity against the side of the containing vessel to form an adhering film, that film has every character of lustre, colour, &c. in the parts differing in thickness, that a film formed at the surface has. Whilst the particles are diffused through the fluid it is difficult to deal with them by tests and reagents; for their absolute quantity is very small and their physical characters are very changeable, chiefly as I believe by aggregation; still there are some expedients which enable one to submit even the finest of them to proof. In several cases particles from ruby and amethystine fluids adhered to the sides of the bottles or flasks in which the fluids had been preserved, and the process of boiling seemed to favour such a result; the adhesion was so strong, that when the fluid contents were removed and the bottles well washed, the glass remained tinged of a ruby or a violet colour. These films, in which the fine particles were fixed mechanically apart and in place, were then submitted to the action of various chemical agents. Drying and access of air did not cause any marked alterations in them. Strong nitric acid produced no change, nor hydrochloric acid, nor sulphuric acid. Neither did a solution of chloride of sodium, even up to brine, cause any alteration in the colour or any other character of the deposit. A little solution of chlorine or of nitromuriatic acid dissolved them at once, producing the ordinary solutions of gold. I can see no other mode of accounting for these effects (which are in strong contrast with what happens when ruby fluid is acted on by these agents), than to suppose that the gold particles, being in a high state of division, were retained in that state for the time by their adhesion to the glass. Of course chemical change was free to occur, but not a change dependent upon their mutual aggregation; yet they were not

held by any special chemical attraction to, or combination with, the glass; for a touch with a card, a feather, or the finger, was sufficient to remove them at once, and if rubbed off with a point of wood, they coated it with brilliant metallic gold.

Again, though these particles are so finely divided that they pass easily through ordinary filters, still a close filter catches some; and if a ruby fluid be passed through again and again, the paper at last becomes of a rosy hue, because of the gold which adheres to it; being then well washed, and, if needful, dried, the gold is again ready for experiment. Such gold paper, placed across the middle of the dark tube and examined by transmitted light, was of the same ruby tint as when looked through in the open air. It was unaffected by salt or brine, though these, added to the rosy fluid which had passed the filter, instantly changed it to violet-blue. Portions of the paper were put into separate glasses with brine, solutions of hydrochloric, nitric and sulphuric acids, ammonia, potassa, soda and sulphuretted hydrogen, but no change occurred with any of them in two days. On the other hand, a very dilute solution of chlorine immediately turned the ruby to blue, and then gradually dissolved the gold. A piece of the ruby paper immersed in a strong solution of cyanide of potassium suffered a very slow action, if any, and remained unaltered in colour; being brought out into the air, the gold very gradually dissolved, becoming first blue. A portion of the ruby paper was dried and heated in oil until the oil and the paper began to change their hue; the gold had not altered in its colour or character. Another portion was heated in the vapour of alcohol and also of ether until the paper began to alter; the gold remained unaltered. A blue fluid being passed oftentimes through a filter gave a blue paper, which, being washed and tried in the same manner, was found to contain particles unchanged by the simple acids or alkalies, or by heat or vapours, but dissolving, as gold would do, in chlorine or nitro-muriatic acid. These tests are, I think, sufficient to prove the metallic nature and permanence of the gold as it exists in the ruby, amethystine, violet, and other coloured fluids.

The production by such different agents as phosphorus, sulphide of carbon, ether, sugar, glycerine, gelatine, tartaric acid, protosulphate of iron and protochloride of tin, of gold fluids all more or less red or ruby at the commencement, and all passing through the same order of changes, is again a proof that only gold was separated; no single one or common compound of gold, as an oxide or a phosphide, could be expected in all these cases. Many of the processes, very different as to the substances employed to reduce the gold, left good ruby films adhering to the glass vessels used, presenting all the characters of the gold described already; this was the case with phosphorus, sugar, tartaric acid, protosulphate of iron, and some other bodies.

Again, the high reflective power of these particles (unalterable by acids and salts), when illuminated by the sun's rays and a lens, and the colour of the light reflected, is in favour of their metallic character. So also is their aggregation, and their refusal to return from blue, violet or amethystine to ruby; for the cohesive and adhering force of the gold particles and their metallic nature and perfect cleanliness is against such a

reverse change. Particles transmitting blue light could be obtained in such quantity as to admit of their being washed and dried in a tube, and being so prepared they presented every character of gold: when heated, no oxygen, water, phosphorus, acid of phosphorus, nor any other substance was evolved from them: they changed a little, as the film when heated changed, becoming more reflective and of a pale brown colour, and contracted into aggregated porous masses of pure ordinary gold.

Gold is reduced from its solution by organic tissues; and *stained gut* has been quoted as a case. I have a very fine specimen which by transmitted light is as pure a ruby as gold-stained glass, and I believe that the gold has been simply reduced and diffused through the tissue. The preparation stood all the trials that had been applied to the ruby films on glass or the gold deposit on filtering-paper. Portions of it remained soaking in water, solution of chloride of sodium and dilute sulphuric acid for weeks, but these caused no change from ruby to blue, such as could be effected on loose ruby particles. Strong hydrochloric acid caused no change as long as the tissue held together; but as that became loose the gold flowed out into the acid in ruby-amethystine streams, finally changing to blue. Caustic potassa caused no change for days whilst the tissue kept together, but on mixing all up by pressure the loosened gold became at last blue. Strong nitric acid caused no change of colour until, by altering the tissue, the gold particles first flowed out in ruby and amethystine streams, and then were gradually changed to the condition of common aggregated gold. All these effects, and the actions on light, accord with the idea that the stain was simply due to diffused particles of finely-divided gold; and I am satisfied that all such stains upon the skin, or other organic matter, are of exactly the same nature.

As to the gold in ruby glass, I think a little consideration is sufficient to satisfy one that it is in the metallic condition. The action of heat tends to separate gold from its state of combination, and when so separated from the chloride, either upon the surface of glass, rock-crystal, topaz, or other inactive bodies, a ruby film of particles is frequently obtained. The sunlight and lens show that in ruby glass the gold is in separated and diffused particles. The parity of the gold glass, with the ruby-gold deflagrations and fluids described, is very great. These considerations, with the sufficiency of the assigned cause to produce the ruby tint, are strong reasons, in the absence of any to the contrary, to induce the belief that finely-divided metallic gold is the source of the ruby colour.

When a pure, clean, stiff jelly is prepared, and mixed, whilst warm and fluid, with a little dilute chloride of gold, as if to prepare a ruby fluid, it gelatinizes when cold, and if left for two or three days may become a ruby jelly; sometimes, however, the gold in the jelly changes but little or changes to blue, or it may happen that it is reduced on the surface as a film, brilliant and metallic by reflected light, and blue-grey by transmitted light. I have not yet ascertained the circumstances determining one or the other state. If a trace of phosphorus in sulphide of carbon be added to the solution of gold in a dilute state, and some salt be added to the warm jelly, and the latter be then mixed gradually and with agitation with the gold solution, a ruby jelly is generally produced.

In such ruby jelly the reduced particles of gold preserve their state and relative place, and the tint does not pass to blue, even though a considerable proportion of salt be present. Such jelly will remain in the air for weeks before it decays, and has every character, in colour and appearance, of gold ruby glass. It is hardly possible to examine the series of ruby glass, ruby membrane, ruby jelly cold and gelatinous, ruby jelly warm and fluid, and the ruby fluids, to consider their production, and then to conclude that the cause of their common ruby colour is not the same in all.

When the warm ruby jelly is poured into a capsule or on to a plate, allowed to gelatinize and then left in the air, it gradually becomes dry. When dry, some of these jellies remain ruby; others will probably be of an amethystine violet colour, or perhaps almost blue. When one of the latter is moistened with water, and has absorbed that fluid, it becomes gelatinous, and whilst in that state resumes its first ruby colour; but on being suffered to dry again it returns to its amethystine or blue colour. This change will occur for any number of times, as often as the jelly is wetted and dried. Here the gold remains in the same metallic state through this great change of colour, the association or the absence of water being the cause: and the effect strengthens in my mind the thought before expressed, that in the ruby fluids the deposited particles are frequently associates of water and gold. It is a striking case of the joint effect of the media and the gold in their action on the rays of light, and the most striking case amongst those where the medium may be changed to and fro.

When a ruby jelly is prepared with salt, and being warm is poured out in thin layers on to glass or porcelain, it first gelatinizes and then dries up; in which case the salt is excluded and crystallizes. When the dry jelly is put into cold water, the salt dissolves and can be removed. The jelly then swells to a certain amount, after which it can be left soaking in water for a week or longer, until everything soluble is separated. No change takes place in the ruby tint, no gold is removed. When the last water is poured off and the remaining jelly warmed, it melts, forming a fine ruby fluid, which can either be dissolved in more water, or regelatinized, or be dried and preserved for any length of time. It is perfectly neutral; gives no signs of dissolved gold by any of the tests of the metal; is not changed by sulphuretted hydrogen, gallic acid, pyrogallic acid, dilute caustic alkalies, or carbonated alkalies or lime-water; or by dilute sulphuric, hydrochloric or nitric acids, the actions being continued for fourteen days:—being boiled with zinc filings it does not change; and even when dilute sulphuric or hydrochloric acid is added to evolve nascent hydrogen, still the ruby character undergoes no alteration. Strong sulphuric, or nitric, or hydrochloric acid do not alter it whilst cold; but when warmed, the first causes the gold to separate as dark aggregated metallic particles, and the two latter gradually cause the change to amethyst and blue formerly described. Chlorine, or a mixture of hydrochloric and nitric acids, dissolves the gold, the ruby colour disappears, and the ordinary solution of gold is produced. In all these cases the ruby gold behaves exactly as metallic gold would do with the same agents, and quite unlike what would be expected from any possible combination of oxygen and gold.

In some of these jellies the ruby particles are so determinate as to give the brown reflexion by common observation; in others they are so fine as to look like ruby solutions, unless a strong sunlight and a lens be employed; and the impression again arises, that gold may exist in particles so minute as to have little or no power of reflecting light. Ruby particles of extreme fineness, when present in small amount in water, appear to remain equally diffused for any length of time; if in larger amount, that which settles to the bottom will remain for weeks and months as a dense ruby fluid, but without coming together: both circumstances seem to imply an association of the particles of gold with envelopes of water. Many circumstances about the ruby jellies imply a like association with that animal substance, and many of the stains of gold upon organic substances probably include an affinity of the metal of the like kind.

Relations of Gold (and other metals) to polarized Light.

It has been already stated, that when a ray of common light passes through a piece of gold-leaf inclined to the ray, the light is polarized. When the angle between the leaf and the ray is small, about 15°, nearly all the light that passes is polarized; but as the leaf is really very irregular in thickness, and ill-stretched as a film, parts inclined at different angles are always present at once. The light transmitted is polarized in the same direction as that transmitted by a bundle of thin plates of glass, inclined in the same direction. The proportion of light transmitted is small, as might be expected from the high reflective power of the metal. The polarization does not seem due to any constrained condition of the beaten gold, for it is produced, as will be shortly seen, by the annealed colourless leaf-gold, and also by deposits of gold particles; but is common to it with other uncrystallized transparent substances. It would seem that a very small proportion of the gold-leaf can be occupied by apertures, since the light which passes is nearly all polarized. On subjecting thin gold-leaf, or heated gold-leaf, or films of gold, or any preparations which required the support of glass, results of polarization were obtained, but the observations were imperfect because of the interfering effect of the glass.

Proceeding to employ a polarized ray of light, it was found that a leaf of gold produced generally the same depolarizing effect as other transparent bodies. Thus, if a plate of glass be held perpendicular to the ray, or inclined to it either in the plane of polarization or at right angles to it, there is no depolarization; but if inclined in the intermediate positions, the ray is more or less depolarized. So it is with gold-leaf; the same effects are produced by it. Further, the depolarization is accompanied by a rotation of the ray, and in this respect the quadrants alternate, the rotation being to the right-hand in two opposite quadrants, and to the left in the intervening quadrants. So it is with gold-leaf; the same effects are produced by it, and the rotation is in the same direction with that produced by glass, when inclined in the same quadrant.

As further observation in this direction was stopped by the necessity of employing glass supports for the leaves, films, &c., I sought for a medium so near glass in its cha-

racter, as should either reduce its effect to nothing, or render it so small as to cause its easy elimination. Either camphine or sulphide of carbon was found to answer the purpose with crown-glass; but the latter, as it possesses no sensible power of rotation under ordinary circumstances, is to be preferred. Should a medium of higher optic force be required, it would probably be supplied by the use of that dangerous fluid, phosphorus dissolved in sulphide of carbon. A rectangular glass cell being provided, which did not itself affect the polarized ray, was placed in its course and filled to a certain height with sulphide of carbon. A plate of crown-glass was then introduced perpendicularly to the ray; it did not affect it; being inclined as before described, the effect on the ray was still insensible, the glass appearing to be, for all ordinary observations such as mine, quite as the medium about it. I could now introduce gold-leaf attached to glass into the course of the polarized ray, its condition as a flat film or plane being far finer than when stretched on a wire ring as before. It proved to be so far above the sulphide of carbon, as to have powers of depolarization apparently as great as those it had in air, and being inclined, brought in the image at the analyser exceedingly well. It was indeed very striking to see, when the plate was moved parallel to itself, the darkness when mere glass intervened, and the light which sprung up when the gold-leaf came into its place; the opake metal and the transparent glass having apparently changed characters with each other. By care I was able to introduce a stretched piece of gold-leaf (without glass) into the sulphide of carbon; its effects were the same with those just described.

In all the experiments to be described, the plane of polarization and the plane of inclination had the same relation to each other: the figure shows the position of the polarizing NICOL prism, as the eye looks through it at the light, and *a*, *b* represents the vertical axis, about which the plates were inclined. Whether they were inclined in one direction or the other, or had the glass face or the metal face towards the eye, made no difference. In all cases with gold-leaf, it was found that the ray had been rotated; that it required a little direct rotation of the analyser to regain the minimum light; that short of that red tints appeared, and beyond it blue or cold, these being necessarily affected in some degree by the green colour of the gold-leaf. Thinned gold-leaf produced the same results; but as holes appeared in those that were thinnest, the results were interfered with, because the light passing through them was affected by the analyser in a different manner, and yet mingled its result with that of the light which had passed through the gold.

The gold-leaf plates, deprived of green colour by heating in oil, were found with the glass in such good annealed condition, as not to affect the ray; but when they were moved, until the oblique colourless gold came into the course of the ray, it was depolarized; a red image appeared; direct rotation of the analyser reduced this a little in intensity and then changed the colour to blue. The reduction was not much, and both in that and the first appearance of the red image, there is a difference between the heated and the unheated gold: probably the green tint of the latter, which would tend to extinguish the red and produce a minimum, may be sufficient to account for the effect. Gold

which had been re-greened by agate pressure acted in like manner on the polarized ray, but the experiments were imperfect.

A glass plate having gold-leaf on one part of it, had a second glass plate put over it and gummed at the edges. In the sulphide of carbon, therefore, it represented in one part a plate of air, and in another a compound plate of air and gold; both acted in the same direction, but the air and gold much more than the air. Gold on glass in this medium, or gold in air, or glass in air, all gave results in the same direction, *i. e.* required direct rotation of the analyser to compensate for them.

I proceeded to examine the other forms of gold; and first, the deposits on glass obtained by *electric deflagration*. These affected the ray of polarized light exactly in the manner of gold-leaf, and that even at the distant parts of the deposit. It was most striking to contrast the thinnest and faintest portion of such a film with the neighbouring parts of the glass from which it had been wiped off. It must be remembered that such a preparation is a layer of separate particles; that these particles are not like those of starch or of crystals, for they have no action whilst in a plane perpendicular to the polarized ray; nor have they a better action for being in a thick layer, as in the central parts of the deposit. The particles seem to form the equivalent of a continuous plate of transparent substance; and as in such a plate it is the two surfaces which act, so there appears to be the equivalent of these two surfaces here; which would seem to imply that the particles are so small and so near, that two or more can act at once upon the individual atoms of the vibrating ether. Their association is such as to present as it were an optical continuity.

The gold films by phosphorus were then submitted to experiment, and gave exactly the same result. All of them depolarized, and required direct rotation of the analyser to arrive at a minimum, or to pass from the red to the blue tints. Graduated films, of which I should judge from the depth of tint that one place was at least twenty times as thick as another, gave the effect as well in the thinnest as the thicker or any intermediate part; indicating that thickness of the plate, and therefore any quality equivalent to crystalline force of the particles, had nothing to do with the matter. A glass beaker, which had been employed to contain ruby fluid, had a film of gold deposited on its inner surface so thin, as to be scarcely perceptible either by reflexion or otherwise, except by a ruby tint which appeared upon it in certain positions; but being examined by a polarized ray, it gave an effect as strong and as perfect as gold-leaf, showing how thin a film of gold was sufficient for the purpose. This thin film appeared to be almost perfect in its continuity, for when the red image was brought in, direct rotation of the analyser reduced it to a minimum which was quite dark; after which, further rotation brought in a good blue image. The least touch of the finger removed the film of gold and all these effects with it. These films, though they are certainly porous to gas, and to water in some form, for it can evaporate from beneath them through its body, have evidently optical continuity.

In order to submit the gold fluids to experiments, cells were made of two glass plates,

separated by the thickness of a card, and fastened at the edges by varnish internally and gum externally. These being filled with dense ruby or blue fluid, gave no indication of action on the ray, showing that the diffused particles were inoperative. The same fluids, dried on plates of glass so as to leave films, did act just as the gold deflagrations had done; for though the particles were very irregularly spread, parts of the general deposit, and these not the thickest, could be selected, which produced the effect excellently well.

When the coloured jellies are laid upon glass plates and allowed to dry, the plates introduced obliquely into the sulphide of carbon affect the ray, but not as gold films; the light image becomes visible, but the plane of polarization is not changed; the light is coloured by the ruby or blue tint of the gold present, but a film of jelly without gold makes it visible to the same extent. In this case the gold is not in one plane, but diffused through the dry jelly, and the effect is the same as if it were diffused through water, being negative.

Such are the effects with the various preparations of divided gold. I will hastily notice what occurs with some other metals. *Platinum deflagrated* in hydrogen: it depolarized the ray, required direct rotation of the analyser to attain a minimum, therefore rotated the plane of polarization; but did not present sensible colour on either side of the minimum of light. *Palladium deflagrated* in hydrogen: it depolarized, producing a red image; direct rotation of the analyser lessened the light to a minimum, and then brought in a blue image. The films of palladium obtained by phosphorus acted well in the same manner. These films appear to be exceedingly continuous, and it could be observed in them, that though the thickest were not the best, yet films could be obtained so thin as to be distinctly inferior to other parts a little thicker; also that where the brilliancy of reflexion which indicates perfect smoothness passed in any degree into dulness, the action of the film was injured: the perfect condition of the surfaces of the films seems to be essential to their good action. *Rhodium films* by phosphorus gave good actions, like those produced by gold. *Silver deflagrations*, either in air or hydrogen, gave depolarizing results like those with gold. Silver films also gave excellent results of the like kind. A thin pale brown film was much better than a thicker one. *Copper* deflagrated in hydrogen: depolarized, bringing in a red image, which by direct rotation of the analyser was lowered a little and then converted to blue. The copper films obtained from oil acted in the same manner; the red and blue images appeared in their order; but very little direct rotation of the analyser was required to produce the minimum of light. *Tin* deflagrated in hydrogen: depolarized and rotated the ray, as with gold; the images were only feeble in colour. *Lead* deflagrated in hydrogen: acted as tin. *Iron* deflagrated in hydrogen: acted as tin. *Zinc* deflagrated in hydrogen: acted as tin. *Aluminium* deflagrated in hydrogen: had like action with the rest; the image brought in by it was red, which direct revolution of the analyser reduced at a little distance to a minimum and then converted to blue. A film of *mercury* produced by sublimation, a film of *arsenic* produced in like manner, and a film of *smoke* from a candle, though all of them sufficiently pervious to light, did not produce any result of depolariza-

tion. Films of the smoke of burning *zinc*, of *antimony*, or of *oxide of iron* produced no effect.

I placed some metallic solutions in a weak atmosphere of sulphuretted hydrogen. Gold and platinum gave no films; silver so poor a film as to be of no use; and lead one so brittle as to be unserviceable. That obtained with palladium I believe to be the metal itself. The films of sulphuret of mercury, sulphuret of antimony which was orange, and sulphuret of copper which was pale brown, all acted on the light and depolarized it. The sulphuret of copper presented a difference from the metals generally, worth recording; it depolarized the light, producing an image which, if not blue at once, was rendered blue by a little direct rotation of the analyser; after which the same motion brought in a minimum and then produced an orange or red tint, *i. e.* with the sulphuret of copper the warm and cold tints appear on opposite sides of the minimum to those where they occur when films of the metals are employed, though the minimum in both cases is in the same direction.

Many of the results obtained in the sulphide of carbon were produced also in camphine, the analyser being in each case adjusted to the minimum of light before the metallic plate or film was introduced. I pass, however, to a very brief account of some polarizations effected by the metals themselves in the sulphide of carbon, in which case the polarizing NICOL prism was dispensed with. The results show that all the dry forms of gold accord in giving the same manifestation of action on light, whatever the state of their division, provided they be disposed in a thin regular layer, equivalent to a continuous film. It was first ascertained that a plate of crown-glass in an inclined position in sulphide of carbon gave no signs of polarity to a ray of light passing through it. When fine gold-leaf was on the glass and inclined to the ray, it polarized the light, and exactly in the same manner and direction as a bundle of glass plates in the same position in the air. More light passed than when the gold-leaf was in air, but it could not be so completely polarized; the minimum light was of a pale bluish colour. A thinned gold-leaf produced the same effect, but let more common light through. I think the difference between gold-leaf and sulphide of carbon is sensibly less than that between the metal and air. The depositions of deflagrated gold, the films of gold obtained by phosphorus, and even the heated deflagrated gold, produced polarizing effects, which, though not large, were easily recognized and distinguished from the non-action of the glass. Gold-leaf and gold films on glass produced a like effect in a camphine-bath, the results being easily distinguished from those of the glass and camphine only, in places where the glass had been cleared from gold.

Films of palladium, rhodium, silver, a plate with deposited gold particles, and a layer of deflagrated silver particles gave a like result, the effect varying in degree. The sulphuret of copper before spoken of as in contrast with the metals, gave only a doubtful result, if any.

Before concluding, I may briefly describe the following negative results with the preparations of gold. I prepared a powerful electro-magnet, sent a polarized ray across

the magnetic field, parallel to the magnetic axis, and then placed portions of the ruby and violet fluids, also of their deposits wet and dry, also portions of the gold films, of gold-leaf, the results of deflagrations, &c., in the course of the ray; but on exciting the magnet could not obtain any effect beyond that due to the water or glass, which in any case accompanied the substance into the magnetic field. In some cases very dense preparations of the ruby and blue deposits were employed, the intense electric lamplight being required to penetrate them.

I passed the coloured rays of the solar beam through the various gold fluids and films that have been described. For this purpose a beam of sunlight entering a dark room through an aperture $\frac{1}{4}$th of an inch in width, was sent through two of BONTEMPS's flint-glass prisms, and its rays either separated, or at once thrown on to a pure white screen; the different objects were then interposed in the course of the ray, but I could not perceive when any portion of a ray passed (and that was generally the case) that it differed sensibly in colour or quality from the ray passing into the preparation. In like manner, the objects were put into the differently coloured rays and observed by the reflected light, a lens being sometimes employed to concentrate the light; but I could not find any marked difference between the colour or character of the ray reflected and the impinging ray, except in quantity.

XI. *On Autopolar Polyedra.*

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Communicated by ARTHUR CAYLEY, *Esq., F.R.S.*

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I.* As any face of a p-edron N is named from the number m of its edges an m-*gon*, so from the number n of its edges I call a summit of N an n-*ace*; and a polyedron having k faces and l summits may be denominated a k-*edron* or an l-*acron*, as may be most convenient.

By an *autopolar polyedron*, I mean one in which the number, rank, and collocation with respect to an a-gon A of its summits and remaining faces, are exactly the number,

* The following note, "On the Analytic Problem of the Polyedra," is kindly placed at my disposal by ARTHUR CAYLEY, Esq., who has wisely judged that to this investigation of a subject so new and intricate, some such statement should appear by way of introduction. I doubt not that the reader will approve of my appending it as he has written it. The note comprises a clearer statement of some things which may be found in my memoir "On the Representation and Enumeration of Polyedra," in the twelfth volume of the Memoirs of the Literary and Philosophical Society of Manchester, with some matter which has arisen in our correspondence; and particularly it supplies a defect in my statement of analytic conditions in art. 22 of that memoir, which Mr. CAYLEY with his rare penetration was the first to point out and amend. I have there laid down, that "multiplets are to be made with a symbol, under these two conditions: first, that every contiguous pair of symbols in any multiplet shall be a contiguous pair in some one other; and secondly, that no three symbols in any multiplet shall occur in any other." It should have been laid down, as Mr. CAYLEY here states it, that no contiguous duad shall occur non-contiguously, and that no non-contiguous duad shall be twice employed. By the words, *in some one other*, any reader of my memoir will see that I meant, *in some one other only*.

"*The Problem of Polyedra.*

"Let $a, b, c, d, e, f, g, h,$ &c. represent the vertices of a polyedron, then a face will be represented *e. g.* by *abcde*, where the contiguous duads, viz. ab, bc, cd, de, ea are the edges of the face; and calling the face K, we may write

$$K = abcde. \quad \ldots \quad (1.)$$

"It is to be noticed that the letters of a face-symbol may be taken forwards or backwards from any letter without altering the meaning of the symbol. Thus, *abcde*, *bcdea*, &c., *edcba*, &c. might any of them be taken to denote the face K. The diagonal of a face cannot be either an edge or a diagonal of any other face, *i. e.* a non-contiguous duad such as ac in face-symbol K cannot be a duad contiguous or non-contiguous of any other face-symbol. But each edge of a face must be an edge of one and only one other face, *i. e.* each contiguous duad such as ab in the face-symbol K must be a contiguous duad of one and only one other face-symbol L. And moreover two faces cannot have more than a single edge in common, *i. e.* two face-symbols cannot contain more than a single contiguous duad the same in each symbol.

"The face K contains the edges $ab, ac,$ i. e. the edge ab is contained in the face K; it will also be contained in one and only one other face, suppose L; this face will contain another edge through the vertex a, suppose

rank and collocation with respect to at least one α-ace a in it, of its faces and remaining summits. This summit a may be considered the pole of that face A. A p-edral p-acron, whose p summits are the poles of its p faces, is an autopolar polyedron.

To every edge AB, between the α-gon A and β-gon B, in such a figure, corresponds an edge ab, between the α-ace a and the β-ace b, and *vice versâ*. Two such edges I call a *gamic pair*, or a pair of gamics, either being the gamic of the other. If these edges

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the edge af, and so on, until we arrive at a face containing the edge ae; we have, for example,

$$\begin{aligned} K &= eabcd \\ L &= baf\ldots \\ M &= fag\ldots \\ N &= gah\ldots \\ P &= hai\ldots \\ Q &= iae\ldots \end{aligned}$$

and we thence derive the vertex-symbol

$$a = KLMNPQ, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.)$$

where the contiguous duads KL, LM, MN, NP, PQ, QK represent in order the edges through the vertex a. The remarks before made with respect to the face-symbols apply to the vertex-symbols. A non-contiguous duad such as KN of the vertex-symbol a cannot be a duad contiguous, or non-contiguous of any other vertex-symbol, but each contiguous duad such as KL of the vertex-symbol a must be a contiguous duad of one and only one other vertex-symbol b. And the symbols of two vertices cannot contain more than one contiguous duad the same in each symbol.

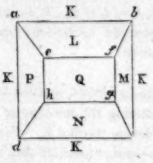
"Any edge of the polyedron admits of a double representation; it is the junction of two vertices, or the intersection of two faces. Thus ab and KL will represent the same edge, or we may write

$$ab = KL. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.)$$

"It is to be remarked, that in this system each to each equation ab=KL, there corresponds one and only one equation of the form ae=KQ, *i. e.* to an edge considered as drawn from a given vertex in a given face there corresponds one and only one other edge from the same vertex in the same face.

"It has been shown how the system of face-symbols (1) leads to the system of vertex-symbols (2) and the system of edge symbols (3); and generally, any one of the three systems leads to the other two; and the three systems conjointly, or each system by itself, is a complete representation of the polyedron. As an example, take the hexaedron: the three systems are,—

K=*abcd* (1)	*a*=LPK (2)	
L=*abfe*	*b*=LMK	
M=*bfge*	*c*=MNK	
N=*gedh*	*d*=NPK	
P=*dhea*	*e*=LPQ	
Q=*hefg*	*f*=LMQ	
	g=MNQ	
	h=NPQ	



ab=KL	*ae*=PL	*ef*=LQ (3)
bc=KM	*bf*=LM	*fg*=MQ
cd=KN	*cg*=MN	*gh*=NQ
de=KP	*dh*=NP	*he*=PQ

"Consider, now, two polyedra having the same number of vertices and also the same number of faces.

ab and AB meet at a summit *b*, they are *nodal gamics* in the *nodal summit* b and the *nodal face* B. From the first notions of poles and polars, it is evident that no edge can be its own gamic.

Every polyedron is either *autopolar* or *heteropolar*. A heteropolar *p*-edral *q*-acron H is the *sympolar* of a *q*-edral *p*-acron H′, whose summits are the poles of the faces of H,

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And let the vertices and faces of the first polyedron taken in any order be represented by

abcde..KLM...,

and the vertices and faces of the second polyedron taken in a certain order be represented by

a′b′c′d′e′..K′L′M′..

Then, forming the substitution symbol

a′b′c′d′e′..K′L′M′..*abcde*..KLM...,

which denotes that *a′* is to be written for *a*, *b′* for *b*... K′ for K, &c., if, operating with this upon the symbol system of the first polyedron, we obtain the symbol system of the second polyedron, the second polyedron will be syntypic with the first. It should be noticed, that there may be several modes of arrangement of the vertices and faces of the second polyedron, which will render it syntypic according to the foregoing definition with the first polyedron, *i. e.* the second polyedron may be syntypic in several different ways with the first polyedron. This is, in fact, the same as saying that a polyedron may be syntypic with itself in several different ways. Suppose, next, that the number of vertices of the second polyedron is equal to the number of faces of the first polyedron, and the number of faces of the second polyedron is equal to the number of vertices of the first polyedron. And let the vertices and faces of the first polyedron in any order be represented by

abcde..KLM...,

and the faces and vertices of the second polyedron in a certain order be represented by

A′B′C′D′E′..*k′l′m′*..

Then, forming the substitution symbol

A′B′C′D′E′..*k′l′m′*..*abcde*..KLM...,

if, operating with this upon the symbol system of the first polyedron, we obtain the symbol system of the second polyedron, the second polyedron is said to be polar-syntypic with the first; and, as in the case of syntypicism, this may happen in several different ways.

"Lastly, if there be a polyedron having the same number of vertices and faces, and if the vertices and faces in any order be represented by

abcd..KLMN...,

and the faces and vertices in a certain order be represented by

ABCD..*klmn*..

Then, forming the substitution symbol

ABCD..*klmn*..*abcd*..KLMN...,

if, operating with this upon the symbol system of the polyedron, we reproduce such symbol system, *i. e.* in fact, if the polyedron be polar-syntypic with itself, the polyedron is said to be *autopolar*; and in accordance with a preceding remark, this may happen in several different ways. It is clear that the substitution symbol, operating on the symbol system of the vertices, must give the symbol system of the faces, and conversely; but operating on the symbol system of the edges, it must reproduce such symbol system of the edges: and this last condition will by itself suffice to make the polyedron autopolar, *i. e.* the polyedron will be autopolar if the substitution symbol, operating on the symbol system of the edges, reproduces such symbol system."

and whose faces are polar to the summits of H. Every edge of H is the gamic of an edge of H′, and H and H′ are either the *sympolar* of the other, and form a *sympolar pair*.

II. An edge of a polyedron is said to *convanesce*, when its two summits run into one, and it is said to *evanesce*, when its two faces revolve into one.

An edge AB is said to be *convanescible*, when neither A nor B is a triangle, and AB joins two summits, which have not besides A and B two faces, one in each summit, collateral nor having a common summit.

An edge *ab* is said to be *evanescible*, when neither *a* nor *b* is a triace, and the two faces about *ab* are not, one in each, in two summits, besides *a* and *b*, collateral nor in one face.

If AB be convanescible, its gamic *ab* is evanescible, and *vice versâ*; and the pair AB, *ab* are *vanescible gamics*, when in an autopolar polyedron.

III. Theorem. *No polyedron, not a pyramid, has every edge both in a triangle and in a triace.*

For let it be supposed that P, which is not a pyramid, has every edge both in a triangle and in a triace, and let E be a face of P not less than the greatest; the faces collateral with E are by hypothesis all triangles. Let A and B be two of these having two contiguous edges of E. There will be a summit of E not a triace, otherwise P would be a pyramid; let this be *d*, a summit of E, A and V (EAV), V being a face collateral with A, but not with E. As one extremity of the edge AV is not in a triace, the other is; therefore AV will end at the triace AVW, the vertex of A. As P is no pyramid, it has more summits than one not in its base E, which are all connected by edges; therefore the edge AW must pass from AVW to some summit not in E.

The edge EA having an extremity *d* not in a triace, terminates at the triace EAB, where AB is the only edge meeting E, and the triangles A and B have a common vertex AVB, the intersection of AV and AB. But this is the triace AVW; therefore AW is AB, an edge passing through a summit of E, contrary to what has been proved. Q. E. A.

Therefore P has an edge either not in a triace, or not in a triangle. Q. E. D.

IV. Theorem. *Every polyedron* P, *not a pyramid, has either a convanescible or an evanescible edge.*

For P has either the edge AB, whose faces A and B are neither of them triangles, or the edge *cd*, whose summits are neither of them triaces. Let it have AB, an edge not in a triangle, lying between the summits *e* and *f*. By definition of a convanescible edge, if *e* and *f* have not besides A and B two collateral faces, C and D, AB is a convanescible edge of P. C and D may have a common edge *gh*, of any length, or a point.

If *e* and *f* have two collateral faces, C through *e*, and D through *f*, intersecting in *gh*, then neither *e* nor *f* will be a triace, or else one of them, *e*, will be a triace. If, firstly, neither be a triace, AB=*ef* is not in a triace, and is therefore evanescible, unless A and B about it are in two other summits *k* and *l* collateral or in one face. Now if a section be made through *ef*, and *m*, any point of *gh*, it will cut any edge *kl* drawn or drawable upon P, from *k* in A to *l* in B, and will divide P into P′, containing A, and P″, contain-

ing B. But this section is the triangle *efm*, whose sides are *em* in C and *fm* in D; therefore *m*, any point of *gh*, is a point of *kl*; which is absurd, unless *kl* is *gh*. And if *kl* is *gh*, *m* may be a point indefinitely near to *k*, and the section *efm* will be the face A, which is impossible, because A is no triangle. Wherefore the absurdity remains, and *kl* has no existence. Therefore P has the evanescible edge AB.

And if, secondly, *e* be a triace, AC and BC are edges of C, *ep* and *ep'*, and C having three summits in A and B, *epp'*, and one or two in *gh*, is no triangle; therefore *ep'* is an edge of P not in a triangle, which is convanescible, unless *e* and *p'* are in two covertical faces besides C and B. Now if they are, A is one of those faces, *e* being the triace ACB. Let G be a face through *p'* meeting A on P. The triangular section *efm* being made as above, cuts no face besides C and D; and divides P into P' containing A, and P'' containing B with its edge *ep'*. Therefore G, being uncut and covertical with A, is entirely in P'; and being uncut and having the summit *p'*, is entirely in P''. Which is absurd; wherefore G has no existence, and P has the convanescible edge *ep'*.

Thus it is proved that in any case P, being not a pyramid, and having an edge AB, not in a triangle, has either a convanescible or an evanescible edge.

Next, let P have an edge *cd*, not in a triace. Then will P_1, the sympolar of P, have an edge CD, the gamic of *cd*, not in a triangle, and also either a convanescible or an evanescible edge *s*; wherefore P has the gamic of *s*, which is either evanescible or convanescible.

Therefore in all cases P, not being a pyramid, has a vanescible edge. Q. E. D.

V. Theorem. *Any* p-*edral* q-*acron* P, *not a pyramid, can be reduced by the vanishing of an edge to either a* (p−1)-*edral* q-*acron or a* p-*edral* (q−1)-*acron.*

For P must have either a convanescible edge AB, or an evanescible edge *ab*. Let it have the former (AB).

If AB convanesces by the union of its summits, P becomes P', losing the faces A and B, and receiving two others, A' and B', in their place, A' being A with one edge less, and B' being B with one edge less. At the same time the two summits *dd'* of (AB) disappear from P, which in their stead and at their union, receives a new summit *h* having two edges less than the sum of theirs. All the faces of P containing *d* or *d'* remain unchanged, except that each now contains the summit *h*, whose edges are those of *d* and *d'* together save two; and no two of these faces thus brought to have a common summit *h* can coincide, because, by the definition of the convanescible *e*, they had not in P a common edge. Hence all the other edges of P remain undisturbed in the result P'. Or if *ab* be an evanescible edge of P, and evanesces by the revolution of DD' its two faces into one plane, giving rise to a new face H, this will have the edges of D and D' together save two; the summit *a* becomes *a'* with one edge less, and *b* becomes *b'* with one edge less. All the summits of P containing D and D' remain unchanged, except that each now contains the face H; and no two of these summits, thus brought into a common face H, coincide, because by the definition of (*ab*) they were not before in any line drawn or drawable on P. Hence all the other edges of P remain unaltered.

It is evident that P by the vanishing of AB, becomes P′ a p-edral $(q-1)$-acron; or by the vanishing of ab becomes P′ a $(p-1)$-edral q-cron. Q. E. D.

If P′ be not a pyramid, it can further be reduced by the evanescence or convanescence of an edge, until the result of the reduction is a pyramid Π. If there be *no pyramid of higher rank than* Π, to which P can be thus reduced, P will be said to be *generable from* Π, and can evidently be *generable* from none but Π.

When P is autopolar, it has both a convanescible and its evanescible gamic; and thus, from being a p-edral p-acron, it reduces to P′, a $(p-1)$-edral $(p-1)$-acron, which will also be autopolar; and P is *generable* only from one pyramid Π.

VI. The problem of the enumeration of N-edra is thus reduced to this: *To find how many* N*-edra are generable from the* K*-edral pyramid* Π. The solution of this question is to be founded on the consideration that if P is generable from Π, it can be constructed by the introduction into Π of new convanescible and evanescible edges; whether P be autopolar or heteropolar. I shall first consider the *autopolars*, and afterwards the *heteropolars* generable from the pyramid.

Problem. *To find the number of nodally autopolar* $(r+2)$*-edra generable from the* $(r+1)$*-edral pyramid.* (See below the definition of nodally autopolar.)

Let the r triangles and r triaces of the pyramid Π be numbered thus,

$$\begin{matrix} 1\ 2 & & 3 & & \dots (R-3) & (R-2) & (R-1) & R \\ r & (r-1) & (r-2) \dots & & 4 & 3 & 2 & 1 \end{matrix},$$

the upper line denoting the faces as they are read by an eye within the pyramid, and the lower the summits.

If E_e and E_ϵ signify the two summits on the right and left of the eth triangle, and e_r, e_l be the faces on either side of E,

$$E_\epsilon = 2 - e \pm$$
$$E_\epsilon = 1 - e \pm,$$

where e and E are the same number, as are r and R.

The $\pm$ in this and all functions of these signatures, denotes that such a multiple of r is to be added or subtracted as shall cause any signature to have an integer value between 1 and r inclusive.

This arrangement is autopolar; for the vertex ω is the pole of the base Ω, and the nth triangle is placed with respect to the adjoining faces and summits and their signatures, as the nth summit with respect to the adjoining summits and faces.

If two gamic edges meet in the kth face at the kth summit, we must have either

$$k = 2 - k \pm \text{ or } k = 1 - k \pm;$$

i. e. $k = 1$, or $k = \frac{1}{2}(r+2)$, or $k = \frac{1}{2}(r+1)$.

This shows that there are only two *nodal summits*, which have the signatures 1 and $\frac{1}{2}(r+2)$ when r is even, and 1 and $\frac{1}{2}(r+1)$ when r is odd.

It will be convenient to call the line joining the nodal summits, *nodal diagonal* or *nodal line*. When r is even, the nodal diagonal is a diameter; but not when r is

odd. When r is odd, the line which bisects the nodal diagonal *at right angles* will be known to us as the *nodal axis*, or the *axis of symmetry*. It passes through one summit of the base, and bisects the side opposite to that summit.

When r is even, there is no method of autopolar signature of the pyramid, which shall not exhibit a nodal diagonal. This pyramid is only *nodally autopolar*. But when r is odd, the signatures may be *enodal*, if every summit e is made the pole of the face E opposite to it. This odd-angled pyramid is either *nodally* or *enodally autopolar*. The autopolarity of the even-angled pyramid is *purely nodal*.

If, now, any two non-contiguous summits numbered e and f of this base be joined by the diagonal ef, and the two portions O and O′ of Ω be considered as faces about ef, the summits e and f are tessaraces in the edge OO′. The vertex ω must, therefore, if autopolarity is to be kept, become two summits o and o', and the triangles numbered EF will be quadrilaterals intersecting in the edge oo'. E and F, be it remembered, are the two numbers e and f.

The two edges, thus added to Π, are evidently a gamic pair, of which OO′ is evanescible, and oo' convanescible, and the result P is an autopolar $(r+2)$-edron.

VII. The question to be answered is, how many such results P can be made by drawing a *generator* OO′ between two summits of Ω, such that no two P shall be either identical, or one the reflected image of the other?

If ef and $e'f'$ be two generators OO′ of the same P, it is evident that $f-e=\pm(f'-e')$, since Ω is divided alike by both ef and $e'f'$. And the figure made by the points of the base $E_1E_2ef F_1F_2$ will be exactly like that presented by the points $E_1'E_2'e'f'F_1'F_2'$, or its reflected image.

For the only difference possible between the two results P, P′, generated by ef and $e'f'$, will consist in the mutual arrangement of the introduced 4-gons and 4-aces; and any difference herein will prevent P from being the same with P′.

The four following equations, along with

$$f-e=\pm(f-e'),$$

comprise all the possible conditions of similarity between P and P′:—

$$E_1-e=E_1'-e'\pm$$
$$E_1-e=e'-E_2'\pm$$
$$E_1-e=E_1'-f'\pm$$
$$E_1-e=e'-F_2'\pm.$$

The first affirms that the distance from the left-hand summit of E to e is the same as from the left of E′ to e', and measured in the same direction.

The second affirms the said distance is that from the right summit of E′ to e', measured in the opposite direction. In the first case looking at the configuration in Ω, the system $(e'f')$ is a direct repetition of the system (ef). In the other it is its reflected image.

The configuration of the third condition is that of the first, and of the fourth that of

the second, with the difference that the signatures e' and f' have changed places in $e'f'$ in the third, and E' and F' have changed places about E'F' in the fourth.

VIII. First, let, along with $e-f=\pm(e'-f')$,

$$E_r-e=E_r-e'\pm, \text{ i. e.}$$
$$2-2e=2-2e'\pm.$$

This, since $e=e'$ and $f=f'$ is contrary to hypothesis (for we are seeking a diagonal $e'f$ different from ef), means

$$2-2e=2-2e'\pm r,$$

or

$$e'=e\pm\tfrac{1}{2}r;$$

that is, r is even, and e' is the opposite extremity to e of the diameter of Ω through e. Hence f' is opposite f in the diameter through f.

It thus appears that when r is even, for every diagonal ef (not a diameter), there can be drawn a diagonal $e'f'$ in Ω, that generates the same $(r+2)$-edron P which is generated by ef; and that e' and f' are diametrically opposite to e and f. When ef is a diameter, it does not hence appear whether there is any $e'f'$ *a fellow-generator* with ef.

IX. Secondly, let

$$E_r-e=e'-E_r\pm, \text{ i. e.}$$
$$2-2e=e'-(1-e')\pm$$
$$2e+2e'=3+r\pm.$$

We gather that r is odd; for no multiple of r even can here be added to make this equation true. Hence

$$e'=\tfrac{1}{2}(r+3-2e+)$$

and

$$f'=\tfrac{1}{2}(r+3-2f+).$$

The diagonal $e'f'$ thus found is different from ef, unless $e'=f$, and $f'=e$, i. e. unless

$$2f+2e=3+r+,$$

the additional $+$ denoting either zero or $2r$. When this relation between f and e holds, it does not thus far appear whether a pair of generators, $e'f'$ and its gamic, can be drawn to produce the same P with ef and its gamic. This relation is otherwise thus:

$$e-1=-\left(f-\tfrac{1}{2}(r+1)\right),$$

or else

$$e-1=-\left(f-\tfrac{1}{2}(3r+1)\right),$$

showing that f is at the same distance measured backwards from the nodal summit $\frac{1}{2}(r+1)$ which is also $\frac{1}{2}(3r+1)=r+(\frac{1}{2}r+1)$, that e is measured forwards from the nodal summit 1. The line ef is either the *nodal diagonal* through 1 and $\frac{1}{2}(r+1)$, or it is parallel thereto. We may call all these diagonals *the nodal parallels*. It is evident that when ef and $e'f'$ are fellow-generators, ee' and ff' are nodal parallels; since

$$2e+2e'=2f+2f'=r+3+.$$

X. Thirdly, let

$$E_i - e = E_i - f' \pm,$$

or

$$2 - 2e = 2 - e' - f' \pm,$$

which gives

$$2e = e' + f' \pm$$

$$2f = f' + e' \pm.$$

The only interpretation of this, consistent with $e'f'$ not ef, is

$$2e = e' + f'$$

$$2f = e' + f' \pm,$$

which can be true only when

$$2e - 2f = \pm kr;$$

but

$$2e \not> 2r, \text{ hence } k < 2,$$

and

$$2e - 2f = \pm r,$$

or

$$e = f \pm \tfrac{1}{2}r,$$

is the only possible relation; *i. e.* ef is the diameter of an even-angled base.

The equation $2e = e' + f'$ is $2e = 2e' \pm \frac{1}{2}r$, giving

$$e' = e \pm \tfrac{1}{4}r$$

$$f' = e' \pm \tfrac{1}{4}r = e \pm \tfrac{3}{4}r.$$

Therefore $r = 4m$.

By this we see that, when $r = 4m$, the diameters all pair themselves m summits apart into generators ef and $e'f'$, of the sum $(r+2)$-edron; for no value of e' and f' can coincide with e and f from which it is obtained, since

$$e' = f \text{ or } e' = e,$$

along with

$$e' = e \pm \tfrac{1}{4}r$$

and

$$f' = e \pm \tfrac{1}{4}r$$

cannot coexist.

XI. Lastly, let

$$E_i - e = e' - F_r \pm,$$

or

$$2 - 2e = e' + f' - 1 \pm;$$

whence comes

$$2e + e' + f' = 3 +$$

and

$$2f + f' + e' = 3 +.$$

Of these, the only meaning that consists with $e'f'$, different from ef, is

$$2e + e' + f' = 3 + kr$$

$$2f + f' + e' = 3 + k'r,$$

$$2(e - f) = (k - k')r.$$

Now as neither e nor $f > r$, $(k-k')=\pm 1$, and

$$e=f\pm\tfrac{1}{2}r,$$

showing that ef is a diameter of an even-angled base, as is also $e'f'$, one case of ef. Putting $e'=f'\pm\frac{1}{2}r$, we deduce, from $k=1$, $k'=2$,

$$2e+2f'-\tfrac{1}{2}r=3+r$$
$$2e+2e'-\tfrac{1}{2}r=3+2r$$
$$2e+2f'=3+\tfrac{3}{2}r;$$

proving that $r=4h+2$, and

$$2e+2e'=3+\tfrac{5}{2}r.$$

These equations give e' and f' different from e and f, except when $e'=f=e\pm\frac{1}{2}r$, or $e'=e$; *i. e.* when

$$4e=\frac{6+3r}{2},$$

and

$$e=\tfrac{3}{4}(2+r);$$

or

$$4e=\tfrac{1}{2}(6+5r),$$

and

$$e=\tfrac{1}{4}(6+5r).$$

Hereby we learn that when $r=8h-2$, or $r=8h+2$, all the diameters of Ω pair themselves into generators, ef and $e'f'$, of the same $(r+2)$-edron P, except only that drawn in the first case through $e=\frac{3}{4}(2+r)$, and that drawn in the second through $e=\frac{1}{4}(6+5r)$.

XII. We can now easily enumerate the autopolar $(r+2)$-edra generable from the pyramidal $(r+1)$-edron (II).

When $r=4h$, all the diagonals of Ω pair themselves into fellow-generators ef and $e'f'$. The number of diagonals is $\frac{1}{2}r.(r-1)-r=\frac{1}{2}(r^2-3r)$. Hence the number of autopolars (P) required is $\frac{1}{4}(r^2-3r)$.

When $r=4h+2$, all the diagonals but one, namely one of the diameters (XI.), pair themselves; consequently, the number of autopolars (P) sought is

$$\tfrac{1}{2}\{\tfrac{1}{2}(r^2-3r)+1\}=\tfrac{1}{4}(r^2-3r+2).$$

The question left unsettled in (IX.), as to whether the nodal parallels pair themselves into fellow-generators, is decided in the negative by silence of our formulæ in X. and XI. on the subject of $r=2h+1$. The diagonals of Ω for r odd pair themselves, unless when

$$2f+2e=3+r,$$

or

$$2f+2e=3+3r.$$

For every value of e, whether $r=4k+1$, or $4k-1$, these equations give f on a nodal and unpaired parallel, except when they give $e=f$, or $e=f+1$; *i. e.* unless

$$4e=3+r,$$

or

$$4e=3+3r,$$

which give $e=f$; and unless

$$4e=1+r,$$

or

$$4e=1+3r,$$

which give

$$f=e+1.$$

When $e=f$, the nodal parallel is simply the point e, an evanescent parallel. When $e=f+1$, it is an edge of Ω. From every other point e of Ω a nodal parallel can be drawn, which has no fellow-generator, and the number of these is $\frac{1}{2}(r-3)$. Therefore the number of diagonals of Ω when r is odd, all different generators, is

$$\tfrac{1}{2}\{\tfrac{1}{2}(r^2-3r)+\tfrac{1}{2}(r-3)\}=\tfrac{1}{4}(r^2-2r-3),$$

the number of autopolars (P), when r is odd, constructed.

XIII. But it is next to be determined, how many times P, thus generated by a diagonal ef, is identical with P′, generated by a diagonal hk different from ef*.

As no two operations on the base Ω have been alike, it is certain that P′ cannot be reduced to a pyramid on the exact base Ω by the operation whereby P is so reduced; but it is possible that P′ may be reducible to an r-gonal based pyramid, by the union of some two faces which are not O and O′. If so, P′ will have a convanescible edge different from oo'. This can be none else than OH or OK, or else O′H or O′K, as H and K are the only faces not triangles distinct from O and O′, which have not a common convanescible, but an evanescible edge.

If OH be convanescible, O is not a triangle; therefore $r>4$, and the two summits of OH, when united, must give a summit at least a pentace. Greater than a pentace it cannot be, since there can be no summit of P, neither o nor o', ampler than a tessarace, and this must adjoin a triace, because no diagonal hk in Ω can make two collateral tessaraces. The edge OH must then both be convanescible and have a tessarace, and be an edge of the base of a 6-edral pyramid. It is easily seen, either by trial or by inspection of

$$\mathbf{1_5 2_4 3_3 4_2 5_1},$$

that the only diagonals fulfilling these conditions are 13 and 24 (or its fellow 25). The first gives a 7-edron P with three convanescible edges, oo', 15, 34, which reduces to a pyramid either on the base $12oo'5$, or on $234o'o$. The second gives a 7-edron P′ having two convanescible edges, oo' and 45, reducing to a pyramid on the base $123o'o$. And P, P′, are plainly not repetitions of each other. Therefore no deduction is to be made from the number of $(r+2)$-edra (P) constructed from the pyramid. If Π_1 be this number,

$$\Pi_1=\tfrac{1}{4}\{(r^2-3r)4_r+(r^2-3r+2)4_{r-2}+(r^2-2r-3).2_{r-1}\};$$

the number of $(r+2)$-edra (P) generable from the $(r+1)$-edral pyramid, where the circulator $4_r=1$ or 0, as r is or is not $4m$.

* It is clearly impossible that any of these P can be either a pyramid or reducible to a pyramid of higher rank than the $(r+1)$-edral Π.

In all these (P) there is but one *leading system*, i. e. system of vanescible gamic pairs whereby P reduces to a $(r+1)$-edral pyramid; except only the two just found, of which one has *two*, and the other *three*, *leading systems*.

These Π, $(r+2)$-edra (P), it is to be kept in mind, are all *nodally autopolar*. Whether there be any other *enodally autopolar* $(r+2)$-edra generable from the same pyramid, when r is odd, remains to be hereafter determined.

XIV. Problem. *To determine the number of nodally autopolar* $(r+3)$*-edra generable from the* $(r+1)$*-edral pyramid, by introduction of gamic pairs.*

For the solution of this problem it is necessary *to determine the number of pairs of diagonals not crossing each other that can be drawn in an* r*-gon.*

This number is less than that of pairs of diagonals; and of diagonals there are $\frac{1}{2}(r^2-3r)$. Therefore $R_{r,2}$ the function of r required is of an order not higher than the fourth. It is evident that the function is of the form

$$R_{r,2}=r.(r-3)(r-4)(ar+b),$$

and by trial,

$$R_{5,2}=5.2.1.(5a+b)=5,$$

and

$$R_{6,2}=6.3.2.(6a+b)=21,$$

viz. three pairs at every angle, and three parallel pairs; therefore

$$a=b=\tfrac{1}{12},$$

and

$$R_{r,2}=\tfrac{1}{12}.r.(r-3).(r-4)(r+1);$$

which can be generalized into

$$R_{r,k}=\frac{r^{\underline{k+1}}}{\lfloor k+1}\cdot\frac{(r-3)^{\underline{k}-1}}{\lfloor k+2},$$

the number of ways in which k diagonals, none crossing each other, can be drawn in an r-gon.

Let ef and hk be one of these R_r pairs of diagonals of Ω which do not cross each other. If we draw them and consider them as new edges, Ω is divided into three faces $OO'O''$, and the summits $efhk$ become tessaraces if they are all different, or if $e=h$, e is a pentace and f and k are tessaraces. The summit ω of the pyramid, if the result is to be autopolar, is broken into three summits $oo'o''$, and the faces EFHK are either four 4-laterals, or a pentagon E and two 4-laterals F and K. The result Q is an autopolar $(r+3)$-edron.

XV. We have to determine how many different $(r+3)$-edra Q can thus be generated by a pair of diagonals ef and hk of Ω.

Of two diagonals not crossing each other in an even-angled base, it is plain that both cannot be diameters. Let ef and hk not cross, and let not these be fellow-generators; then if neither be a diameter, there is another pair $e'f'$ and $h'k'$, the fellow-generators of the former two (VIII.), such that the points $e'f'h'k'$ are diametrically opposite to the points $efhk$. And if one of the former two ef be a diameter, and $h'k'$ be the fellow of

hk, the points $feh'k'$ are diametrically opposite to $ef'hk$. Consequently all distances measured on the summits of Ω between the signatures $ef'hk$, $E_r E_r F_r$, &c. will correspond exactly with distances between the signature $ef'h'k'E_r'$... or else between the signatures $feh'k'F_r$,.....; for if $\alpha-\beta$ be one of these distances, $(\alpha+\frac{1}{2}r)-(\beta+\frac{1}{2}r)$ is the corresponding distance. Consequently Q generated by ef and hk will not differ from Q' generated by ef' and $h'k'$, when ef and hk are neither of them diameters, nor from Q' generated by fe and $h'k'$, when ef is a diameter of Ω. Let, when r is even,

$$R_r = R_r' + R_r'',$$

R_r' being the number of pairs ef and hk which are not, and R_r'' that of the pairs ef and ef' which are, fellow-generators. It is evident that with the pairs R_r' we cannot obtain more than $\frac{1}{2}R_r'$ autopolars Q. Nor fewer. For let $e''f''$ and $h''k''$ give the same Q with the pairs ef and hk, and ef' and $h'k'$. Q'' from the first has the same faces OO'O'' with Q from the second; and if hk and its gamic be erased in Q, and $h''k''$ and its gamic be erased in Q'', the results P and P'' will be identical; *i. e.* ef will be the fellow of $e''f''$. In like manner hk is proved to be the fellow of $h''k''$; but ef' and $h'k'$ are the fellows of ef and hk; wherefore ef' and $h'k'$ are not different from $e''f''$ and $h''k''$; contrary to hypothesis, which is absurd. Therefore $\frac{1}{2}R_r'$ is exactly the number of $(r+3)$-edra Q generable by the pairs R_r'.

Next, let ef and ef' be one of the pairs R_r'', the number of which is easily seen to be $\frac{1}{4}r.(r-4)$, since $r-4$ lines not diameters can be drawn from e.

There is no pair hk and $h'k'$ of these R_r'' that can give the same Q with ef and ef', else ef would be proved the fellow either of hk or $h'k'$. Consequently the number of different Q obtainable from these R_r'' pairs is R_r''. It follows that the number $\Pi_{,,}$ of $(r+3)$-edra Q generable when r is even from all the R_r pairs is

$$\Pi_{,,} = \tfrac{1}{2}(R_r' + 2R_r'') = \tfrac{1}{2}\left(R_r + \tfrac{1}{4}(r^2.(r-4)(r-2)\right) = \tfrac{1}{24}\left(r^3.(r-4)(r-2)\right).$$

But yet this number is subject to a doubt, which will be discussed presently.

XVI. When r is odd and >3, we have seen (IX.) that if ef and hk be any two diagonals whose fellow-generators are ef' and $h'k'$, ee', ff', hh', kk' are nodal parallels. If then ef and hk do not meet each other, neither will ef' and $h'k'$ meet each other. Both these pairs have therefore been counted in the R_r pairs of not-crossing diagonals. And these pairs give the same result Q; for if B—A be any distance measured on the summits of Ω in one between summits affected by the first pair, $\left(A-\frac{1}{2}(r+3)\right)-\left(B-\frac{1}{2}(r+3)\right)$ is the corresponding distance in the summits affected by the second pair.

Next, let ef and ef' be fellow-generators, which do not meet each other in the base Ω. They are opposite sides of a quadrilateral $ee'ff'$; and the resulting Q cannot be identical with that made by any other pair gh ij, for this would require that gh should be fellow to one and ij to the other of ef and ef'.

The number of pairs ef ef' comprises, first, that of the quadrilaterals whose opposite sides are non-contiguous nodal parallels ee' and ff', of which nodal parallels there are $\frac{1}{2}(r-3)$ (XII.); and secondly, every pair of lines ef, ef' drawn from the point $(e=f)$,

the evanescent parallel; and thirdly, every pair drawn from the extremities of that nodal parallel which is a side of Ω. That is, there are $\frac{1}{2}(r+1)$ of these parallels whose non-contiguous pairs are $\frac{1}{2}(r-3)(r-1)$, which is the number of pairs ef and $e'f'$, of fellow-generators which do not cross each other, and of the $(r+3)$-edra Q generated by such pairs.

We suppose next that ee' is one of the $\frac{1}{2}(r-3)$ nodal parallels, and that hk, having a fellow-generator $h'k'$, does not meet ee'. Then $h'k'$ will not meet ee'; for ee', parallel to two sides hh' and kk' of the 4-lateral hk, $h'k'$, does meet $h'k'$ unless it meets hk. And for a reason given in this article the $(r+3)$-edron generated by ee' and hk cannot differ from that obtained from $e'e$ and $h'k'$. Every Q obtained by a pair ee', hk, is obtained also from $e'e$, $h'k'$.

Now let ee' and ff' be two of the $\frac{1}{2}(r-3)$ nodal parallels. They do not meet; and they generate a $(r+3)$-edron Q which cannot be generated by any other pair ee', hh' or hh', ii'; for, if it could, this other pair would be two fellows of ee' and ff', which have no fellows. Thus each of the $\frac{1}{8}.(r-3)(r-5)$ pairs of nodal parallels will generate a distinct $(r+3)$-edron Q.

From all this it appears that the number Π_r of different (Q) obtained from the R_r pairs of diagonals that do not cross each other, is ($r>3$ and $=2m+1$)

$$\Pi_r=\tfrac{1}{2}\{R_r+\tfrac{1}{8}.(r-3)(r-1)+\tfrac{1}{8}.(r-3)(r-5)\}$$
$$=\tfrac{1}{48}\{2r.(r-3)(r-4)(r+1)+3(r-3)(r-1)+3(r-3)(r-5)\}.$$

XVII. It appears, I say: and it is certain, that none of these Q obtained from operations in the base Ω, are alike in the relation between the faces $OO'O''$ and the other faces of the figure. Any two of them, Q' and Q'', present different arrangements of the circuit of base summits $1\,2\ldots r$, which enclose those three faces; so that Q' and Q'' cannot be reduced to a pyramid having the base Ω whose summits shall be those signed $1\,2\ldots r$, by similar operations with vanishing gamic pairs. This is plain, because Q' and Q'' have not been constructed by similar operations upon the base Ω and the vertex ω.

But it remains to be proved that Q'' has not three faces different from $OO'O''$, which by proper selection of vanescible gamic pairs may be brought to form an r-gonal base Ω''. If such a thing is possible, this Q'' may have the relation to the contour of that Ω'', which Q' has to that of Ω; and thus the operations that gave us Q' will have generated the same $(r+3)$-edron with those which gave us Q'', and *vice versâ*.

We are then to examine whether Q'' has three summits, not $oo'o''$, which can by convanescing edges be united to form an r-gonal summit ω''.

Now Q'' has no summits besides $oo'o''$, except $1\,2\ldots r$. Of the three summits of which we are in search, two, one, or none may be among $oo'o''$. The three may be $oo'e$, oee' or $ee'e''$.

There may be a convanescible edge oe, the edge F.(F+1). Of these two faces, F and F+1, neither can be a pentagon; for the lines fg, fh, evanescibles making the pentace f, can neither of them be $f(f+1)$, a side of Ω. If, then, oe convanesces, F and F+1

become triangles, and oo', oo'', which are their edges out of the base, are neither of them convanescible. Therefore $oo'e$ cannot be united.

There may be a convanescible edge ee', which is $e.(e+1)$, or FO'. If F be a pentagon, for a reason just given, it adjoins two triangles upon $(e-1)e$ and $(e+1)(e+2)$, so that oe nor oe', one of the sides of it, cannot be a convanescible edge. If F be a quadrilateral $oo'ee'$, its sides oe', $o'e$ will be in the triangle $oo'e$ when ee' convanesces. Therefore oee' cannot be united into one summit.

Our three sought summits will therefore be $(e-1)$, e, $(e+1)$. That $(e-1)e$ may convanesce, its faces F and O must not be triangles, neither can $(F-1)$ and O' about $e(e+1)$ be triangles. Therefore O and O' together have at least six summits of Ω, which, supplying at least one summit more to O'', cannot be *simpler* than a heptagon. Neither can it be *ampler*; for e is at most a pentace, adjoining two triaces $(e-1)$, $(e+1)$, for a reason above given. And no diagonals have been drawn to make a third tessarace among $(e-1)$, e, $(e+1)$, supposing two of them tessaraces. That is, these three can unite at most to form a heptace ω''. We are then to examine whether two diagonals can be drawn in the heptagonal Ω to make two adjoining convanescible edges. The only way of doing this is by drawing either 62 and 63, or 72 and 53: in either case 65 and 67 are both convanescible, as well as $o'o''$ and $o'o$. The faces of the decaedron in the first result are

$$3456,\ 362,\ 2671,\ 1o''2,\ 2o''o'3,\ 3o4,\ 4o5,\ 5oo'6,\ 6o'o''7,\ 7o''1;$$

and in the second,

$$345,\ 35672,\ 721,\ 7o''1,\ 1o''o'2,\ 2o'3,\ 3o'o4,\ 4o5,\ 5oo'6,\ 6o'o''7.$$

These are identical, if we exchange the signatures in the edges $5o$, $6o'$, and $7o''$.

XVIII. This decaedron is the only one of the autopolar $(r+1)$-edra Q above constructed which has two *leading systems*, i. e. two pairs of leading convanescible edges, and two pairs their gamics evanescible; and in this case alone have the operations (62 and 63) that gave us Q′ brought out the same polyedron with the operations (72 and 53) that gave us Q″. It is therefore necessary to deduct one from the (Q) enumerated, namely, one for the value $r=7$.

This number is consequently $(r>3)$, (XV. XVI.),

$$\Pi_u+\Pi_v=\tfrac{1}{24}.r^3.(r-4)(r-2).2,$$
$$+\tfrac{1}{48}\{2.r.(r-3)(r-4)(r+1)+3(r-3)(r-1)+3(r-3)(r-5)\}.2_{r-1}-0^{(r-7)^3}.$$

XIX. Our next step is to take up one of the $(r+2)$-edra P (XIII.), obtained by drawing a diagonal ef in the base Ω of the pyramid. P has two quadrilaterals E and F, in either of which F we can draw a diagonal, making a triangle F^t, a side of which is oo', the gamic of OO' in the base. This new diagonal in F is either $F_o\rho$ or $(F_o\rho')$; and the triangle is either $F_o o'$ or $F_o o'$. Thus a new triangle is introduced between F and $F+1$, or else between F and $F-1$, about the line oo'. Consequently there must be a new triace between f and $(f+1)$, or between f and $(f-1)$ on the edge OO'. Thus the gene-

rator OO' is drawn from a summit of Ω to the mid-point of a side of Ω, dividing the faces OO' whose summits together make $r+3$.

We are to consider how many $(r+3)$-edra S can be obtained by such an operation in Ω.

The diagonal ef being drawn, becomes by the motion of the extremity f, $e(f\pm\frac{1}{3})$. If $e'f'$ be the fellow-generator of ef, $e'(f'+1)$ is that of $e(f+1)$ when r is even, and $e'(f'-1)$ that of $e(f+1)$ when r is odd. Two fellow-generators when drawn are symmetrical with respect to the nodal line, which is a diameter when r is even, but not when r is odd. Consequently, $e'(f'\pm\frac{1}{3})$ will be the fellow-generator of $e(f\mp\frac{1}{3})$ when r is even, and of $e(f\mp\frac{1}{3})$ when r is odd. Even when fe is the fellow of ef, or ef is an unpaired generator, $f(e\pm\frac{1}{3})$ is a different line from either $e(f\pm\frac{1}{3})$ or $e(f\mp\frac{1}{3})$. Consequently, every line λ that can be drawn in Ω from a summit to the mid-point of an edge, has a fellow-generator, except only when $e(f+1)$ is the fellow of ef, which happens when r is odd, and $e(f+\frac{1}{3})$ bisects Ω, being the nodal axis perpendicular to the nodal line. And there is plainly but one such fellow of λ, which being drawn shall be symmetrical with it as to the nodal line. Therefore every $(r+3)$-edron S, except one, that is generated by a line λ, will be generated also by a line λ'. Of lines λ there are $r.(r-2)$; hence the number of $(r+3)$-edra S is not greater than $\frac{1}{3}\{r.(r-2)+2_{r-1}\}$. Nor are there fewer that can be reduced to a pyramid on the exact base Ω. It is, however, to be inquired, whether some of these (S) may not have two faces not O and O', which can by the vanishing of two edges be united into an r-gon Ω', giving a pyramid on the base Ω'.

But before we examine whether any of these $(r+3)$-edra S are repetitions of each other, it is desirable to ascertain how many of them are reducible to the $(r+2)$-edral pyramid. If any one of them, s, is so reducible by the vanishing of a single gamic pair, it will be a repetition of a $(r+3)$-edron P, generated by a diagonal in the $(r+1)$-gonal Ω; and will have a convanescible edge ϵ between summits whose united ranks x and y give $x+y=r+3$. That these two summits in ϵ cannot be o and o' is certain; for the edge oo' is in a triangle. Then one only, or neither of these x-ace and y-ace, will be o or o'.

XX. First, let neither of them be o or o'; i. e. let ϵ be an edge of Ω. The greatest of x and $y \geqslant 5$; for ef_1 being drawn in s to the mid-point of a face H, E becomes a 4-lateral containing o and o', and η the pole of H becomes a 4-ace containing $f_1 o$ and o', unless E=H, in which case it is a 5-gon, containing o, o', f_1 and two summits of Ω. As no summit or face, except e and E, is affected by ef_1 in this case, all the faces but E about Ω remain triangles, and therefore there is no convanescent edge ϵ. Wherefore $x \geqslant 4$, and y is either 4 or 3. First, let $x=4$ and $y=4$; whence $r=5$. We are to look for an edge convanescible and between two 4-aces in s made from the pentagonal Ω. The only 4-aces in s, neither o nor o', are e and η. Therefore $e\eta$ is the edge ϵ, and $e=\eta\pm 1$. We see from

$$1_5 2_4 3_3 4_2 5_1,$$

that in order to have e and η contiguous 4-aces, we must either draw the generator

$(1, 5-\frac{1}{3})$ or $(2, 5+\frac{1}{3})$. In the first case, 1 is a 4-ace and 2 a 4-lateral; in the second, 2 is a 4-ace and 1 a 4-lateral. In either case 1 and 2 are 4-aces, but in both (12) is an edge of the triangle 5, and therefore not convanescible. Therefore $\mathfrak{s}$ has no existence when $r=5$. Let, next, $x=4$ and $y=3$; then $r=4$.

We are to look in $\mathfrak{s}$ made from the 5-edral pyramid for a convanescent $\mathfrak{s}$ not passing through o or o'. The simplest way of doing this is to write out the four results of drawing the four generators λ in the 4-gonal Ω. The eight edges of the pyramid are represented thus:

$$\begin{array}{cccc} 4_0\mathbf{3}_1\mathbf{3}_4\mathbf{3}_2 & 4_0\mathbf{3}_2\mathbf{3}_3\mathbf{3}_3 & 4_0\mathbf{3}_3\mathbf{3}_2\mathbf{3}_4 & 4_0\mathbf{3}_4\mathbf{3}_1\mathbf{3}_1 \\ 3_1\mathbf{3}_4\mathbf{3}_2\mathbf{4}_0 & 3_2\mathbf{3}_3\mathbf{3}_3\mathbf{4}_3 & 3_3\mathbf{3}_2\mathbf{3}_4\mathbf{4}_0 & 3_4\mathbf{3}_1\mathbf{3}_1\mathbf{4}_0, \end{array}$$

where the first and third places in the quadruplet show the rank and signature of the left and right summits, and the second and fourth, those of the upper and lower faces of the edge; and each is written over its gamic.

If we draw 12_i to the mid-point of the edge 23, the face 3 becomes a 4-gon, the summit 3 a 4-ace, and we have the 4-gon 1 and the 4-ace 1. The 4_0 of the base and summit becomes 4_0 and 3_{0_i}, and the new edges oo' and its gamic appear; while instead of 23, the second edge and its gamic, we have 22_i and 2_i3 with their gamics. The result is

$$\left.\begin{array}{cccccc} 3_0\mathbf{4}_1\mathbf{3}_4\mathbf{3}_2 & 3_0\mathbf{3}_2\mathbf{4}_3\mathbf{3}_{2_i} & 4_0\mathbf{3}_{2_i}\mathbf{4}_3\mathbf{4}_3 & 4_0\mathbf{4}_3\mathbf{3}_2\mathbf{3}_4 & 4_0\mathbf{3}_4\mathbf{4}_1\mathbf{4}_1 & 3_0\mathbf{3}_{2_i}\mathbf{4}_0\mathbf{4}_1 \\ 4_1\mathbf{3}_4\mathbf{3}_2\mathbf{3}_{0_i} & 3_2\mathbf{4}_3\mathbf{3}_{2_i}\mathbf{3}_{0_i} & 3_{2_i}\mathbf{4}_3\mathbf{4}_3\mathbf{4}_0 & 4_3\mathbf{3}_2\mathbf{3}_4\mathbf{4}_0 & 3_4\mathbf{4}_1\mathbf{4}_1\mathbf{4}_0 & 3_{2_i}\mathbf{4}_0\mathbf{4}_1\mathbf{3}_{0_i} \end{array}\right\} \quad (12_i)$$

The results of drawing 13_i to the mid-point of 34, 23_i to the middle of 34, and 24_i to the middle of 14, are in order following:—

$$\left.\begin{array}{ccccc} 4_0\mathbf{4}_1\mathbf{3}_4\mathbf{4}_2 & 4_0\mathbf{4}_2\mathbf{3}_3\mathbf{3}_3 & 4_0\mathbf{3}_3\mathbf{4}_2\mathbf{3}_{3_i} & 3_0\mathbf{3}_{3_i}\mathbf{4}_2\mathbf{3}_4 & 3_0\mathbf{3}_4\mathbf{4}_1\mathbf{4}_1 & 3_0\mathbf{4}_1\mathbf{4}_0\mathbf{3}_{3_i} \\ 4_1\mathbf{3}_4\mathbf{4}_2\mathbf{4}_0 & 4_2\mathbf{3}_3\mathbf{3}_3\mathbf{4}_0 & 3_3\mathbf{4}_2\mathbf{3}_{3_i}\mathbf{4}_0 & 3_{3_i}\mathbf{4}_2\mathbf{3}_4\mathbf{3}_{0_i} & 3_4\mathbf{4}_1\mathbf{4}_1\mathbf{3}_{0_i} & 4_1\mathbf{4}_0\mathbf{3}_{3_i}\mathbf{3}_{0_i} \end{array}\right\} \quad (13_i)$$

$$\left.\begin{array}{ccccc} 4_0\mathbf{3}_1\mathbf{3}_4\mathbf{5}_2 & 3_0\mathbf{5}_2\mathbf{3}_3\mathbf{3}_3 & 3_0\mathbf{3}_3\mathbf{5}_2\mathbf{3}_{3_i} & 4_0\mathbf{3}_{3_i}\mathbf{5}_2\mathbf{3}_4 & 4_0\mathbf{3}_4\mathbf{3}_1\mathbf{3}_1 & 4_0\mathbf{5}_2\mathbf{3}_0\mathbf{3}_{3_i} \\ 3_1\mathbf{3}_4\mathbf{5}_2\mathbf{4}_0 & 5_2\mathbf{3}_3\mathbf{3}_3\mathbf{3}_{0_i} & 3_3\mathbf{5}_2\mathbf{3}_{3_i}\mathbf{3}_{0_i} & 3_{3_i}\mathbf{5}_2\mathbf{3}_4\mathbf{4}_0 & 3_4\mathbf{3}_1\mathbf{3}_1\mathbf{4}_0 & 5_2\mathbf{3}_{0_i}\mathbf{3}_{3_i}\mathbf{4}_0 \end{array}\right\} \quad (23_i)$$

$$\left.\begin{array}{ccccc} 3_0\mathbf{4}_1\mathbf{3}_4\mathbf{4}_2 & 4_0\mathbf{4}_2\mathbf{3}_3\mathbf{3}_3 & 4_0\mathbf{3}_3\mathbf{4}_2\mathbf{3}_4 & 4_0\mathbf{3}_4\mathbf{4}_1\mathbf{3}_{4_i} & 3_0\mathbf{3}_{4_i}\mathbf{4}_1\mathbf{4}_1 & 3_0\mathbf{4}_2\mathbf{4}_0\mathbf{3}_{4_i} \\ 4_1\mathbf{3}_4\mathbf{4}_2\mathbf{3}_{0_i} & 4_2\mathbf{3}_3\mathbf{3}_3\mathbf{4}_0 & 3_3\mathbf{4}_2\mathbf{3}_4\mathbf{4}_0 & 3_4\mathbf{4}_1\mathbf{3}_{4_i}\mathbf{4}_0 & 3_{4_i}\mathbf{4}_1\mathbf{4}_1\mathbf{3}_{0_i} & 4_2\mathbf{4}_0\mathbf{3}_{4_i}\mathbf{3}_{0_i} \end{array}\right\} \quad (24_i)$$

In all these, the only convanescible edges, not through o or o_i, are $3_2\mathbf{4}_3\mathbf{4}_3\mathbf{4}_0$ and $3_4\mathbf{4}_1\mathbf{4}_1\mathbf{4}_0$ in (12_i), either of which is in two 4-laterals, and the summits of neither are in two faces having any other common edge. Either therefore is convanescible; but with their present signatures they will neither of them vanish so as to give a 5-ace by the union of their summits; nor is either of them a generator introduced into the 5-gonal Ω, in our construction of the $(r+2)$-edra P. For they are both nodal gamics; the first meeting its gamic at 3, and the second at 1, while the generators of the class P never met their gamics. The first edge cannot vanish without the loss of two edges at the

summit 3, so that its union with 2_i will give only a 4-ace: the like remark is to be made on the second. Yet nothing but the signatures prevents either of these lines from vanishing so as to yield a pentace. Now as these signatures are not essential to any polyedron, it is necessary, before we pronounce positively that the s before us (12_i) cannot be generated from the 6-edral pyramid, to satisfy ourselves whether, by any rearrangement of the signatures, the autopolar character being preserved, this vanescible edge s cannot be made to stand as the gamic of one not meeting it.

If we exchange the signatures of the 4-aces 1 and 3, of the triaces o, and 2, and of the triaces 4 and 2_i, we have the result following, still nodally autopolar:—

$$3_3\mathbf{4}_1 3_2 3_2 \quad 3_3 3_2 4_1 \mathbf{3}_2 \quad 4_0 3_2 4_1 \mathbf{4}_3 \quad 4_0 \mathbf{4}_3 3_0 3_4 \quad 4_0 3_4 4_3 \mathbf{4}_1 \quad 3_2 3_2 4_0 \mathbf{4}_1$$
$$4_3 3_4 3_0 \mathbf{3}_0 \quad 3_0 \mathbf{4}_3 3_4 3_0 \quad 3_4 4_3 4_1 \mathbf{4}_0 \quad 4_1 3_2 3_2 \mathbf{4}_0 \quad 3_2 \mathbf{4}_1 4_3 4_0 \quad 3_4 4_0 4_3 3_0,$$

which is, placing the gamics one over the other,

$$3_3\mathbf{4}_1 3_2 3_2 \quad 4_3 3_4 3_0 \mathbf{3}_0 \quad 4_0 \mathbf{4}_3 3_0 3_4 \quad 4_0 3_4 4_3 \mathbf{4}_1 \quad 4_0 3_2 4_1 \mathbf{4}_3 \quad 3_2 3_2 4_0 \mathbf{4}_1$$
$$4_1 3_2 3_3 3_2 \quad 3_4 3_0 3_0 \mathbf{4}_3 \quad 4_3 3_0 3_4 \mathbf{4}_0 \quad 3_4 4_3 4_1 \mathbf{4}_0 \quad 3_2 \mathbf{4}_1 4_3 4_0 \quad 3_2 \mathbf{4}_0 4_1 3_2,$$

the heptaedron (P) made from the 6-edral pyramid, which has the two convanescibles $3_4\mathbf{4}_3 4_1 \mathbf{4}_0$ and $3_2 \mathbf{4}_1 4_3 \mathbf{4}_0$.

It is thus proved that one $(r+3)$-edron s, namely, when $r=4$, must be rejected as reducible to the $(r+2=)$6-edral pyramid, by the convanescence of an edge not in o. Wherefore $-0^{(r-4)^2}$ is to be added to the number found in XIX.

XXI. Next, let o' be the x-ace in s. As the united rank of o and $o'=r+3$, $o' \not> \frac{1}{2}(r+3)$; then $y \not< \frac{1}{2}(r+3)$. Now $y \not> 5$, and can be $=5$ only when there is no convanescible s, as it has been just proved: therefore $y \not> 4$ and $r \not> 5$. Then o' is either a 4-ace or a triace, and a 4-ace only when $o=o'$, in which case it is indifferent which of the two is called o. That is, o must be this x-ace in s wherever $x>3$. We are then to look in our $(r+3)$-edron s for a convanescible edge through o. This can be none other than the common edge of the only 4-laterals, H and E, which are contiguous only when $H=E\pm 1$. The y-ace in s cannot be a summit of O, because s, being in the two faces O and E that have the edge OE, could not be convanescible. Therefore the y-ace is neither e nor f_i, these being both in O, but is another summit of O′, which must be a triace. And as $x+y=$the united rank of o and o', of which o is the x-ace, o' is a triace, and O′ is a triangle. Therefore s is the edge $o(e\pm 1)$ through $(e\pm 1)$ a summit of the triangle O′, *i. e.* through (OEH), *i. e.* through $\big(\mathrm{OE}(\mathrm{E}\mp 1)\big)$, s being $\mathrm{E}(\mathrm{E}\mp 1)$. Therefore O′E and O′$(\mathrm{E}\mp 1)$ are edges of O′. The signatures of the pyramid about e are thus exhibited:

$$\ldots \mathrm{G}-2 \quad \mathrm{G}-1 \; \mathrm{G} \quad \mathrm{G}+1 \ldots$$
$$\ldots \; : \; e+1 \quad e \; e-1 \; : \; \ldots$$

the colon being the point f_i. The edges of $(e, e+1, f_i)$ are OO′, O′$(\mathrm{G}-1)$ and O′$(\mathrm{G}-2)$, the two last being O′E and O′$(\mathrm{E}+1)$; therefore $\mathrm{G}-1=\mathrm{E}$, and e is nodal. Two edges

of $(e, e-1, f_i)$ are O'G and O'(G+1), which are O'E and O' (E−1), wherefore G=e and e is nodal. In the first case, s or (G−1)(G−2) is between two 4-laterals and is convanescible; in the second, s or G(G+1) is for a like reason convanescible, so that its summits, $(e\pm1)$ and the r-ace o, will reduce s to a $(r+2)$-edron with a $(r+1)$-ace, *i. e.* to a pyramid. Consequently s, having the triangle $(e, e\pm1, f_i)$ when e is nodal, and EO' is an edge of s, is to be rejected when $r>3$, because of the vanescible pair O(E±1) and $o(e\pm1)$. And this is the only s to be rejected in our enumeration for any value of r because of such a pair. Adding to this the one above rejected for another reason, for $r=4$, we have $1+0^{(r-4)^2}$ to deduct from the $\frac{1}{2}\{r.(r-2)+2_{r-1}\}$ $(r+3)$-edra (S), for every value of r, on the account of being generable from the $(r+2)$-edral pyramid. This makes the number of (S) thus far ascertained to be

$$\tfrac{1}{2}\{r.(r-2)-2+2_{r-1}\}-0^{(r-4)^2}.$$

XXII. It is necessary that we inquire how many of these are repetitions of each other, or of those enumerated in the class (Q).

The leading system of every S is the two edges $o\eta$ and oo' in the triangle $o'o\eta$; $o\eta$ being the diagonal in the face F, from the summit η=F$_r$ or F$_i$, as the case may be. This $o\eta$ is evanescible, and by evanishing, makes oo' convanescible, so that the union of its summits gives the pyramidal summit ω. Of course the gamics of $o\eta'$ and oo' (OH' and OO') are the first convanescible, and the second, thereby made evanescible.

We are then first to examine whether any of these (S) have a second *triangle* $\alpha\beta\gamma$, *containing a leading system like that in* $oo'\eta$, whereby S can be reduced to a pyramid, on some r-gonal base different from Ω; and then to decide whether this S has been twice enumerated above.

Let ef_i be the generator OO', f_i being the point $f\pm\frac{1}{2}$. Let the summit F$_r$ (or F$_i$) be η; then the triangle $oo'\eta$ contains the evanescible $o\eta$ (or $o'\eta$) and the convanescible oo'. Let $r>5$ for the present: then, if o be not the simplest of the two summits oo', there is no pair of summits in S excluding o, whose united rank $=r+3$. For in general, η and e, the only summits not triaces, except o and o', are tessaraces; and if η is e, it is a pentace containing ef_i, two sides of the triangle $oo'\eta$, and two edges of Ω. That is, two tessaraces, or a pentace and a triace, are the amplest pair in S, not o' or o; whose united rank makes only 8, $<r+3$.

Now if there be a triangle $\alpha\beta\gamma$ containing a leading system different from that in $oo'\eta$, α the greatest of its summits must be o, because $\alpha+\beta$ gives the same sum with o and o' by a convanescing edge. This triangle is therefore $o\beta\gamma$. And as $o\gamma$ is evanescible, γ, being neither o' nor η, must be e, the only summit besides these not a triace; therefore this triangle is $oe\beta$: and β being not o', nor η nor e, is a triace; wherefore, since $o+\beta=o+o'$, o' is a triace, and O' is the triangle $e(e\pm1)f_r$.

Again, β cannot be a summit of O, for oo' is an edge of E, because e is in OO'; and OE is an edge of S, because oe is. If, then, β were in O, $o\beta$ could not become convanescible by the evanescence of oe, because its summits $o\beta$ are in O and E collateral in

OE. Therefore β is in O′, and is $e\pm1$. Now $o(e\pm1)$ being convanescible after the evanescence of oe, must before that be in a face not triangular. Of such faces S has only E, O, and H the polar of η. But oo' is in E; therefore $o(e\pm1)$ is in H.

The conditions, therefore, that the triangle $oe(e\pm1)$ shall contain a second leading system, are, that O′ shall be the triangle $e(e\pm1)f_i$ having the triaces $e\pm1$ and f_i, that oe shall be an edge distinct from $o\eta$, and that $o(e\pm1)$ shall be an edge of H, no triangle.

XXIII. The signatures of the pyramid about e are

$$\begin{array}{ccccccc} \ldots(\mathrm{G}-2) & & (\mathrm{G}-1)\ \mathrm{G} & & (\mathrm{G}+1)\ldots \\ \ldots\ : & (e+1) & e & (e-1) & :\ \ldots \end{array}$$

where the colon is the point f_i. As the only summits of the figure S not in the contour of Ω are o and o', e has the edge oe, or $o'e$, or both. The edges of $(e, e+1, f)$ (the triangle O′) are O′(G−1) and O′(G−2), and of $(e, e-1, f_i)$ are O′G and O′(G+1); of which none is O′E unless G−1=E, or G−2=E; or G=E, or G+1=E.

If the first or third be true, e is nodal.

If G−2=E, G−1=E+1, and $e+1$ is nodal.

If G+1=E, G=E−1, and $e-1$ is nodal.

Therefore O′E is never an edge of $(e, e\pm1, f_i)$, unless one summit of this triangle is nodal; and OE is always an edge of S, if no summit of O′ is nodal.

When no summit of O′ is nodal, $o(e\pm1)$ is always an edge of a quadrilateral. For in the triangle $(e, e+1, f_i)$ neither O′(G−1) nor O′(G−2) is O′(E+1), because $e+1$ not being nodal, E+1 is neither G−1 nor G−2. As then O′(E+1) is no edge, $o'(e+1)$ is none, and $o(e+1)$ is in G−2, whose summits are $e+1, f_i, e+2, o$.

And in the triangle $(e, e-1, f_i)$ neither O′G nor O′(G+1) is O′(E−1); wherefore $o(e-1)$ is in G+1, whose summits are $e-1, f_i, e-2, o$.

Consequently, when no summit of the triangle O′ is nodal, S has two leading systems, and may be a repetition of some other of the $\frac{1}{2}\{r.(r-2)-2+2_{r-1}\}$ S above (XXI.) constructed.

XXIV. Let e be nodal, and G=E. The triangle $(e, e-1, f_i)$ has the edge O′E; but the triangle $(e, e+1, f_i)$ has not the edge O′E. In the latter case OE is an edge of S; but not in the former, unless e be a pentace containing ef, eo and eo', with two sides of Ω. But G=E has only the summits e, $e-1$, o, o'; therefore e is no pentace, and OE is not an edge of S in the former case. Hence when e is nodal, and G=E, one line ef_i can be drawn to make one triangle O′ or $(e, e-1, f_i)$ such that O′E is an edge of S, and OE is not; and another, $(e, e+1, f_i)$ such that OE is an edge and O′E is not.

When O′E is no edge, O′(G−1) and O′(G−2) are neither of them O′(E+1) in $(e, e+1, f_i)$; and neither O′G nor O′(G+1) is O′(E−1) in $(e, e-1, f_i)$. Therefore O(E+1) is an edge in one case, and O(E−1) in the other; and $o(e+1)$ is in (G−2), whose summits are $e+1, f_i, e+2, o$ in the former case, and $o(e-1)$ is in G+1, whose summits are $e-1, f_i, e-2, o$, in the latter. Hence $e\pm1$ is a triace in O′.

The like conclusion follows from supposing e nodal, and G−1=E, that one generator

ef_i can be drawn such that OE and not O'E shall be an edge of S, and another ef_i such that O'E and not OE shall be an edge. And when O'E is no edge, $o(e\pm1)$ is an edge of a quadrilateral. And $(e\pm1)$ is a triace in O'.

Thus whenever e is nodal in $(e, e\pm1, f_i)$ and O'E is no edge, S may be a repetition of some other of the (S) above constructed, having the triangle $(o, e\pm1, e)$ which contains a second leading system.

Let now $e+1$ be nodal in the triangle $(e, e+1, f_i)$, and let $e+1=G-2$. Here $e+1$ is a tessarace, because $G-2$ is a quadrilateral $(e+1, f_r, e+2, o')$; for $o'(e+1)$ is an edge because $O'(G-2)$ is. In this case therefore O' has two tessaraces, and there is no second leading system in S.

Let then $e+1$ be nodal and $e+1=G-1$, or $e=G-2$. Here e is a pentace; for E has the point f_r. In this case oe is $o\eta$, and there is no second leading system.

In like manner it can be proved that if $(e-1)$ is nodal in the triangle $(e, e-1, f_i)$, either $e-1$ is a tessarace, as well as e, in O', or that e is the pentace η.

Consequently S is never repeated when O' has a nodal summit, except when e is nodal, and O'E is no edge of S.

Thus we are tempted to conclude that S is always twice counted among those of the $\frac{1}{2}\{r.(r-2)-2+2_{r-1}\}$ which have a triangular O', except when $(e\pm1)$ in O' is nodal, or when e is nodal and O'E is an edge.

XXVI. There are two triangles O' in which ef_i is drawn from a nodal summit e to make an edge O'E, i. e. e may be either of the nodal summits of the pyramid, but as ef_i drawn thus from one nodal summit is the fellow-generator in Ω of the other, only one of the resulting S was enumerated in XIX. And this is the s rejected in XXI. Also there are two lines ef_i which can be drawn about either nodal summit to cut off a triangle $(e, e\pm1, f_i)$ having $(e\pm1)$ nodal; but the two lines so drawn about one nodal summit of Ω are fellow-generators of those so drawn about the other. We have therefore among the $\frac{1}{2}\{r(r-2)+2_{r-1}\}$ counted in XIX., three of those *which have a triangular* O', which have not two leading systems: and we have not more than three.

XXVII. We have now to determine how many of these (S) which have a triangle O' and two leading systems are repetitions of each other.

Let S' be one of these. Its base O is r-gonal, having all the summits of Ω except $(e\pm1)$, and instead of this the summit f_r. By the vanishing of $o\eta$ and oo', the pyramid is restored, having the r-gonal base

$$\ldots(e+2)(e+1)e(e-1)(e-2)\ldots.(g+1)g(g-1)(g-2)\ldots. \quad (\Omega)$$

If O' is $(e, e-1, f_i)$, the r-gon O is

$$\ldots(e+2)(e+1)e.f_i(e-2)\ldots(g+1)g(g-1)(g-2)\ldots \quad (O)$$

If O' is $(e, e+1, f_i)$, the r-gon O is

$$\ldots(e+2)f_ie(e-1)(e-2)\ldots(g+1)g(g-1)(g-2)\ldots \quad (O)_i$$

When O is (O), S' has the triace $(e-1)$, which is (G, G+1, O'), and has also the triangle

$(g, g+1, o')$, and (O) has all the summits of S′ except o, o' and $e-1$. By the vanishing of one of the leading systems of this S′, S′ becomes the pyramid, and (O) becomes (Ω); losing the triace f_i and receiving instead the triace $(e-1)$. By the vanishing of the other leading system, the same S′ becomes a pyramid on the r-gonal base $(\Omega)'$,

$$..(e+2)(e+1)e.f(e-2)\ldots(g+1)o'(g-1)(g-2)\ldots \quad . \quad . \quad (\Omega)'$$

losing the triace g, and receiving instead the triace o'. Observe that $g+1$ is a tessarace and g a triace, as is evident from their polars G and G+1, by inspection of

$$\ldots G-2,\ G-1,\ G,\ G+1\ldots$$
$$:e+1,\ e\ \ e-1:$$

where G is $\left(o\ e(e-1)\right)$ and G+1 is $\left(o(e-1)f(e-2)\right)$.

When O is $(O)_i$, $(O)_i$ becomes (Ω) by losing the triace f_i and receiving in its stead the triace $(e+1)$. Or it may become a pyramid on the r-gonal base

$$..(e+2)f_i\ e(e-1)(e-2)\ldots(g+1)g\ o'(g-2)\ldots \quad . \quad . \quad (\Omega)''$$

by the vanishing of its second leading system, receiving the triace o' and losing the triace $(g-1)$.

It is evident that the pyramids whose bases are Ω, $(\Omega)'$ and $(\Omega)''$ are identical, having the same r signatures with two slight changes of name, and the same nodal line through the summit 1. Now S′ having the triangle $\left(e(e-1)f''\right)$ is generated by drawing in the base $(\Omega)'$ either the generator $e(e-1)$ from e to a point between f_i and $(e-2)$, or the generator $(g+1)g$ from $g+1$ to a point between o' and $g-1$. And S′ having the triangle $\left(e(e+1)f_i\right)$ is obtained by drawing in $(\Omega)''$ either $e(e+1)$ from e to a point between f_i and $(e+2)$, or the generator $(g-2)(g-1)$ from $(g-2)$ to a point between o' and g.

XXVIII. Whenever these two generators drawn in $(\Omega)'$, or $(\Omega)''$, are not fellows, we have constructed S′ twice and twice counted it; but when they are fellows, we have constructed and counted it only once; for (XIX.) we have never used two fellow-generators in forming the $(r+3)$-edra S.

To examine this, let first r be odd. If fellow-generators can be drawn from e and $(g+1)$ in $(\Omega)'$, we must have, since e is G_i in

$$\begin{array}{c} \ldots \quad (G-1)\ G \quad\quad G(+1), \\ \ldots(e+1) \quad\quad e\ \ (e-1) \quad : \\ 2G_i+2(G+1)=r+3+ \quad \text{(VI.) (X.)}, \end{array}$$

or

$$4-2G+2G+2=r+3+,$$

whence

$$3 \quad\quad =r,$$

or the pyramid is a tetraedron, contrary to hypothesis. And if fellow-generators can be drawn from e and $g-2$ in $(\Omega)''$, we have, since e is G_i in

$$\begin{array}{c} ..(G-2) \quad\quad (G-1)\ G.. \\ .. \quad : \quad (e+1) \quad\quad e \quad .., \\ 2G_i+2(G-2)=r+3,+ \end{array}$$

or

$$4-2G+2G-2=r+3+,$$

whence

$$-1 \quad =r+,$$

which is still more absurd.

Therefore S′ is always made when r is odd from two different generators, and has been twice constructed among the $(r+3)$-edra S.

Let now r be even. If fellow-generators can be drawn in $(\Omega)'$ from e and $g+1$, we have

$$G_i=G+1\pm\tfrac{1}{2}r,$$

or

$$2-G=G+1\pm\tfrac{1}{2}r,$$

whence

$$\tfrac{1}{2}r=2G-1,$$

so that

$$r=4m-2.$$

If fellow-generators can be drawn from e and $g-2$ in $(\Omega)'$, we have (VIII.),

$$G_i=G-2\pm\tfrac{1}{2}r,$$

or

$$2-G=G-2\pm\tfrac{1}{2}r,$$

so that

$$r=4m$$

and

$$g-2=\frac{r}{4},$$

e being the opposite end of the diameter through $\frac{1}{2}r$.

It is thus proved that a single S′ having two leading systems is made by fellow-generators only, when r is even, and has been constructed and counted only once among the $(r+3)$-edra S. Every other S, having two leading systems, has been twice constructed from the even-angled Ω.

XXIX. The preceding reasoning from XXII. proceeds on the hypothesis that $r>5$. This restriction was made in order the more readily to prove that one summit of the triangle $\alpha\beta\gamma$ must be o. The same necessity is proved for $r=5$ thus.

In this triangle $\alpha\beta\gamma$ there is an evanescible edge, which is not $o\eta$, but equal in the united rank of its summits with $o\eta$. The only summits not triaces are the 4-aces e and η, and either two 4-aces o and o', or the pentace o.

The evanescible edge must either be oe or $e\eta$. Now when $e\eta$ is an edge $\eta=e\pm1$; that is H, the face in which f_i is $=e\pm1$. We see from the signatures

$$\mathbf{1_5 2_4 3_3 4_2 5_1},$$

that either

$$H=3 \text{ and } e=2, \text{ or } H=2 \text{ and } e=1,$$

or else

$$H=2 \text{ and } e=3, \text{ or } H=1 \text{ and } e=2,$$

in all which cases O is a pentagon and O′ a triangle. Now no pentagon but O exists in the figure; therefore no triangle $\alpha\beta\gamma$ of which o is not a summit has an edge $\alpha\gamma$, the

united rank of whose summits equals $5+4$, the united rank of o and η in the triangle $oo'\eta$. Therefore $\alpha=o$ also when $r=5$, and all the reasoning from article XX. is applicable to that value of r.

When $r=4$, we see from

$$1_4 2_3 3_2 4_1$$

that $e\eta$ is not an edge, unless either

$$H=2 \text{ and } e=1,$$

or

$$H=4 \text{ and } e=3,$$

two equivalent conditions, either of which gives the heptaedron (12_1) of art. XX., which is to be rejected on account of its being reducible to the 6-edral pyramid. The only other system, $H=2=e$, and $H=4=e$, both alike, is the S′ having two leading systems, made with fellow-generators, m being $=1$ in the formula $r=4m$. No other has two leading systems. Therefore all the reasoning of articles XXII....XXVIII. applies equally to every value of $r>3$.

The number of $(r+3)$-edra (S′) having a triangular O′ which we constructed in XIX. is exactly r, namely, half the number of generators λ, which all pair themselves into fellow-generators $e(e\pm\frac{3}{2})$. Of these we have proved that there are always three which have only a single leading system; and it has just been shown that, when r is even, there is always a fourth, having two leading but fellow systems. The number to be deducted from our enumeration on the score of their being repetitions of some other S′, is therefore

$$\tfrac{1}{2}(r-3-2_r)$$

(XXI.), to be subtracted from

$$\tfrac{1}{2}\big(r.(r-2)-2+2_{r-1}\big)-0^{(r-4)^2}.$$

The remainder is

$$\tfrac{1}{2}(r^2-3r+2)-0^{(r-4)^2}=\Pi'_r,$$

the number of $(r+3)$-edra (S) thus far known, which is to be added to $\Pi_r+\Pi'_r$ in XVIII.

XXX. It remains that we inquire how many of these (S) are repetitions of those (Q) enumerated in XVI. If Q′ one of this class (Q) is one of (S), it has an evanescible edge the united rank of whose summits is $r+3$. The highest rank of o o' or o'' in Q′, the summits into which the vertex of the pyramid was broken, is $r-2$; for the united rank of the three $\geqslant r+4$, and the smallest have together at least six edges. There must therefore be in Q′ a pentace distinct from o o' and o''; for $r+3-(r-2)=5$. Now Q has this pentace only when the generators ef and eh are drawn from one summit e. Then this evanescible edge, if it passes through one of o o' o'', the greatest of which is o, must be oe, and o' and o'' must be triaces. Again, if oe in Q′ be the evanescible edge of the leading system of S, there will be a convanescible edge through either o or e, which vanishing after oe, will make o or e an r-ace. The only summits not triaces in Q′ are o, e, h, and f; and it is impossible that the summit o after the loss of the edge oe, can unite with either h or f, which are both 4-aces, to make an r-ace; for $r-3+4<r+2$.

Therefore the convanescible edge of the S-system of Q′ must be $e(e\pm1)$. But $e\pm1\geqslant3$;

so that $r+3 \not> 5+3$, and $r \not> 5$. Also $r \not< 5$, because Q′ cannot be generated, when $r<5$. We have then to endeavour to draw *ef* and *eh* in the base of the 6-edral pyramid, so that *ef* shall be evanescible, and $e(e\pm1)$ then convanescible (for *eo* cannot vanish, since $o\,o'$ and o'' are all triaces of necessity). It would be trifling with the reader to write a demonstration of what inspection of the pentagon proves in a moment, that of the three summits which *e* may be, 1, 2, and 5, 2 and 5 alone fulfil the conditions. Draw 25 and 24; then 25 is evanescible, and by its evanescence 21 becomes convanescible. Again, draw 53 and 52; then by the evanescence of 53, 54 becomes convanescible. These two Q have therefore been enumerated both in the class (Q) and the class (S), from which latter we have then to subtract two when $r=5$. That is, we must add $-2.0^{(r-3)^2}$, to Π'_r before found in XXIX. This makes up

$$\Pi''_r=\tfrac{1}{6}(r^3-3r+2)-0^{(r-4)^2}-2.0^{(r-5)^2},$$

the correct number of the $(r+3)$-edra (S) generable from $(r+1)$-edral pyramid with nodal signatures. And thus the Problem (XIV.) is solved.

XXXI. It is yet required that we consider (VI.) the pyramid on an odd-angled base *with enodal signatures*, and determine whether, by operating upon it, we can obtain $(r+2)$-edra or $(r+3)$-edra which we have not already constructed with nodal signatures.

The nodal arrangement becomes enodal in Ω the base of the pyramid, if without disturbing those of the faces we exchange the signatures at the extremity of every nodal parallel, including among them that edge of Ω which is bisected by the *nodal axis* (VI.). After these exchanges every summit stands as the pole of the face opposite to it. And, conversely, an enodal signature of a pyramid becomes nodal, if all the diagonals parallel to any side of the enodal Ω be drawn, and then the signatures be exchanged both in that side and in these diagonals. The line bisecting that side, and passing through the opposite summit, becomes the *nodal axis*. All this will be clear to the reader, if he will have the goodness to draw a pentagon or a heptagon, and make these exchanges.

Let $r=2k+1$. All the single generators that can be drawn in the enodal Ω to generate a $(r+2)$-edral autopolar are among the parallels to the side $(k,\ k+1)$. Let $(e,\ r-e)$ be one of these: it is perpendicular to the axis of symmetry $(r,\ k+\frac{1}{2})$, which passes through *r* and bisects Ω. If E and (R−E) become 4-laterals, opposite to the 4-aces, *e*, and $(r-e)$, the edge $\big(\mathrm{E},\ (\mathrm{R}-\mathrm{E})\big)$ is the gamic of $\big(e,\ (r-e)\big)$, and the result is autopolar.

Let now the signatures be exchanged in $\big(e(r-e)\big)$ and on both sides the axis in lines parallel to $\big(e(r-e)\big)$. The result is still autopolar with nodal signatures; from which the nodally autopolar pyramid may be obtained by the evanescence of $(e,\ r-e)$ and the convanescence of (E, R−E); its nodal axis will be that axis of symmetry.

Thus it appears that every enodally autopolar $(r+2)$-edron generable from the enodal $(r+1)$-edral pyramid is also nodally autopolar, and generable from the same nodal pyramid. That is, it is one of the $(r+2)$-edra P already enumerated.

XXXII. Suppose that instead of drawing one diagonal $(e,\ r-e)$ perpendicular to the

axis of symmetry, we had drawn any pair of diagonals so as to preserve symmetry, such as two $(e, r-e)$ and $(e_1, r+e_1)$, or the pair (r, e) $(r, r-e)$, or the pair $(e, e+f)$, $(r-e, r-e-f)$; of which the first pair are parallels, the second pair meet in the pentace r, and the third pair are equidistant from the centre of the axis of symmetry on opposite sides of it; or suppose that we had drawn any number $2k$, of such lines, making k pairs, each preserving the symmetry about the axis through r. The gamic operations being performed in the faces opposite the affected summits, the symmetry would remain, and the result would be autopolar. If, next, the summits e and $(r-e)$, $(e+f)$ and $(r-e-f)$, &c., be exchanged in the parallels to $(k, k+1)$, the arrangement will become nodal and remain autopolar. The pair $(e, r-e)$ and $(e_1, r-e_1)$ do not change their names; the pair (r, e) and $(r, r-e)$ make a simple exchange of names with each other, as do also the pair $(e, e+f)$ and $(r-e, r-e-f)$. The line (r, e) of the enodal arrangement, by changing its name to $(r, r-e)$ in the nodal arrangement, is no longer the gamic to (R, E); but to (R, R−E) an edge symmetrically placed with (R, E) about the axis of symmetry through r. From these considerations, which need not be further dwelt upon, it is plain that all the enodally autopolar $(r+3)$-edra, generable by symmetrically drawn pairs of generators in the enodal Ω, are also nodal, and are consequently among the class (Q) which have been already enumerated. For every one of them, in its nodal shape, can be reduced to the nodally-signed pyramid from which the (Q) were generated.

As any bisector of the enodal Ω from any summit is an axis of symmetry, it is clear that all the parallel or symmetric pairs of generators that can be drawn in Ω, are among those just considered about the axis through r. And, for the same reason, of any pair whatever that can be drawn in the enodal Ω, one may be always assumed to be drawn from r; or, if it be more convenient, perpendicular to the axis through r; and if two are drawn from one point, or to meet in a point, this point may be assumed to be r.

It may be hereafter convenient to define autopolarity as of three kinds, nodal, enodal, and utral; understanding by the first, purely nodal, or incapable of signature without two nodal summits, and by the second, purely enodal, or incapable of signature with a nodal summit or summits. The autopolars just reviewed, which are capable of both nodal and enodal signature, are then utrally autopolar.

XXXIII. Now let two non-parallel and non-symmetrical generators in the enodal Ω be drawn. We may assume one (r, e) to be drawn from r; the other is not parallel to (r, e), nor of the same length with it, whether it meets it at r or not. If this other be (r, f), whether f be on the same side or not of the axis of symmetry with (r, e), it is impossible that by exchanging the signatures of the diagonals perpendicular to that axis, the result should be autopolar. For e and f being 4-aces not in the same perpendicular, will after the exchanges be triaces, while E and F remain 4-laterals. And if (r, e) (k, l) be the two generators, whether (k, l) is or is not perpendicular to the axis of symmetry, e, k and l cannot all, after the exchanges, be 4-aces; but one of them (e or k) will appear as a triace, while E or K remains a 4-lateral. The enodal Q made by this (r, e) and (k, f), or by this (r, e) and (k, l), is then not capable of being made nodal, and thus

reducible to the nodally-signed pyramid: it must therefore be a different $(r+3)$-edron from all the nodal (Q) before constructed. We may call it *purely enodal*.

We have then to add to the nodally autopolars (Q) all those enodally autopolars generable by a pair of diagonals of the enodal Ω, which are neither a parallel nor an equal pair. And we have to enumerate these pairs, taking care that no one is a repetition or a reflexion of another in the enodal Ω.

As any of the summits of Ω may be r, the number we are seeking cannot be greater than $\frac{1}{r}$th of all the possible pairs of not-crossing diagonals of Ω; that is, the function of r, V_r required, is not of a degree higher than the third. And it is evident that it must vanish, for $r=3$ and $r=5$, and must be

$$V_r=(r-3)(r-5)(ar+b).$$

By trial we find readily that

$$V_7=4.2.(7a+b)=2,$$

and

$$V_9=6.4.(9a+b)=8,$$

V_7 being the two pairs (72, 73) and (72, 74); and V_9 being (92, 93), (92, 94), (92, 95), (92, 96), (92, 84), (92, 85), (93, 94), (93, 95). Hence comes

$$V_r=\tfrac{1}{24}.(r-3)(r-5)(r-1).$$

And this is the number of *purely enodal* $(r+3)$-edra (Q), which are generable from the $(r+1)$-edral pyramid (when r is odd only). That is, we have to join to our previous enumeration the number

$$(Q)''=\tfrac{1}{24}.(r-3)(r-5)(r-1).2_{r-1},$$

to be added to $\Pi_{ii}+\Pi'_{ii}+\Pi''_{ii}$ of XVIII., XXIX., XXX.

XXXIV. We have yet to determine whether any purely enodal $(r+3)$-edra S can be generated by a line drawn from any summit to the mid-point of any edge of Ω.

When $r=4k+1$, the enodal signatures read thus:

$$\begin{array}{llllll} \ldots\,\mathrm{R} & (\mathrm{R}-1), & \ldots & (\mathrm{R}-\mathrm{K}) & (\mathrm{R}-\mathrm{K}-1) & \ldots \\ \ldots\ 2k & (2k-1)\ldots(k+1) & k & : & k-1 & \ldots \end{array}$$

Let an axis of symmetry be drawn through r; and let the generator $(k+1, f_i)$ be drawn from $(k+1)$ to the mid-point of $(k, k-1)$. $\big(\mathrm{R}-(\mathrm{K}+1)\big)$ and $(\mathrm{K}+1)$ are now quadrilaterals, and $\big(r-(k+1)\big)$ and $(k+1)$ are tessaraces, if the gamic operations be completed; and these tessaraces are upon the diagonal perpendicular to the axis of symmetry, being at equal distances from r. We have the triangles and triaces

$$\mathrm{O}'=(k+1,\ k,\ f_i), \qquad \mathrm{K}=(o',\ r-k,\ r-k-1),$$
$$o' =(\mathrm{K}+1,\ \mathrm{K},\ \mathrm{F}_i), \qquad k =(\mathrm{O}',\ \mathrm{R}-\mathrm{K},\ \mathrm{R}-\mathrm{K}-1),$$

F_i being the triangle polar to f_i, introduced between K and K-1. Leaving now the signatures of the faces undisturbed, let $r-x$ be exchanged for x, upon the diagonals

perpendicular to the axis of symmetry, except that $r-k$ shall be exchanged for f_i; and let o' be exchanged for k. The result will be a nodally autopolar $(r+3)$-edron S, having the triangles and triaces

$$\begin{aligned} O'&=(r-k-1,\ o',\ r-k), & K&=(k,\ f_i,\ k+1),\\ k&=(K+1,\ K,\ F_i), & o'&=(O',\ R-K,\ R-K-1), \end{aligned}$$

in which o' and k are evidently nodal summits.

When $r=4k+3$, the enodal pyramid reads thus:

$$\underset{2k+1}{\ldots R} \quad \underset{2k\ldots}{(R-1)} \quad \underset{\vdots}{\ldots(R-K)} \quad \underset{k+1}{(R-K-1)} \quad \underset{k\ldots}{\ldots}$$

The axis of symmetry being drawn through r, let the generator (k, f_i) be drawn from k to the mid-point of $(k+1, k+2)$. We have now the 4-laterals K and R−K, and the triaces k and $r-k$. The figure has the triangles and triaces

$$\begin{aligned} O'&=(k,\ k+1,\ f_i), & K+1&=(o',\ r-k,\ r-k-1),\\ o'&=(K,\ K+1,\ F_i), & k+1&=(O',\ R-K,\ R-K-1). \end{aligned}$$

If we now exchange upon the diagonals perpendicular to the axis of symmetry $r-e$ for e, except only that $r-k-1$ is to be exchanged for f_i, and at the same time exchange o' and $k+1$, we have again a nodally autopolar (S), having the triangles and triaces

$$\begin{aligned} O'&=(r-k,\ o',\ r-k-1), & K+1&=(k+1,\ k,\ f_i),\\ o'&=(R-K,\ O',\ R-K-1), & k+1&=(K+1,\ K,\ F_i); \end{aligned}$$

in which o' and $k+1$ are the nodal summits.

Since all summits of the enodal Ω are alike, as origins of generators, no generators can be drawn to make an r-gonal O and a triangular O′ different from these; therefore no *purely enodal* $(r+3)$-edron can be constructed to have such an O and O′.

If O′=O, the generator $(e, f+\frac{1}{2})$ is the axis of symmetry through e, and all the summits are triaces, except the pentace e. In this case the resulting $(r+3)$-edron is instantly made nodally autopolar, by the exchange of signatures on each side of the axis, which becomes the nodal axis. The gentle reader will kindly assist my demonstrations in this new and somewhat intricate subject, by drawing a 7-gon and a 9-gon, and joining the angles in the required way to two included summits o and o'.

XXXV. But when O′ is not a triangle, nor of equal rank with O, it becomes impossible to give to the S generated by the line $(e, f+\frac{1}{2})$ a nodal arrangement; for there is no summit with which o' can be exchanged. If f_i is a point of the face H, f_i and η are tessaraces which may take each other's place, but o', whether tessarace or m-ace, stands alone and immoveable. Consequently, the resulting $(r+3)$-edron is of *purely enodal autopolarity*, and, with all the other enodal (S) having no triangular O′, is to be added to our enumeration. The number of different generators $(r, f+\frac{1}{2})$ is $\frac{1}{2}(r-5)$, f receiving every value from $f=2$ to $f=\frac{1}{2}(r-3)$: this $\frac{1}{2}(r-5)$, the number of purely enodal auto-

polars (S), having no r-gonal O, is to be added to (Q)$''$ in XXXIII., completing the list of purely enodal autopolars. And thus the question of (XXXI.) is determined.

We have exhausted all the methods of adding two faces and two summits to the $(r+1)$-edral pyramid. By adding two faces in the base Ω, and their two poles about the vertex ω, we formed the $(r+3)$-edra (Q). By adding a face and a summit in Ω, and their polars about ω, we formed the $(r+3)$-edra (S).

Collecting, now, our results from XIII., XVIII., XXIX., XXX., XXXIII., we find the numbers Π, of $(r+2)$-edra and Π_3 of $(r+3)$-edra, nodal, enodal, and utral, which are generable from the $(r+1)$-edral pyramid, to be the following:

$$\Pi_1=\tfrac{1}{4}\{r^3-3r+2.4_{r-3}+(r-3).2_{r-1}\};$$

$$\Pi_2=\tfrac{1}{24}\{r^3(r-4)(r-2)\}2_r$$
$$+\tfrac{1}{48}\{2r.(r-3)(r-4)(r+1)-3.(r-3)(r-1)+3.(r-3)(r-3)\}2_{r-1}$$
$$+\tfrac{1}{3}(r^3-3r+2)+\tfrac{1}{24}(r-3)(r-5)(r-1).2_{r-1}+\tfrac{1}{8}(r-5).2_{r-1}-0^{(r-4)^3.(r-7)^3}-2.0^{(r-5)^3},$$

or

$$\Pi_2=\tfrac{1}{24}\{(r^4-6r^3+20r^2-36r+24)2_r+(r-3)(r^3-2r^2+5r+8)2_{r-1}-0^{(r-4)^3(r-7)^3}-2.0^{(r-5)^3}\}.$$

It has been proved that these Π_1 and Π_2, and these only, are the autopolar $(r+2)$-edra and $(r+3)$-edra generable from the $(r+1)$-edral pyramid, that none of them is reducible by vanescible pairs to a higher pyramid, nor any one a repetition of another.

The problem of enumeration of the x-edra may, by a slight extension of the meaning of the word partition, be stated thus: *to find the* k-*partitions of a pyramid.* This depends on another: *to find the* k-*partitions of a polygon*; which is also thus: *to find the* k-*partitions of a pencil.* By the k-partitions of a p-gon, I mean the number of ways, none a repetition or reflexion of another, in which k lines can be drawn in a p-gon, none crossing another, so as to make the system of 1 face and p summits into a system of $h+1$ faces and $p+i$ summits $(h+i=k)$, the k lines being terminated either by summits of the p-gon, or by i points chosen either on its edges or within its area; with the understanding that at least three lines shall meet in each of the i points, two of which will always be a side and its segment, when i is chosen on a side, the segment counting among the k lines.

Whenever a p-gon and a p-ace are similarly k-partitioned, a certain number of autopolar $(p+k+1)$-edra are obtained by different ways of applying the p-ace to the p-gon; when they are dissimilarly k-partitioned, or when the p-gon is k-partitioned and the p-ace $(k\pm l)$-partitioned, a certain number of heteropolars will arise from different ways of applying the p-ace to the p-gon. The direct or reverse manner of applying the p-ace to the p-gon will give nodal or enodal autopolars.

The way seems now clearly indicated, and partly laid open to the solution of a geometrical problem, which, while it seems at first sight almost elementary, has lain for centuries before mathematicians unanswered. *The enumeration of the* x-*edra* is a question of

partition, and the *k*-partitions of a polygon, a pencil and a pyramid, will probably be found still more inaccessible than our new and unamiable acquaintances, *the commensurable* k*-partitions of a line.*

XXXVI. It may be useful to give a list of the autopolar *n*-edra as far as $n=8$, which are not pyramids.

The only such autopolar hexaedron is obtained by drawing 13 in the 4-diagonal Ω, dividing it into the triangles 6 and 5. Its ten edges are

$$\begin{array}{lllll} 4_1\mathbf{3}_4\mathbf{3}_2\mathbf{3}_6 & 3_2\mathbf{4}_2\mathbf{4}_3\mathbf{3}_6 & 4_3\mathbf{3}_2\mathbf{3}_4\mathbf{3}_5 & 3_4\mathbf{4}_1\mathbf{4}_1\mathbf{3}_5 & 3_6\mathbf{4}_3\mathbf{3}_5\mathbf{4}_1 \\ \mathbf{4}_1\mathbf{3}_4\mathbf{3}_2\mathbf{3}_6 & \mathbf{3}_2\mathbf{4}_2\mathbf{4}_3\mathbf{3}_6 & \mathbf{4}_3\mathbf{3}_2\mathbf{3}_4\mathbf{3}_5 & \mathbf{3}_4\mathbf{4}_1\mathbf{4}_1\mathbf{3}_5 & \mathbf{3}_6\mathbf{4}_3\mathbf{3}_5\mathbf{4}_1, \end{array}$$

where the heavier type denotes the faces, and the lighter the summits. The leading system are the final pair of gamics, of which the first is convanescible and the second evanescible.

There are three autopolar 7-edra of the class P (VII.), obtained by drawing the generators 13, 14 and 25 in the pentagonal base Ω, all of the class P. These are—

$$\left.\begin{array}{lllll} 4_1\mathbf{3}_5\mathbf{3}_2\mathbf{3}_7 & 3_2\mathbf{3}_4\mathbf{4}_2\mathbf{3}_7 & 4_2\mathbf{4}_3\mathbf{3}_4\mathbf{4}_6 & 3_4\mathbf{3}_2\mathbf{3}_5\mathbf{4}_6 & 3_5\mathbf{4}_1\mathbf{4}_1\mathbf{4}_6 \quad 3_7\mathbf{4}_3\mathbf{4}_6\mathbf{4}_1 \\ \mathbf{4}_1\mathbf{3}_5\mathbf{3}_2\mathbf{3}_7 & \mathbf{3}_2\mathbf{3}_4\mathbf{4}_2\mathbf{3}_7 & \mathbf{4}_2\mathbf{4}_3\mathbf{3}_4\mathbf{4}_6 & \mathbf{3}_4\mathbf{3}_2\mathbf{3}_5\mathbf{4}_6 & \mathbf{3}_5\mathbf{4}_1\mathbf{4}_1\mathbf{4}_6 \quad \mathbf{3}_7\mathbf{4}_2\mathbf{4}_6\mathbf{4}_1 \end{array}\right\} \quad (13)$$

$$\left.\begin{array}{lllll} 4_1\mathbf{3}_5\mathbf{3}_2\mathbf{4}_6 & 3_2\mathbf{4}_4\mathbf{3}_3\mathbf{4}_6 & 3_3\mathbf{3}_3\mathbf{4}_4\mathbf{4}_6 & 4_4\mathbf{3}_2\mathbf{3}_3\mathbf{3}_7 & 3_3\mathbf{4}_1\mathbf{4}_1\mathbf{3}_7 \quad 4_6\mathbf{4}_4\mathbf{3}_7\mathbf{4}_1 \\ \mathbf{4}_1\mathbf{3}_5\mathbf{3}_2\mathbf{4}_6 & \mathbf{3}_2\mathbf{4}_4\mathbf{3}_3\mathbf{4}_6 & \mathbf{3}_3\mathbf{3}_3\mathbf{4}_4\mathbf{4}_6 & \mathbf{4}_4\mathbf{3}_2\mathbf{3}_3\mathbf{3}_7 & \mathbf{3}_3\mathbf{4}_1\mathbf{4}_1\mathbf{3}_7 \quad \mathbf{4}_6\mathbf{4}_4\mathbf{3}_7\mathbf{4}_1 \end{array}\right\} \quad (14)$$

$$\left.\begin{array}{lllll} 3_1\mathbf{4}_5\mathbf{4}_2\mathbf{3}_7 & 4_2\mathbf{3}_4\mathbf{3}_2\mathbf{4}_6 & 3_3\mathbf{3}_3\mathbf{4}_4\mathbf{4}_6 & 3_4\mathbf{4}_2\mathbf{4}_5\mathbf{4}_6 & 4_5\mathbf{3}_1\mathbf{3}_1\mathbf{3}_7 \quad 3_6\mathbf{4}_2\mathbf{4}_7\mathbf{4}_5 \\ \mathbf{3}_1\mathbf{4}_5\mathbf{4}_2\mathbf{3}_7 & \mathbf{4}_2\mathbf{3}_4\mathbf{3}_2\mathbf{4}_6 & \mathbf{3}_3\mathbf{3}_3\mathbf{4}_4\mathbf{4}_6 & \mathbf{3}_4\mathbf{4}_2\mathbf{4}_5\mathbf{4}_6 & \mathbf{4}_5\mathbf{3}_1\mathbf{3}_1\mathbf{3}_7 \quad \mathbf{3}_6\mathbf{4}_2\mathbf{4}_7\mathbf{4}_5 \end{array}\right\} \quad (25)$$

the last two of which are in order the 7-edra $(13_{,})$ and $(12_{,})$ of art. XX. They have all three quadrilaterals and four triangles.

Two more of the class S are generated from the 5-edral pyramid by drawing 25 to the mid-point of 14, and 25 to that of 34; viz.

$$\left.\begin{array}{lllll} 4_1\mathbf{3}_4\mathbf{4}_2\mathbf{3}_7 & 4_2\mathbf{3}_3\mathbf{3}_2\mathbf{4}_6 & 3_3\mathbf{4}_2\mathbf{3}_4\mathbf{4}_6 & 3_4\mathbf{4}_1\mathbf{3}_5\mathbf{4}_6 & 3_5\mathbf{4}_1\mathbf{4}_1\mathbf{3}_6 \quad 3_7\mathbf{4}_2\mathbf{4}_6\mathbf{3}_5 \\ \mathbf{4}_1\mathbf{3}_4\mathbf{4}_2\mathbf{3}_7 & \mathbf{4}_2\mathbf{3}_3\mathbf{3}_2\mathbf{4}_6 & \mathbf{3}_3\mathbf{4}_2\mathbf{3}_4\mathbf{4}_6 & \mathbf{3}_4\mathbf{4}_1\mathbf{3}_5\mathbf{4}_6 & \mathbf{3}_5\mathbf{4}_1\mathbf{4}_1\mathbf{3}_6 \quad \mathbf{3}_7\mathbf{4}_2\mathbf{4}_6\mathbf{3}_5 \end{array}\right\} \quad (2, 4\tfrac{1}{2})$$

$$\left.\begin{array}{lllll} 3_1\mathbf{4}_5\mathbf{5}_2\mathbf{4}_6 & 5_2\mathbf{3}_3\mathbf{3}_3\mathbf{3}_7 & 3_3\mathbf{5}_2\mathbf{3}_5\mathbf{3}_7 & 3_5\mathbf{5}_2\mathbf{3}_4\mathbf{4}_6 & 3_4\mathbf{3}_1\mathbf{3}_1\mathbf{4}_6 \quad 4_6\mathbf{5}_2\mathbf{3}_7\mathbf{3}_5 \\ \mathbf{3}_1\mathbf{4}_5\mathbf{5}_2\mathbf{4}_6 & \mathbf{5}_2\mathbf{3}_3\mathbf{3}_3\mathbf{3}_7 & \mathbf{3}_3\mathbf{5}_2\mathbf{3}_5\mathbf{3}_7 & \mathbf{3}_5\mathbf{5}_2\mathbf{3}_4\mathbf{4}_6 & \mathbf{3}_4\mathbf{3}_1\mathbf{3}_1\mathbf{4}_6 \quad \mathbf{4}_6\mathbf{5}_2\mathbf{3}_7\mathbf{3}_5 \end{array}\right\} \quad (2, 3\tfrac{1}{2})$$

Of these two, which are identical with (23′) and (24′) of art. XX., the first has three quadrilaterals and four triangles, while the second has a pentagon, a quadrilateral, and four triangles. In all these 7-edra 6 and 7 are the faces into which the base Ω is divided.

XXXVII. Of autopolar 8-edra there are five generable from the 7-edral pyramid, by drawing 13, 15, 35, 14, 36, in the 6-gonal Ω, all of the class P.

$4_1 3_6 3_2 3_8 \quad 3_2 3_5 4_2 3_8 \quad 4_2 3_3 4_5 5_7 \quad 3_4 4_3 3_5 5_7 \quad 3_5 3_2 3_6 5_7 \quad 3_8 4_1 4_1 5_7 \quad 5_7 4_1 3_8 4_3$
$4_1 3_6 3_2 3_8 \quad 3_2 3_5 4_2 3_8 \quad 4_2 3_3 4_5 5_7 \quad 3_4 4_3 3_5 5_7 \quad 3_5 3_2 3_6 5_7 \quad 3_8 4_1 4_1 5_7 \quad 5_7 4_1 3_8 4_3$ (13)

$4_1 3_6 3_2 5_7 \quad 3_2 4_2 3_5 5_7 \quad 3_3 3_4 3_4 5_7 \quad 3_4 3_3 4_2 5_7 \quad 4_2 3_2 3_6 3_8 \quad 3_6 4_1 4_1 3_8 \quad 3_8 4_1 5_7 4_2$
$4_1 3_6 3_2 5_7 \quad 3_2 4_2 3_2 5_7 \quad 3_3 4_3 3_4 5_7 \quad 3_4 3_3 4_2 5_7 \quad 4_2 3_2 3_6 3_8 \quad 3_6 4_1 4_1 3_8 \quad 3_8 4_1 5_7 4_2$ (15)

$3_1 3_6 3_2 5_7 \quad 3_2 4_2 4_2 5_7 \quad 4_2 3_4 3_4 3_8 \quad 3_4 4_2 4_2 3_8 \quad 4_2 3_2 3_6 5_7 \quad 3_6 3_1 3_1 5_7 \quad 5_7 4_2 3_8 4_2$
$3_1 3_6 3_2 5_7 \quad 3_2 4_2 4_2 5_7 \quad 4_2 3_4 3_4 3_8 \quad 3_4 4_2 4_2 3_8 \quad 4_2 3_2 3_6 5_7 \quad 3_6 3_1 3_1 5_7 \quad 5_7 4_2 3_8 4_2$ (35)

$4_1 3_6 3_2 4_8 \quad 3_2 3_5 3_2 4_8 \quad 3_2 4_2 4_2 4_8 \quad 4_2 3_3 3_2 4_7 \quad 3_5 3_2 3_6 4_7 \quad 3_6 4_1 4_1 4_7 \quad 4_7 4_1 4_8 4_4$
$4_1 3_6 3_2 4_8 \quad 3_2 3_5 3_2 4_8 \quad 3_2 4_2 4_2 4_8 \quad 4_2 3_3 3_2 4_7 \quad 3_5 3_2 3_6 4_7 \quad 3_6 4_1 4_1 4_7 \quad 4_7 4_1 4_8 4_4$ (14)

$3_1 4_6 3_2 4_7 \quad 3_2 3_5 4_2 4_7 \quad 4_2 3_4 3_4 4_8 \quad 3_4 4_2 3_2 4_8 \quad 3_5 3_2 4_6 4_8 \quad 4_6 3_1 3_1 4_7 \quad 4_7 4_2 4_8 4_6$
$3_1 4_6 3_2 4_7 \quad 3_2 3_5 4_2 4_7 \quad 4_2 3_4 3_4 4_8 \quad 3_4 4_2 3_2 4_8 \quad 3_5 3_2 4_6 4_8 \quad 4_6 3_1 3_1 4_7 \quad 4_7 4_2 4_8 4_6$ (36)

XXXVIII. There are also seven autopolar 8-edra generable from the 6-edral pyramid, viz. three of the class Q, generated by drawing 25 and 24, 13 and 14, 52 and 53, in the pentagon Ω, and four of the class S. The Q are these:—

$3_1 4_5 5_2 3_6 \quad 5_2 4_4 3_2 3_8 \quad 3_3 3_4 4_3 3_8 \quad 4_4 5_2 4_2 3_7 \quad 4_5 3_1 3_1 3_6 \quad 3_8 4_4 3_7 5_2 \quad 3_7 4_5 3_6 5_2$
$3_1 4_5 5_2 3_6 \quad 5_2 4_4 3_2 3_8 \quad 3_3 3_4 4_3 3_8 \quad 4_4 5_2 4_2 3_7 \quad 4_5 3_1 3_1 3_6 \quad 3_8 4_4 3_7 5_2 \quad 3_7 4_5 3_6 5_2$ (24, 25)

$5_1 3_5 3_2 3_6 \quad 3_2 4_4 4_2 3_6 \quad 4_2 3_4 4_3 3_7 \quad 4_4 3_2 3_5 3_8 \quad 3_5 5_1 5_1 3_8 \quad 3_5 5_1 3_7 4_4 \quad 3_7 5_1 3_6 4_3$
$5_1 3_5 3_2 3_6 \quad 3_2 4_4 4_2 3_6 \quad 4_2 3_4 4_3 3_7 \quad 4_4 3_2 3_5 3_8 \quad 3_5 5_1 5_1 3_8 \quad 3_8 5_1 3_7 4_4 \quad 3_7 5_1 3_6 4_3$ (13, 14)

$3_4 5_5 4_2 3_6 \quad 4_2 3_4 4_2 3_7 \quad 4_2 4_3 3_4 3_8 \quad 3_4 4_2 5_2 3_8 \quad 5_2 3_1 3_1 3_6 \quad 3_6 4_2 3_7 5_5 \quad 3_7 4_2 3_8 5_5$
$3_1 5_5 4_2 3_6 \quad 4_2 3_4 4_2 3_7 \quad 4_2 4_3 3_4 3_8 \quad 3_4 4_2 5_2 3_8 \quad 5_2 3_1 3_1 3_6 \quad 3_6 4_2 3_7 5_5 \quad 3_7 4_2 3_8 5_5$ (52, 53)

The autopolar 8-edra of the class S, which are distinct from the preceding, are the following four, obtained by the generators 16, 46, 56, all drawn to the mid-point of 23, and 16 drawn from 1 to the mid-point of 34, in the pentagon Ω.

$4_1 3_5 3_2 3_7 \quad 3_2 4_2 3_6 3_7 \quad 3_6 4_2 3_5 5_8 \quad 3_3 3_2 4_4 5_8 \quad 4_4 3_2 3_5 5_8 \quad 3_5 4_1 4_1 5_8 \quad 3_7 3_6 5_8 4_1$
$4_1 3_5 3_2 3_7 \quad 3_2 4_4 3_6 3_7 \quad 3_6 4_4 3_5 5_8 \quad 3_3 3_2 4_4 5_8 \quad 4_4 3_2 3_5 5_8 \quad 3_5 4_1 4_1 5_8 \quad 3_7 3_6 5_8 4_1$ (1, $2\frac{1}{2}$)

$3_1 3_5 3_2 5_8 \quad 3_2 5_4 3_6 5_8 \quad 3_6 5_4 3_3 3_7 \quad 3_3 3_2 5_4 3_7 \quad 5_4 3_2 3_5 5_8 \quad 3_5 3_1 3_1 5_8 \quad 5_8 3_6 3_7 5_4$
$3_1 3_5 3_2 5_8 \quad 3_2 5_4 3_6 5_8 \quad 3_6 5_4 3_3 3_7 \quad 3_3 3_2 5_4 3_7 \quad 5_4 3_2 3_5 5_8 \quad 3_5 3_1 3_1 5_8 \quad 5_8 3_6 3_7 5_4$ (4, $2\frac{1}{2}$)

$3_1 4_5 3_2 4_8 \quad 3_2 4_4 3_6 4_8 \quad 3_6 4_4 3_3 4_7 \quad 3_3 3_2 4_4 4_7 \quad 4_4 3_2 4_5 4_7 \quad 4_5 3_1 3_1 4_8 \quad 4_8 3_6 4_7 4_5$
$3_1 4_5 3_2 4_8 \quad 3_2 4_4 3_6 4_8 \quad 3_6 4_4 3_3 4_7 \quad 3_3 3_2 4_4 4_7 \quad 4_4 3_2 4_5 4_7 \quad 4_5 3_1 3_1 4_8 \quad 4_8 3_6 4_7 4_5$ (5, $2\frac{1}{2}$)

$4_1 3_5 3_2 4_8 \quad 3_2 3_4 4_2 4_8 \quad 4_2 4_3 3_6 4_8 \quad 3_6 4_2 3_4 4_7 \quad 3_4 3_2 3_5 4_7 \quad 3_5 4_1 4_1 4_7 \quad 4_7 4_1 4_8 3_6$
$4_1 3_5 3_2 4_8 \quad 3_2 3_4 4_2 4_8 \quad 4_2 4_3 3_6 4_8 \quad 3_6 4_2 3_4 4_7 \quad 3_4 3_2 3_5 4_7 \quad 3_5 4_1 4_1 4_7 \quad 4_7 4_1 4_8 3_6$ (1, $3\frac{1}{2}$)

In all these 7 and 8 are the faces into which Ω is divided.

There are yet three autopolar 8-edra not included in the classes discussed in this memoir, being $(r+4)$-edra generable by gamic pairs from the $(r+1)$-edral pyramid. The first is obtained by drawing in the 4-gonal Ω, 42, and then 45 to the middle of 12; the second by drawing 24 and then 38 to the middle of 24; the third by joining 65, in the mid-points of 14 and 12, as follows:—

$$\left.\begin{array}{llllllll} 3_1\mathbf{6}_4 3_5 3_6 & 3_5\mathbf{6}_4 4_2 3_7 & 4_2 3_3 3_3 3_8 & 3_3\mathbf{4}_2 6_4 3_8 & 6_4 3_1 3_1 3_6 & 3_6 3_5 3_7 6_4 & 3_7\mathbf{4}_2 3_8 6_4 \\ 3_1 6_4 3_5 3_6 & 3_5 6_4 \mathbf{4}_2 3_7 & \mathbf{4}_2 3_3 3_3 3_8 & 3_3 4_2 \mathbf{6}_4 3_8 & \mathbf{6}_4 3_1 3_1 3_6 & 3_6 3_5 \mathbf{3}_7 6_4 & 3_7 4_2 \mathbf{3}_8 6_4 \end{array}\right\}(42, 41\tfrac{1}{4})$$

$$\left.\begin{array}{llllllll} 3_1\mathbf{4}_4 4_2 \mathbf{4}_5 & 4_2\mathbf{4}_2 4_3 3_6 & 4_3\mathbf{4}_2 4_4 3_7 & 3_6\mathbf{4}_2 3_7 3_8 & 3_7\mathbf{4}_4 4_5 3_8 & 4_5\mathbf{4}_2 3_6 3_8 & 4_4\mathbf{3}_1 3_1 \mathbf{4}_5 \\ 3_1 4_4 \mathbf{4}_2 4_5 & 4_2 4_2 \mathbf{4}_3 3_6 & 4_3 4_2 \mathbf{4}_4 3_7 & 3_6 4_2 3_7 3_8 & 3_7 4_4 4_5 3_8 & 4_5 4_2 3_6 3_8 & 4_4 3_1 3_1 4_5 \end{array}\right\}(83,\ 82,\ 84)$$

$$\left.\begin{array}{llllllll} 4_1\mathbf{4}_4 3_5 3_7 & 3_5\mathbf{4}_4 3_2 5_8 & 3_2 3_3 3_3 5_8 & 3_3 3_2 4_4 5_8 & 4_4\mathbf{4}_1 3_6 5_8 & 3_6\mathbf{4}_1 4_1 3_7 & 3_7 3_5 5_8 3_8 \\ \mathbf{4}_1 4_4 3_5 3_7 & 3_5 4_4 3_2 5_8 & 3_2 3_3 3_3 5_8 & 3_3 3_2 4_4 5_8 & 4_4 4_1 3_6 5_8 & 3_6 4_1 4_1 3_7 & 3_7 3_5 5_8 3_6 \end{array}\right\}(1\tfrac{1}{4},\ 4\tfrac{1}{4})$$

Thus we find that there are one autopolar 6-edron, five autopolar 7-edra, and fifteen autopolar 8-edra, besides the three pyramids.

XXXIX. We may, for example, effect the reduction of the three last written 8-edra to the 5-edral pyramid. The only vanescible pair in the first is the seventh pair. As it stands, the summits and their signatures read thus,

$$3_1 4_2 3_3 6_4 3_5 3_6 3_7 3_8 :$$

after the disappearance of that pair, which unites the seventh and eighth summits and degrades the second and fourth, they read

$$3_1 3_2 3_3 5_4 3_5 3_6 \mathbf{4}_{7\,8}.$$

The second pair are now vanescible and next vanish, and we read thus,

$$3_1 4_{2\,5} 3_3 4_4 3_6 3_{7\,8},$$

after which the vanishing of the sixth pair gives us

$$3_1 3_{2\,5} 3_3 3_4 4_{6\,7\,8},$$

the pyramid on the base 1234.

But if we do not insist on the vanescence of gamic pairs, and on an autopolar result, we can reduce the 8-edron readily to the 7-edral pyramid, by the evanescence of $\mathbf{3}_7 4_2 \mathbf{3}_8 6_4$, which makes the face $\mathbf{4}_{7\,8}$, and then the convanescence of $3_3 \mathbf{4}_2 6_4 \mathbf{4}_8$; but the first of these two steps gives a heteropolar result. Generally any x-edron that has at most a p-ace or a p-gon can be reduced by single vanescences to the $(p+1)$-edral pyramid.

In the second of the above three 8-edra, the first pair, though at first sight it appears a vanescible pair, is not so. For putting s_n and f_n for the summit and face signed n, we see in the eighth and second pairs, or the nodal pairs, s_1 in f_1, and s_2 in f_2, s_3 in f_3, and s_4 in f_4. If now $\mathbf{3}_1 4_4 \mathbf{4}_2 4_5$ evanesces, we have $f_1 = f_2$, whence, s_3 is in f_3 through s_2, and s_2 in f_1 through s_1, so that $3_1 \mathbf{4}_4 4_2 \mathbf{4}_5$ is no longer convanescible, f_1 and f_3 through s_1 and s_2 being covertical, and in fact collateral faces.

Hence the only vanescible pairs are the second and third. As they stand, the summits read thus with their signatures,

$$3_1 4_2 4_3 4_4 4_5 3_6 3_7 3_8.$$

The vanescence of the third pair gives us the reading

$$3_1 5_{27} 3_3 3_4 4_5 3_6 3_8;$$

and now the fourth pair has become the only vanescible one, giving the reading

$$3_1 4_{27} 3_3 4_{48} 3_5 3_6,$$

which by the vanescence of the sixth pair becomes

$$3_1 3_{27} 3_3 3_{48} 4_{56},$$

the pyramid on the base 1234.

If we had begun by making the second pair vanish,

$$3_1 4_2 4_3 4_4 4_5 3_6 3_7 3_8$$

would have reduced to

$$3_1 3_2 4_{36} 4_4 4_5 3_7 3_8,$$

for the third summit losing both the edges of the nodal angle, 36 and 32, can unite with the sixth to form a 4-ace only. The sixth pair vanishing, gives us

$$3_1 4_{28} 3_{36} 4_4 3_5 3_7,$$

which reduces, as before, to the pyramid on the base 1234 by the vanescence of the fifth pair.

The vanescible pairs of the third 8-edron are the second and the fifth, which give us two ways of reducing it to the pyramid on the base 1234. Or we can reduce it by the evanescence of $\mathbf{3}_5 4_4 \mathbf{3}_2 5_8$, $\mathbf{4}_4 4_1 \mathbf{3}_6 4_8$, and convanescence of $3_4 \mathbf{4}_1 3_6 \mathbf{5}_8$ and $3_3 \mathbf{4}_2 3_4 \mathbf{4}_8$ to the pyramid on pentagonal base.

XL. All the autopolars above given can be represented by square paradigms, showing all the faces, summits, edges and angles of the figure; for a closed polygon can be drawn through the summits of any of them. For example, the three last written have through their summits the circles 15238746, 15672384, 15238746; *i. e.* the duads of these circles occur among the non-contiguous duads of the subindices. If these circles be employed as directed in my paper "On the Representation of Polyedra," in the volume for 1856 of the Transactions of the Royal Society, the paradigms are easily written out. But I do not find that any mode of representation is so simple and distinguishing as that above given in art. XXXVI. and those following it.

The figures in Plates XII., XIII., XIV., are intended to illustrate the Articles of this Memoir to which reference is made in them.

XII. *On the* K-*partitions of the* R-*gon and* R-*ace.*

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I. By the k-*partitions of an* r-*gon*, I mean the number of ways in which it can be divided by $k-1$ diagonals, of which none crosses another; two ways being different only when no cyclical permutation or reversion of the numbers 1 2 3..r at the angles can make them alike: and by the k-*partitions of an* r-*ace* (a pencil of r rays in space or a plane), I mean the number of ways in which it can be divided into k smaller pencils, by the introduction of $k-1$ connecting lines, of which none enclose a space; two ways being different only when by no cyclical permutation or reversion of the numbers 1 2 3..r in the angular spaces of the r-ace they can be made identical. The polygon here considered is the section of a pyramid, and its discussion includes that of the polyace.

The enumeration of the partitions of the polygon and polyace is indispensable in the theory of the polyedra. In a memoir "On the x-edra which have an $(x-1)$-gonal base, and all their Summits Triedral," in the Transactions of the Royal Society, 1856, page 399, I have investigated the $(r-2)$-partitions of the r-ace, or the r-gon; for the number of x-edra there determined is exactly that of these $(r-2)$-partitions. What follows may be considered as a completion of the investigation in that memoir begun; yet not properly a continuation, inasmuch as the results there obtained are here deduced by a different and more general method.

II. A partition of an r-gon is *reversible* or *irreversible*: reversible, when it is symmetrical about a diameter or bisector of the figure, so that the configuration is unaltered by a semirevolution about that line, which is called *an axis of reversion*, of which axes there may be one or many; and *irreversible*, when it is reversible about no axis. An irreversible is about no axis its own reflexion.

An axis of reversion is always a bisector of the r-gon, and is *agonal*, *monogonal*, or *diagonal*, according as it passes through no angle, one angle only, or two angles of the r-gon; and the polygon is said to be about that axis, *agonally*, *monogonally*, or *diagonally reversible*.

A diagonal axis may be drawn or undrawn; a monogonal or agonal axis is always undrawn.

III. A partition is said to be m-ly reversible when it has m axes of reversion. The simple $2n$-gon ($k=1$) is $2n$-ly reversible, having n agonal and n diagonal axes; and its sides may be so loaded with polygons, that this number of axes shall be either retained or diminished. The simple $(2n+1)$-gon has $2n+1$ monogonal axes, on which an r-gon

may be built to have the same or a smaller number. To help our conceptions, we may always suppose our r-gon regularly inscribed in a circle; but it is evident that the *syntypy* of two identically-partitioned r-gons in no wise depends on such *symmetry*, but may remain after any distortion of either r-gon which does not change the angles on any diagonal. So that, if we wish to build an r'-gon on an inscribed r-gon, we need not fear exceeding the limits of the circle by our additions, while we may *suppose* these all contained within it.

A partition is said to be m-ly irreversible, when it has an irreversible sequence of configuration m times repeated in the circuit of the r-gon. This sequence will occupy $\frac{r}{m}$ angles; and, from whatever angle we begin to read, we shall see a sequence of $\frac{r}{m}$ sides irreversible, such that through the mid-point of it no axis of reversion can be drawn.

Obs. 1. Hence a $2m$-ly irreversible has an irreversible sequence, simple if $m=1$, and m-ple if $m>1$, occupying half the circuit of the r-gon; but a $(2m+1)$-ly irreversible has no repeated sequence occupying half its circuit.

IV. THEOREM A. *Every reversible* $(1+k)$*-partition of an* r*-gon has two reversible sequences of configuration which are bisected by alternate and equidistant axes of reversion, and has not more than two, whatever be the number of these axes.*

For, first, let there be only one axis of reversion in the r-gon: there must be two aspects of configuration observable from opposite ends of that axis, otherwise the figure would be reversible about a perpendicular to that axis, *i. e.* there would be two axes, contrary to hypothesis.

Secondly, let there be more axes of reversion than one; any axis a bisects an aspect A, because the figure is unchanged by a semirevolution about that axis; and the axis b next in order to a along the circumference bisects an aspect B. This B is different from A; for if not, the series of configurations read from a to b will be that read from b to a, and there will be either a vertex or a side centrally placed between a and b, having on both sides the same aspect, or an axis of reversion can be drawn between a and b; but b is the next in order to a, which is absurd; therefore B is not A. Now b bisecting the aspect B must have the axial termination a at the same distance on either side of it, and for the same reasons a must have the axial termination b at the same distance on either side of it; so that the terminations of the axes must recur at equal distances in the order $..a\,b\,a\,b\,a...$ bisecting the aspects ..A B A B.. And this series of aspects has as many terms as there are axial terminations, viz. $4m$ terms, if the number of axes is even, and $4m+2$ if it be odd. Wherefore no aspect different from A and B can be bisected by any axis, and A and B, different reversible aspects or sequences, are bisected by alternate equidistant axes. Q. E. D.

We may call A and B the two axial configurations.

V. *Obs.* 2. A reversible partition of the r-gon having more than one axis of reversion, has both reversible and irreversible sequences repeated in the circuit of the r-gon, which occupy an interval equal to that between alternate axes; those being reversible sequences

which begin and end with a side or angle carrying an axial termination, and those being irreversible which begin and end at any other side or angle.

Obs. 3. But a singly reversible partition has no sequence repeated in the circuit of the r-gon; for if it had a repeated sequence reversible as read from no point, the r-gon would not be reversible; and if it had a reversible sequence repeated in the circuit of the r-gon, it would not be singly reversible.

Obs. 4. A $2m$-ly reversible r-gon, if $r>2m$, has beginning at any angle of the r-gon which is not the termination of an axis, an irreversible sequence, simple, if $m=1$, and m-ple, if $m>1$, occupying half the circuit of the r-gon, and repeated in the other half.

VI. THEOREM B. *When the number of axes of reversion is odd in any partitioned* r*-gon, none is perpendicular to another; and when that number is even, every one is perpendicular to some other.*

For when the number is odd, there is on each side of any one a an equal number of terminations of other axes, all equidistant from a and from each other. And when that number is even, there is an odd number there of such terminations. Whence the truth of the proposition is evident.

VII. THEOREM C. *When the axes of reversion are odd in any partitioned* r*-gon, each one bisects both axial configurations; and when they are even in number, each bisects but one, read on it alike at either end, and half the axes carry one, and half the other, axial configuration.*

This is very evident from what is proved in Theorem A, that the axial configurations present themselves alternately upon the axes in order.

Cor. 1. If there be both agonal and diagonal axes, there is an equal number of each kind; and, as this number is even or odd, so is each axis perpendicular to one of its own or of the other kind.

Obs. 5. A $(2m+1)$-ly reversible partition never has an irreversible sequence occupying half the circuit of the r-gon and repeated in the other half; for this would require that every axis of reversion should carry the same configuration at both ends, which are points in those two sequences.

VIII. THEOREM D. *If a diagonal be perpendicular to an axis of reversion in any* h*-ly reversible* r*-gon, it is one of a system of not fewer than* h *diagonals symmetrically placed about the centre. And all diagonals not perpendicular to that axis form pairs making each an angle bisected by that axis, or that produced.*

For if h be odd, every axis carries the same perpendicular; and if h be even, at least $\frac{1}{2}h$ axes carry that perpendicular on opposite sides of the centre; and these axes are equidistant from each other: whence the first part of the theorem is evident. The second part follows from the definition of an axis of reversion.

Cor. The intersection of two produced diagonals equidistant from the centre, in any reversibly partitioned r-gon, is upon an axis of reversion.

IX. THEOREM E. *No* $(2m+1)$*-gon has an agonal or diagonal, and no* $(2m)$*-gon has a monogonal, axis of reversion.*

For a monogonal axis must have an equal number of vertices of the r-gon on each side of it, besides the vertex through which it passes; hence r is odd. And a diagonal axis must have an equal number of vertices on either side, besides the two through which it passes. And an agonal axis, which passes through no angles, must have an equal number of vertices on either side. Hence, in these two latter cases, r is even.

X. Theorem F. *If one axis of reversion is monogonal in a partitioned* r*-gon, all its axes of reversion are monogonal and odd in number.*

For r is odd, and the r-gon cannot have either a diagonal or an agonal axis; and as each axis bisects two aspects, A and B, A must be opposite to B in the circle ..ABABAB.. of axial configurations; *i. e.* the number of its terms is $2(2m+1)$, wherefore the axes are odd in number.

XI. Theorem G. *If there be a drawn axis of reversion,* a, *in a* $(1+k)$*-partitioned* r*-gon, there cannot be more than one other axis. If there be another,* b, *it is undrawn, and perpendicular to the former,* a, *and is either agonal or diagonal, as* $r=4m+2$, *or* $r=4m$.

For, if there be a second axis, b, it cannot meet the drawn one, and must be undrawn. And all the k diagonals are symmetrically placed about or upon this b; therefore a, meeting it and bisecting it in the centre of the r-gon, meets it at right angles; otherwise (Theorem D) two diagonals would meet b in the centre, which is impossible. And no line besides b can so meet a; wherefore a and b are the only axes. As a is not a monogonal axis, neither is b (Theorem F). If $r=4m$, a has on either side an even number of sides of the r-gon, and b, bisecting that system, is diagonal; if $r=4m+2$, a has on either side an odd number of sides of the r-gon bisected by b, which is therefore an agonal axis. Q. E. D.

XII. Theorem H. *If there be more than one undrawn axis of reversion in a partitioned* r*-gon, the* r*-gon is built regularly on a polygonal nucleus* (Q), *which is reversible about all the axes of reversion of the* r*-gon, and has no drawn diagonal.*

For consider the symmetry of the r-gon about any one of its axes, a, which are all undrawn (Theorem G). We see, on each side of a, f *marginal faces*, limited each by one diagonal d and certain sides of the r-gon, the $2f$ diagonals d forming pairs making angles bisected by a. Let these $2f$ faces be erased: the $2f$ lines d are now sides of an r'-gon ($r'<r$), which is still reversible about a, for the symmetry about a is not disturbed by the erasures. This r'-gon has also $2f'$ marginal faces which can be erased, leaving an r''-gon ($r''<r'$) still reversible about a; and thus by erasure of all the pairs of faces about a, the r-gon will be finally reduced to a polygon P, having no diagonals but what are bisected at right angles by a (Theorem D). Let Q be that portion of P which contains the centre of the r-gon, and Q_1, Q_2, &c. the remaining portions of P limited by perpendiculars to a. Then Q has evidently no diagonals.

Next consider the symmetry of the r-gon about any other axis b. We can reduce it by erasure of pairs of faces about b to a polygon P', consisting of a portion Q' about the centre and having no diagonals, and of portions limited by diagonals perpendicular to b. Q and Q' are polygons about the centre having no diagonals; they are therefore one

polygon Q. The polygons Q_1, Q_2, the remaining portions of P, have disappeared in the erasure of pairs of faces about b and Q; that is, every one of them, Q_m, is one of a set of two or more faces standing symmetrically about Q. Therefore Q is a nucleus, reversible about all the axes of the r-gon, free from diagonals, and having everything arranged symmetrically about it in the r-gon. Q. E. D.

XIII. THEOREM K. *If there be no axis of reversion in a partitioned* r*-gon except one undrawn,* a, *the figure is built symmetrically about* a *and about a polygonal nucleus* P, *which has either no diagonals, or only what are at right angles to* a.

This is established by the former part of the preceding demonstration.

But the polygon P, in this singly reversible r-gon, if it has diagonals perpendicular to the axis, is not properly a nucleus; nor is there any reason why one portion of it, Q, should be called a nucleus rather than Q_1 or Q_2; for P can be constructed by loading opposite sides of any of them with proper polygons. But if P is a simple polygon, it is properly the nucleus of the figure, which is made by loading opposite sides of it, on different sides of the axis, with the same polygons, thus preserving the reversibility about the axis.

XIV. THEOREM L. *If a diagonal axis of the nucleus is an axis of reversion of the* r*-gon, it is a diagonal axis thereof.*

For it passes through two angles of the r-gon.

THEOREM M. *An agonal axis of the* r*-gon is an agonal axis of the nucleus.*

For a diagonal or monogonal axis of the nucleus passes through a vertex of the r-gon, and cannot be an agonal axis of it.

THEOREM N. *If an agonal axis of the nucleus is an axis of the* r*-gon, it may be an agonal, or a diagonal, or a monogonal axis of it.*

For the sides of the nucleus through which that agonal axis passes may be sides of the r-gon, or they may be loaded with agonally reversible polygons having the same agonal axis: this will then be an agonal axis of the r-gon.

Or those sides may be loaded both with monogonally reversible polygons, having their axes in prolongation of that agonal axis: this makes it a diagonal axis of the r-gon.

Or one of those sides may be loaded with a monogonally reversible polygon, and the other either not at all, or with an agonally reversible: this, if the axes of the imposed polygon are in direction with that agonal axis, makes it a monogonal axis of the r-gon.

XV. THEOREM O. *A monogonal axis of the nucleus may be either a monogonal or a diagonal axis of the* r*-gon.*

For the side of the nucleus which that axis bisects may be unloaded, or loaded with an agonally reversible polygon, so as to make it a monogonal axis of the r-gon. Or that side can be loaded with a monogonally reversible polygon, so as to make the axis a diagonal one of the r-gon.

It is easily seen, and unnecessary to be formally propounded, how and how far the converses of these latter propositions are to be laid down.

Def. A *clear axis of reversion* meets no diagonal at right angles.

A *scored or loaded axis* of reversion meets one or more diagonals at right angles.

An *e-scored axis* of reversion meets *e* diagonals at right angles, of course bisecting them.

The clear and the loaded axis are both supposed undrawn.

XVI. THEOREM P. *If a clear axis of a* $(2m+3)$*-ly reversible* r*-gon be scored in any way by* e *diagonals at right angles to it, the* r*-gon becomes singly reversible about that scored axis.*

For, let ABC..be the axes of reversion of the *r*-gon N. The scored axis A remains an axis of reversion, because the symmetry about is not disturbed by the perpendicular scores; but none of the other axes BC... is perpendicular to A (Theorem B); wherefore each meets singly all the scores upon A, and is no axis of reversion (Theorem D) of the scored *r*-gon N′. Let then, M, any other diameter of N′ not amongst ABC... be an axis of reversion; it meets all the diagonals to which it is not perpendicular in pairs whose angles it bisects (Theorem D); wherefore these are pairs of equidistants from the centre; now the intersections of all these, except the newly-added scores upon A, are on the axes BC... (*Cor.* Theorem D); wherefore M, passing through the centre and one of these intersections, is one of the lines ABC..., contrary to hypothesis, which is absurd. Therefore A is the only axis of reversion of N′. Q. E. D.

XVII. THEOREM Q. *If any loaded axis* (A) *of a* $(2m+3)$*-ly reversible* r*-gon* (N) *be cleared by erasure of the diagonals perpendicular to* A, *the cleared figure* (N′) *is singly reversible about that axis* A.

For, let ABC...be the axes of reversion of N, on which are the diagonals at right angles to them *abc*...forming a system of lines symmetrically placed about the centre (Theorem D). No one of these B, after the erasure of *a* from A, is an axis of reversion of N′, because the diagonal *b* at right angles to B is not one of a system of lines symmetrically placed about the centre (Theorem D). Let then, M, any other diameter of N′ be an axis of reversion of N′; this line meets all the diagonals of N′ not perpendicular to it in pairs, whose angles it bisects, wherefore these pairs are equidistants from the centre of N′; but all the intersections of these pairs lie on the lines ABC...(*Cor.* Theorem D); wherefore M is one of these lines ABC... Q. E. A. Therefore A is the only axis of reversion of N′. Q. E. D.

XVIII. THEOREM R. *If a clear axis* (A) *of reversion of a* $2m$*-ly reversible partitioned* r*-gon* (N) *be scored by perpendiculars to it symmetrically about the centre, the figure is not made singly reversible about* (A); *but if it be so scored unsymmetrically about the centre, the scored figure* (N′) *is singly reversible about that axis* (A).

For there is an axis perpendicular to A (Theorem B) about which the symmetry is not disturbed by symmetric scores, *i. e.* pairs of parallels to it equidistant from the centre, or a diameter parallel with such pairs.

But when the scores are not such pairs, or a diameter and such pairs, the axis perpendicular to A is no axis of reversion evidently; nor is any other diameter of N′,

for it meets all those scores singly. Wherefore A is the only axis of reversion of N'. Q. E. D.

XIX. THEOREM S. *If a clear axis* A *of a singly reversible partitioned* r*-gon* N' *be scored by diagonals perpendicular to it, the scored figure* N *becomes sometimes singly, and at other times* $(2m+3)$*-ly reversible about the scored axis.*

For the r-gon N' being singly reversible has no axis of reversion perpendicular to A; and no addition of diagonals parallel to the perpendicular diameter can make it an axis of reversion; for that addition cannot alter its intersections with the diagonals of N'; wherefore N is not $2m$-ly reversible (Theorem D and B). If N' should be the N' of Theorem Q, and the scores upon A should be those erased in that theorem, N will be the $(2m+3)$-ly reversible of that theorem: but if this is not the case in both these conditions, N will remain, like N' unscored, singly reversible about A, since the scores do not disturb the symmetry about that axis. Q. E. D.

What precedes about singly reversibles with loaded axes is sufficient for our present purpose, which is to show that before we can determine the number of singly reversibles, with clear and loaded axes, it is necessary that we should know the number of $(2m+1)$-ly reversible $(1+k-e)$-partitions of the r-gon which have a clear axis, *i. e.* which have clear axes; for here the configurations about all the axes are alike (Theorem C); and also that of the $2m$-ly reversible $(1+k-e)$-partitioned r-gons which have one configuration about clear axes, and also of those which have both their configurations about clear axes. This matter will be more evident as we proceed.

XX. Let $R^{2h\cdot agdi}(r,k)_n$, $R^{h\cdot ag}(r,k)_n$, $R^{h\cdot di}(r,k)_n$, $R^{h\cdot mo}(r,k)_n$ denote the whole number of $(1+k)$-partitioned r-gons built on the n-gonal nucleus $(n>2)$, which have (Theorem C)

$R^{2h\cdot agdi}(r,k)_n$, h agonal and h diagonal,

$R^{h\cdot ag}\ (r,k)_n$, h agonal only,

$R^{h\cdot di}\ (r,k)_n$, h diagonal only, and

$R^{h\cdot mo}\ (r,k)_n$, h monogonal axes only, of reversion.

We shall denote those having all their axes clear by c subscript to R; those having no clear axes by zero subscript to R; those of the second and third classes which have half their axes, bearing one configuration, clear, by $\frac{1}{2}c$ subscript, and those of the first, which have half their axes clear, the agonal or the diagonal ones, by ac or dc subscript to R. We write

$$R^{2h\cdot agdi}(r,k)_n = R^{2h\cdot agdi}_{c}(r,k)_n + R^{2h\cdot agdi}_{ac}(r,k)_n + R^{2h\cdot agdi}_{dc}(r,k)_n + R^{2h\cdot agdi}_{0}(r,k)_n,$$

$$R^{h\cdot ag}\ (r,k)_n = R^{h\cdot ag}_{c}\ (r,k)_n + R^{h\cdot ag}_{\frac{1}{2}c}\ (r,k)_n + R^{h\cdot ag}_{0}\ (r,k)_n,$$

$$R^{h\cdot di}\ (r,k)_n = R^{h\cdot di}_{c}\ (r,k)_n + R^{h\cdot di}_{\frac{1}{2}c}\ (r,k)_n + R^{h\cdot di}_{0}\ (r,k)_n,$$

$$R^{h\cdot mo}\ (r,k)_n = R^{h\cdot mo}_{0}\ (r,k)_n + R^{h\cdot mo}_{0}\ (r,k)_n.$$

In the second and third lines the second subclass is of course nothing when h is odd. In the fourth class the number of axes of reversion is always odd. And in all those equations we suppose $n \lessdot 3$.

XXI. Before we can proceed to investigate formulæ for the determination of these numbers, it is necessary,—

Problem a. *To find the* k*-divisions of the* r*-gon* or r*-ace.*

By the $(1+k)$-divisions of an r-gon I mean the entire number of ways in which k diagonals can be drawn in it, none crossing another, all ways being different which occupy different angles $1\,2..r$ of the r-gon. Thus there are five 3-divisions, but only one 3-partition of the pentagon made by drawing a pair of diagonals. And there are five 3-divisions of a pentace made by breaking it into three triaces, but these are all the same 3-partition.

If we call the number sought of $(1+k)$-divisions of the r-gon $\mathrm{D}(r, k)$, we can express it in terms of $\mathrm{D}(r', k')$, $r'<r$, and $k'<k$.

For consider any diagonal b, drawn from any angle β of the r-gon, dividing it into a $(3+h)$-gon and a $(r-h-1)$-gon. This line b will be drawn in the $(1+k)$-divisions along with every $(1+s)$-division of the $(3+h)$-gon, combined with every $(k-s-1)$-division of the $(r-h-1)$-gon, s diagonals being drawn on one side, and $k-s-1$ diagonals on the other side of b.

That is, $$\Sigma_s \mathrm{D}(3+h, s)\times \mathrm{D}(r-h-1, k-s-1)$$

taken from $s=0$ to $s=k-1$, is the number of $(1+k)$-divisions in which that line b will be seen.

If, now, we give to h every value from $h=0$ to $h=r-4$, we shall have counted every $(1+k)$-division in which any line b appears that can be drawn from that angle β. If we put for β each of the r angles in succession, that is, if we multiply by r, we shall have enumerated every $(1+k)$-division in which any line b appears, that is drawn from any angle β. But we have thus handled twice, once from either extremity, every line in every set of k diagonals; that is, we have counted every $(1+k)$-division $2k$ times. Wherefore the correct result is

$$2k.\mathrm{D}(r, k)=r.\Sigma_h\Sigma_s\{\mathrm{D}(3+h, s).\mathrm{D}(r-h-1, k-s-1)\},$$

where every value of h from $h=0$ to $h=r-4$ is to be combined with every value of s from $s=0$ to $s=k-1$. If, then, we know these divisions for all values of r and k up to $\mathrm{D}(r-1, k-1)$, we obtain $\mathrm{D}(r, k)$ by addition. And as $\mathrm{D}(r+0)=1$, and $\mathrm{D}(r, 1)=\frac{1}{2}r.(r-3)$, $\mathrm{D}(r, 2)$ is given, and thence $\mathrm{D}(r, 3)$, and so on, up to $\mathrm{D}(r, k)$, for any given values of r and k.

XXII. To find $\mathrm{D}(r, k)$ in terms of r, for the general value of r, and for a given value of k, we write

$$\mathrm{D}(r, k)=\mathrm{A}r^{2k}+\mathrm{B}r^{2k-1}+..+\mathrm{L}r+\mathrm{M};$$

for we know that this number is not greater than $\{\frac{1}{2}(r.(r-3))\}^{k-1}\times\overline{k+1}^{-1}$, that of all possible sets of k diagonals that cross or not. Then, from the $2k+1$ equations,

$$\begin{aligned}
\mathrm{D}(k+3, k)&=\mathrm{A}(k+3)^{2k}+\mathrm{B}(k+3)^{2k-1}+..+\mathrm{M},\\
\mathrm{D}(k+4, k)&=\mathrm{A}(k+4)^{2k}+\mathrm{B}(k+4)^{2k+1}+..+\mathrm{M},\\
&\vdots\quad=\quad\vdots\\
\mathrm{D}(3k+3, k)&=\mathrm{A}(3k+3)^{2k}+\mathrm{B}(3k+3)^{2k+1}+..+\mathrm{M},
\end{aligned}$$

the left members of which are to be calculated by the preceding article, we can determine the coefficients A, B, .. LM.

So far as I have pursued the inquiry, I find always one factorial form of the function $D(r, k)$; but in order to prove that this is always the form, it is necessary to show that, if

$$D(3+h, s) = \frac{(3+h)^{\underline{s+1}}}{\lfloor s+1} \times \frac{(3+h-s-2)^{\underline{s+2}}}{\lfloor s+2}, \text{ and}$$

$$D(r-h-1, k-s-1) = \frac{(r-h-1)^{\underline{k-s-1}}}{\lfloor k-s} \times \frac{(r-h-k+s-2)^{\underline{k-s-1}}}{\lfloor k-s+1},$$

$$D(r, k) = \frac{r}{2k} \Sigma_h \Sigma_s \{D(3+h, s).D(r-h-1, k-s-1)\} = \frac{r^{\underline{k+1}}}{\lfloor k+1} \times \frac{(r-k-2)^{\underline{k+1}}}{\lfloor k+2},$$

from $h=0$ to $h=r-4$, and from $s=0$ to $s=k-1$; which is the expression that I continually find.

This summation I must leave to the learned and industrious reader; but, meanwhile, I shall venture to enunciate with the best demonstration, such as it is, that occurs to me, the following

THEOREM T. *The number of* (1+k)*-divisions of an* r*-gon*, i. e. *of all the ways in which* k *diagonals can be drawn in it, none crossing another, is*

$$D(r, k) = \frac{r^{\underline{k+1}}}{\lfloor k+2} \times \frac{(r-k-2)^{\underline{k+1}}}{\lfloor k+1}.$$

For first, let the r-gon be divided into triangles, *i. e.* let $k=r-3$. Here the $(3+h)$-gon of the preceding article can have only h diagonals, or $s=h$; and the final result, the last equation of that article, becomes

$$D(r, r-3) = \frac{r}{2(r-3)} . \Sigma_h \{D(3+h, h).D(r-h-1, r-h-4)\},$$

from $h=0$ to $h=r-4$.

Hence we obtain for $r=4$, $r=5$, $r=6$, &c.,

$$D(4, 1) = \tfrac{4}{2} . D(3, 0) . D(3, 0) = 2 = \frac{4^{\underline{4-3}}}{\lfloor 4-1},$$

$$D(5, 2) = \tfrac{5}{4} . \{D(3, 0) . D(4, 1) + D(4, 1) . D(3, 0)\}$$
$$= 5 = \frac{5^{\underline{5-3}}}{\lfloor 5-1},$$

$$D(6, 3) = \tfrac{6}{6} \{D(3, 0) D(5, 2) + D(4, 1) . D(4, 1) + D(5, 2)(D3, 0)\}$$
$$= 14 = \frac{6^{\underline{6-3}}}{\lfloor 6-1},$$

$$D(7, 4) = \tfrac{7}{8} \{D(3, 0) D(6, 3) + D(4, 1) D(5, 2) + D(5, 2) D(4, 1) + D(6, 3) . D(3, 0)\}$$
$$= \tfrac{7}{8} \{14 + 10 + 10 + 14\} = 42$$
$$= \frac{7^{\underline{7-3}}}{\lfloor 7-1};$$

and as we have demonstrated that $D(8, 5)$ is a sum of products of these continuous

functions, it is a continuous function, and has a permanent form: wherefore it is sufficiently evident that

$$D(r, r-3)=\frac{r^{r-3|1}}{\lfloor r-1}.$$

We know that $D(r, k)=0$, if $k>0$, for every value of r from $r=3$ to $r=k-2$; for no $(k-2)$-gon has k diagonals none crossing another. Hence $(r-k-2)^{k|1}$ is a factor of $D(r, k)$. Therefore

$$D(r, k)=(r-k-2)^{k|1}(A'r^k+B'r^{k-1}+..+L'r+M'),$$

$$D(k+3, k)=1.2..k\big(A'(k+3)^k+B'(k+3)^{k-1}+..+L'(k+3)+0\big),$$

$$\frac{(k+3)^{k|1}}{\lfloor k+2\,\lfloor k+1}=A'(k+3)^k+B'(k+3)^{k-1}+..+L'(k+3),$$

$$\frac{(k+3+1)(k+3+2)..(k+3+k-1)}{\lfloor k+2\,\lfloor k+1}=A'(k+3)^{k-1}+B'(k+3)^{k-2}+..+L'.$$

Hence

$$A'=\frac{1}{\lfloor k+2\,\lfloor k+1}$$

$$L'=\frac{1.2.3..(k-1)}{\lfloor k+2\,\lfloor k+1};$$

consequently

$$\frac{D(r, k)}{r}=(r-k-2)^{k|1}\times\frac{(r+1)(r+2)...(r+k-1)}{\lfloor k+2\,\lfloor k+1}, \text{ and } D(r, k)=\frac{r^{k|1}}{\lfloor k+2}\cdot\frac{(r-k-2)^{k|1}}{\lfloor k+1};$$ Q. E. D.

XXIII. THEOREM U. *Every* m*-ly irreversible* k*-partition of an* r*-gon occurs* 2r : m *times among the* k*-divisions, and every* m*-ly reversible* k*-partition of it is found* r : m *times among the* k*-divisions.*

For any m-ly irreversible k-partition has a different configuration about $r:m$ successive angles (III.), and is nowhere its own reflexion; so that $r:m$ more configurations the reflexions of the former are read on the reversed face of the r-gon. And as all these configurations are found among the k-divisions separately enumerated about the same angle, this partition is counted $2r:m$ times among them.

And any m-ly reversible k-partition of the r-gon has $r:m$ different configurations in the interval between the termination of any axis A and its repetition, that is, about $r:m$ angles if A be an agonal axis, and about $r:m$ sides if it be diagonal. The central configuration of these $r:m$ is about the axis alternate with A; the others form pairs of configurations reflecting each other. All these are constructed and counted separately among the k-divisions about the same point of the r-gon; *i. e.* this partition is counted $r:m$ times among the k-divisions. Q. E. D.

Let $R^m(r, k)$ denote the entire number of m-ly reversible $(1+k)$-partitions of the r-gon about all nuclei, and all kinds of axes, and let $I^m(r, k)$ stand for the complete number of m-ly irreversible $(1+k)$-partitions.

The theorem U may be thus expressed:

$$D(r, k)=\Sigma_m\left\{\frac{r}{m}\cdot R^m(r, k)+\frac{2r}{m}\cdot I^m(r, k)\right\}, \text{ or, } (m>0),$$

$$2r.\mathrm{I}(r,\,k)=\mathrm{D}(r,\,k)-\Sigma_m\left\{\frac{r}{m}\mathrm{R}^m(r,\,k)+\frac{2r}{m+1}\cdot\mathrm{I}^{m+1}(r,\,k)\right\};$$

which shows that, if we can find $\mathrm{R}^m(r,\,k)$ and $\mathrm{I}^{m+1}(r,\,k)$ for all values of $m>0$, we can obtain the most numerous of all the classes, $\mathrm{I}(r,\,k)$, by a simple subtraction and division.

XXIV. Our first step towards the actual solution of our problem of partitions, which, after all these tedious prolegomena, we may now think of taking, is to find the numbers in (XX.). We begin with

$$\mathrm{R}^{2h\alpha g\delta i}(r,\,k)_n=\mathrm{R}_v^{2h\cdot\alpha g\delta i}(r,\,k)_n+\mathrm{R}_{vv'}^{2h\alpha g\cdot\delta i}(r,\,k)_n+\mathrm{R}_{\Delta v}^{2h\cdot\alpha g\delta i}(r,\,k)_n+\mathrm{R}_0^{2h\cdot\alpha g\delta i}(r,\,k)_n.$$

Problem b. *To find* $\mathrm{R}_v^{2h\cdot\alpha g\delta i}(\mathrm{r},\,\mathrm{k})_n$, *the number of* (1+k)*-partitioned* r*-gons, built on the* n*-gonal nucleus, and having* h *agonal and* h *diagonal axes of reversion, all clear.*

There are, in every one of these, $2h$ equidistant terminations of diagonal axes, between which intervene $\frac{n}{2h}$ sides of the n-gon; the central side of these $\frac{n}{2h}$ is bisected by an agonal axis, wherefore $\frac{n}{2h}$ is an odd number.

If we mark this central side as the nth of the n-gon, we see on either side of it $\frac{1}{2}\left(\frac{n}{2h}-1\right)$ sides between it and a termination of a diagonal axis. This nth side is one of $2h$ equidistant sides of the n-gon, bisected by terminations of agonal axes, namely the

$$\frac{n}{2h},\ \frac{2n}{2h},\ \frac{3n}{2h},\ \ldots\ n\text{th sides},$$

which are all unloaded, being sides both of the r-gon and the n-gon. The $\frac{1}{2}\left(\frac{n}{2h}-1\right)$ sides on either side of the nth are loaded, the first and $(n-1)$th with a certain e_1-partitioned $(2+a_1)$-gon, so as to form a configuration bisected by the agonal axis between them; the second and $(n-2)$th by a e_2-partitioned $(2+a_2)$-gon placed in like reversible manner, and so on; in such a way as to satisfy the equations following,

$$\left.\begin{aligned} r&=n+4h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{4h}}\right),\\ k&=n-2h+4h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{4h}}\right);\end{aligned}\right\}\quad\ldots\ldots\quad(\mathrm{A.})$$

where $a_m\geqq 0$, and $e_m\not> a_m-1$, since no $(2+a_m)$-gon can have more than a_m-1 diagonals drawn none crossing another; wherefore if $a_m=0$, $e_m=-1$.

These $\frac{n-2h}{4h}$ polygons a_1a_2, &c. are laid on in the same manner to correspond about the $\frac{n}{2h}$th $\frac{2n}{2h}$th, &c. sides; *i. e.* they fill up the $4h$ intervals between agonal and diagonal axes, making a configuration reversible about any one of them.

XXV. The equations (A.) must be satisfied, because the n summits of the n-gon, with the a_1 summits added $4h$ times in the $(2+a_1)$-gon, and the a_m summits added $4h$ times in

the $(2+a_m)$-gon, must make up the r summits of the r-gon. And, supposing that a_m everywhere >0, the $n-2h$ loaded sides of the n-gon, with the e_m diagonals of every $(2+a_m)$-gon $4h$ times laid on, must make up the k diagonals of the r-gon. And if $a_m=0$, in which case the mth and $(n-m)$th sides of the n-gon are unloaded, whereby $4h$ sides besides the $2h$ above-mentioned remain sides common to the r-gon and the n-gon, the appearance of $e_m=-1$ corrects the error made by counting those $4h$ sides for diagonals of the r-gon, and the second equation still remains true, however many of the numbers a_1, a_2, &c. $=0$. But these equations are impossible, and $R^{2h\text{-gon}}_{n}(r, k)_*=0$, unless $r-n$, $n-2h$, and $k-n+2h$ are all multiples of $4h$.

Consider any single solution of the equation (A.), and let N be an r-gon bearing, on the mth side everywhere reckoned in both directions from the one bisected by an agonal axis, a $(2+a_m)$-gon having e_m diagonals. Whatever be the posture of those e_m diagonals, and whatever be the character of the $(2+a_m)$-gon, the figure is reversible at least about h agonal and h diagonal axes. And let C be the configuration read in N about the agonal axis through the nth side of the n-gon.

XXVI. Let, next, any disturbance be made in the arrangement of the e_m diagonals in one or more of these imposed $\frac{n-2h}{4h}$ polygons, so that the reversible character about the $2h$ axes be preserved, and the diagonals of the $(2+a_m)$-gon on both sides of any axis shall form pairs making angles bisected by the intervening axis of reversion. The solution before us of (A.) being undisturbed in both values and order of the numbers $a_1 e_1$, $a_2 e_2$, &c., we can combine every $(1+e_m)$-division of the $(2+a_m)$-gon in the interval with every $(1+e_p)$-division of the $(2+a_p)$-gon, &c.; that is, we can make C take

$$\mathrm{D}(2+a_1, e_1).\mathrm{D}(2+a_2, e_2).\mathrm{D}(2+a_3, e_3)\ldots\mathrm{D}(2+a_{\frac{n-2h}{4h}}, e_{\frac{n-2h}{4h}})=\Delta_{\frac{n-2h}{4h}}$$

different forms, by mere variation of arrangement of diagonals, the same variation being made about every agonal axis, and consequently the same variation occurring about every diagonal axis. Of these $\Delta_{\frac{n-2h}{4h}}$ configurations no two can be alike; for the intervals of $\frac{n-2h}{4h}$ sides between two adjoining axes are irreversible configurations, if we look at the axes which limit them, because one is an agonal and the other a diagonal axis; so that, in fact, no disturbance of one or more diagonals of N can produce N′ a repetition of N. If the limiting axes were alike, it might possibly occur that N′ should be a reflexion of N, the configuration of one read from right to left in the interval being that of the other read from left to right.

It is to be observed here, and in all products that we may handle of the form of $\Delta_{\frac{n-2h}{4h}}$, that when $a_m=0$ and consequently $e_m=-1$, $\mathrm{D}(2+a_m, e_m)$ is either to be omitted, or counted unity, so that $\Delta 0$ is always $=1$.

XXVII. Next, let us make a variation in our solution of equation (A.), either by altering two or more values of $a_1 a_2..$ or of $e_1 e_2..$ or of both; or by exchanging the places of certain values without altering them. It is evident that C′, the result of such

disturbance of the solution, must differ from every C that we have before regarded about the axis through the nth side of the n-gon; for it has no longer the same polygons in the same places of the interval between the axes. We have either changed the order of the imposed polygons, or we have in certain places put a $(1+e_p)$-partitioned $(2+a_p)$-gon which was not employed in the construction of C, for a $(1+e_m)$-partitioned $(2+a_m)$-gon, which was. Whatever be the solution we work from, we shall be able to produce from it, by mere disturbances of diagonals in the imposed polygons, $\Delta_{\frac{n-2h}{ih}}$ distinct configurations, all different from the $\Delta_{\frac{n-2h}{ih}}$ counted before; and all the r-gons thus produced will have the property of being reversible about h agonal and h diagonal axes. And it is evident, that every configuration about an agonal axis of any $(1+k)$-partitioned r-gon, having h agonal and h diagonal axes of reversion, will be produced from some solution of equations (A.), and some arrangement of diagonals in the imposed polygons.

If, then, we denote by $\Sigma.\Delta_{\frac{n-2h}{ih}}$ the sum of the products $\Delta_{\frac{n-2h}{ih}}$ made from every solution of equations (A.), every change, either in value or order, of $a_1 e_1$, $a_2 e_2$, &c. being counted as a solution, we find that $\Sigma\Delta_{\frac{n-2h}{ih}}$, many terms of which will be equal numbers, is the exact number of configurations about an agonal axis which can be seen on any $(1+k)$-partitioned r-gon having h agonal and h diagonal clear axes of reversion.

Now no ago-diagonally reversible has two configurations about agonal axes; and every $(1+k)$-partitioned r-gon having ih agonal and ih diagonal axes, is reversible about h agonal and h diagonal axes, whatever positive number i may be; for it has h equidistant agonal axes, and between every pair of these $(i-1)$ more agonal axes, and it has h diagonal axes, because ih diagonal axes bisect the angles between the ih agonal ones. Hence it follows that

$$\Sigma\Delta_{\frac{n-2h}{ih}}=\Sigma_i R_c^{2ih.agdi}(r,\ k)_n,$$

where i is every whole number giving $\frac{n-2ih}{4ih}$, $\frac{r-n}{4ih}$, and $\frac{k-n+2ih}{4ih}$ positive integers; conditions necessary to the existence of equations (A.).

XXVIII. Consequently $(i>0)$,

$$R_c^{2h.agdi}(r,\ k)_n=\Sigma\Delta_{\frac{n-2h}{ih}}-\Sigma_i R_c^{2(i+1)h.agdi}(r,\ k)_n,$$

for all values of $(i+1)h=h'$, which make $n-2h'$, $r-n$, and all multiples of $4h'$.

As $r>n$, $n-2h>0$ in equations (A.); hence $n\not<6h$, and if $2h>\frac{n}{3}$, $R_c^{2h}=0$; *i. e.*

$$R_c^{n.agdi}(r,\ k)_n=R_c^{\frac{n}{3}agdi}(r,\ k)_n=0.$$

If $n=6h$, the equations (A.) become

$$r-n=\frac{2n}{3}a_1,$$

$$k-\frac{2n}{3}=\frac{2n}{3}e_1,$$

$$\therefore\quad 2+a_1=\frac{3r+n}{2n},\quad e_1=\frac{3k-2n}{2n}$$

$$\Delta_{\frac{n-2k}{4k}}=\mathrm{D}(2+a_1,\, e_1)=\mathrm{D}\left(\frac{3r+n}{2n},\ \frac{3k-2n}{2n}\right),$$

and this is $\Sigma\Delta_{\frac{n-2k}{4k}}$, as there is no other solution of (A.). Wherefore

$$\mathrm{R}_e^{\frac{n}{5}\cdot ogdi}(r,\, k)_n=\mathrm{D}\left(\frac{3r+n}{2n},\ \frac{3k-2n}{2n}\right).$$

Next let $\frac{n-2k}{4k}=2$, or $2k=\frac{n}{5}$; we obtain

$$\mathrm{R}_e^{\frac{n}{5}\cdot ogdi}(r,\, k)_n=\Sigma\Delta_{\frac{n-2k}{4k}}-\mathrm{R}_e^{\frac{n}{5}}(r,\, k)_n,$$

which is a given number. Here, as before, $\Sigma\Delta$ has as many terms as there are solutions of

$$r=n+\frac{2n}{5}(a_1+a_2)$$

$$k=\frac{4n}{5}+\frac{2n}{5}(e_1+e_2).$$

For an example, let $r=78$, $k=48$, $n=30$; $\frac{n}{5}=6$: to find $\mathrm{R}_e^{6\cdot ogdi}(78,\, 48)_{30}$, we form the solutions of

$$4=a_1+a_2 \qquad (a_m\geqq 0,\quad e_m\geqslant a_m-1)$$
$$2=e_1+e_2,$$

of which there are five, namely,

$$4=4+0=\ \ 0+4=3+1=1+3=2+2$$
$$2=3-1=-1+3=2+0=0+2=1+1;$$

whence, disregarding the negative -1, we obtain

$$2.\mathrm{D}(6,\, 3)+2.\mathrm{D}(5,\, 2).\mathrm{D}(3,\, 0)+\mathrm{D}(4,\, 1).\mathrm{D}(4,\, 1)=\Sigma\Delta_{\frac{n-2k}{4k}},$$

or
$$\Sigma\Delta_{\frac{n-2k}{4k}}=2.14+2.5+2^2=42;$$

and
$$\mathrm{R}_e^{6\cdot ogdi}(78,\, 48)_{30}=42-\mathrm{R}_e^{10ogdi}(78,\, 48)_{30}=42,$$

because $\frac{78-30}{20}$ is not a whole number.

We may then put $\frac{30-2k}{4k}=7$, the next possible integral value after 2 of that subindex; and thus finding $k=1$, proceed to enumerate $\mathrm{R}_e^{2ogdi}(78,\, 48)_{30}$. The equations (A.) become

$$\frac{78-30}{4}=12=a_1+a_2+\ldots+a_7$$

$$\frac{48-28}{4}=\ 5=e_1+e_2+\ldots+e_7,$$

and the solutions are already numerous. The sum of products Δ_7 being found, we obtain

$$\mathrm{R}_e^{2\cdot ogdi}(78,\, 48)_{30}=\Sigma\Delta_7-42.$$

Thus it is evident that we can register $R_e^{2h\cdot agdi}(r, k)_n$, from $h=\frac{n}{3}$ to $h=1$, for all values that make $r-n$, $n-2h$, and $k-n+2h$, multiples of $4h$.

For a simple and complete example, to be carried out through all our investigation, we may take the partitions of the octagon. Here we are to seek $R_e^{2hagdi}(8, k)_n$. Evidently $n<8$ (if $k>0$), and $=6$ or $=4$.

When $n=6$, $\frac{6-2h}{4h}=1$, the only value, gives $h=1$, and $R_e^{2agdi}(8, k)_6=0$ whatever be k, because $\frac{8-6}{4}$ is not integer.

When $n=4$, $\frac{4-2h}{4h}$ has no integer value >0; wherefore, since $r>n$, $R_e^{2hagdi}(8, k)_4=0$, for all values of k. Hence

$$R_e^{2hagdi}(8, k)=0,$$

whatever be the nucleus, except that

$$R_e^{2agdi}(8, 0)_8=1 \text{ (art. III.)}.$$

XXIX. Problem c. *To find* $R_{de}^{2hagdi}(r, k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons, built on an* n*-gonal nucleus, having* h *loaded agonal, and* h *clear diagonal, axes of reversion.*

The investigation to be made differs from that in the five preceding articles in that the $2h$ sides of the n-gon, bisected by the agonal axes, are all loaded with a certain $(1+\varepsilon_0)$-partitioned $(4+2\alpha_0)$-gon A_0, having at least one agonal axis of reversion (Theorem G.). This must be so posited that its agonal axis shall be in a line with that of the n-gon (Theorem N.), and must be placed alike on the $2h$ equidistant sides, since all agonal axes of the r-gon must bisect the same configuration at each end (Theorem C.). The remaining sides of the n-gon may be loaded, the mth from A_0, with any $(1+e_m)$-partitioned (or divisioned) $(2+a_m)$-gon so as to form a figure reversible about the $2h$ axes, and so that the equations following be satisfied:—

$$\left.\begin{aligned} r&=n+2h(2+2\alpha_0)+4h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{4h}}\right)\\ k&=n+2h\varepsilon_0+4h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{4h}}\right); \end{aligned}\right\} \quad \ldots\ldots \quad (A'.)$$

where $\quad a_m \geqq 0,\ e_m \geqq a_m-1$; and $e_m=-1$, if $a_m=0$;

the numbers α_0 and ε_0 being any that we may select from our register of polygons having at least one agonal axis, which is supposed complete for values of $(2+2\alpha_0)$ not greater than $\frac{r-n}{2h}$. All that is said of the loading of the $\frac{n-2h}{4h}$ sides between the axes, and about the import and necessity of equations (A.), applies to equations (A'.).

The sides being thus loaded, we have an r-gon which has only clear diagonal axes, and is reversible about $2h$ equidistant and alternate agonal and diagonal axes. Leaving now the $2h$ agonally reversible $(1+2\alpha_0)$-gons undisturbed, we can vary the positions of the diagonals, exactly as in (XXVI.), so as to produce about the agonal axis through the nth side of the r-gon

$$D(2+a_1, e_1).D(2+a_2, e_2)\ldots D\left(2+a_{\frac{n-2h}{4h}}, e_{\frac{n-2h}{4h}}\right)=\Delta_{\frac{n-2h}{4h}}$$

different configurations, the numbers a_1, e_1, &c. here being those of a solution of equations (A'.). And every one of these may be varied again by changing the posture of the $(4+2\alpha_0)$-gon, if it has a second configuration about an agonal axis, so that this axis shall come into a line with that of the n-gon, or by exchanging the $(4+2\alpha_0)$-gon for any other that is also $(1+\epsilon_0)$-partitioned and has an agonal axis. The number of variations in our power by disturbance of the $(4+2\alpha_0)$-gons alone, is

$$\Sigma_m\{2R^{m\text{-}ag}(4+2\alpha_0,\ \epsilon_0)+R^{2m\text{-}agdi}(4+2\alpha_0,\ \epsilon_0)\}=H,$$

because each m-agonally reversible has two configurations about agonal axes and thus admits of two postures, while each ago-diagonally reversible has only one; whatever be the number m of agonal axes (Theorem A.).

Thus we see, that for every solution of equations (A'.) that we can write down, all values and orders of the numbers counting in the solution, we can make

$$H.\Delta_{\frac{n-2k}{4k}}$$

different configurations about the axis through the nth side of the n-gon. The proof that there are no two of them alike need not be repeated here from art. XVI.

If then

$$\Sigma H.\Delta_{\frac{n-2k}{4k}}$$

denote the sum of those products $H\Delta$ made from every solution of equations (A'.), this number is that of the r-gons $(1+k)$-partitioned, and having h agonal and h diagonal axes of reversion, the latter only being clear axes, which are constructible on the n-gonal nucleus. This is shown exactly as in art. XXVII. And by the reasoning of that article we obtain

$$\Sigma.H.\Delta_{\frac{n-2k}{4k}}=\Sigma_i R^{2hi\cdot agdi}_{4r}(r,\ k)_n,$$

hi being any number h' such that

$$\frac{n-2h'}{4h'},\ \frac{r-n-2h'.(2+2\alpha_0)}{4h'} \text{ and } \frac{k-n-2h'\epsilon_0}{4h'}$$

are positive integers, for some values of α_0 and ϵ_0 that are found in our register of $(1+\epsilon_0)$-partitioned $(2+2\alpha_0)$-gons, having an agonal axis of reversion.

Hence $(i>0)$,

$$R^{2k\cdot agdi}_{4r}(r,\ k)_n=\Sigma H.\Delta_{\frac{n-2k}{4k}}-\Sigma_i R^{2k(i+1)\cdot agdi}_{4r}(r,\ k)_n.$$

XXX. The highest value of h in equations (A'.) gives $\frac{n-2h}{4h}=0$, or $2h=n$. In this case A_4 is the only imposed polygon, and the equations (A'.) become

$$\frac{r-3n}{n}=2\alpha_0,\ \frac{k-n}{n}=\epsilon_0,$$

$$H=\Sigma_m\left\{2R^{m\text{-}ag}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right)+R^{m\text{-}agdi}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right)\right\},$$

which $=0$ of course if the fractions are irreducible, or if no such values are in our register of reversible polygons with an agonal axis. $\Delta_{\frac{n-2k}{4k}}=\Delta_0$ is here to be considered

unity; wherefore

$$R^{n\cdot aqdi}_{\Delta c}(r, k)_n = \Sigma_n\left\{2R^{n\cdot aq}\left(\frac{r+n}{n}, \frac{k-n}{n}\right) + R^{2n\cdot aqdi}\left(\frac{r+n}{n}, \frac{k-n}{n}\right)\right\},$$

which is a number given in our register, or else $=0$.

The next step is to put the subindex $\frac{n-2h}{4h}=1$, or $\frac{n}{3}=2h$, and find

$$R^{\frac{n}{3}\cdot aqdi}_{\Delta c}(r, k)_n = 0,$$

because $2h=\frac{n}{2}$ does not give an integer $\frac{n-2h}{4h}$.

$$R^{\frac{n}{2}\cdot aqdi}_{\Delta c}(r, k)_n = \Sigma H.\Delta_1 - R^{\frac{n}{3}\cdot aqdi}_{\Delta c}(r, k)_n,$$

a number readily obtained, and so on through all integer values of $\frac{n-2h}{4h}$, from $2h=\frac{n}{2}$ to $2h=2$.

XXXI. For an example take

$$r=22,\ k=12,\ n=6,\ h=1, \text{ and seek } R^{3\cdot aqdi}_{\Delta c}(22, 12)_6.$$

By equations (A′.) we have

$$22=6+4+4\alpha_0+4a_1$$

or

$$\alpha_0 \geqslant 2,\ 4+2\alpha_0 \geqslant 8.$$

Our register as far as $r=8$ contains, under R^{aqdi},

$$R^{2aqdi}(8, 0)=1,\ R^{5\cdot aqdi}(6, 0)=1,\ R^{3\cdot aqdi}(6, 1)=1,\ R^{2aqdi}(6, 2)_4=1,\ R^{4aqdi}(4, 0)=1,$$

where $\alpha_0=2$, $\alpha_0=1$, $\alpha_0=1$, $\alpha_0=2$, $\alpha=0$

$\epsilon_0=0$, $\epsilon_0=0$, $\epsilon_0=1$, $\epsilon_0=2$, $\epsilon_0=0$.

But as $\frac{12-6-2.\epsilon_0}{4}$ is integer, $\epsilon_0=1=\alpha_0$; and the only solution of (A′.) is

$$22=6+2.(2+2.1)+4.2$$
$$12=6+2.1 \qquad +4.1,$$

where $a_1=2,\ e_1=1.$

$$D(2+2, 1)=2=\Sigma\Delta_1,\ \Sigma H=R^{2aqdi}(6, 1)=1$$
$$R^{2aqdi}_{\Delta c}(22, 12)_6=\Sigma H\Delta_1 - R^{3\cdot 2aqdi}_{\Delta c}(22, 12)_6$$
$$=2-0,$$

because $\frac{r+n}{n}=\frac{22+6}{6}$, in $R^{\alpha}_{\Delta n}(r, k)_n$, is not integer.

We proceed with our partitions of the octagon, and seek $R^{2h\cdot aqdi}_{\Delta c}(8, k)_n$.

Now $R^{4aqdi}_{\Delta c}(8, k)_6=0$, because $\frac{r+n}{n}$ here is not integer: let $n=4$; then, $\frac{n-2h}{4h}=0$, the only integer value, and $R^{4aqdi}_{\Delta c}(8, k)_4=0$, because $\frac{8+4}{4}=3$, a value of r not found in our register under R^{aq} or R^{aqdi}. Therefore $R^{2h\cdot aqdi}_{\Delta c}(8, k)=0$.

XXXII. Problem d. *To find* $R_{\alpha\varepsilon}^{nh\cdot ogdi}(r, k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons built on an* n*-gonal nucleus, to have* h *loaded diagonal and* h *clear agonal axes of reversion.*

All the axes are agonal axes of the nucleus (Theorems M, N). There are $2h$ unloaded sides, carrying the agonal axes, and midway between each pair of these, a side loaded with a $\left(1+\frac{\varepsilon_n}{\alpha h}\right)$-partitioned $\left(3+2\frac{\alpha_n}{\alpha h}\right)$-gon, having at least one monogonal axis (Theorem N), which is laid alike on the $\frac{n}{4h}$th, $\frac{3n}{4h}$th, ... $\frac{2h-1}{4h}n$th sides of the nucleus; $\frac{\alpha_n}{\alpha h}$ and $\frac{\varepsilon_n}{\alpha h}$ being numbers selected from our register of monogonally reversibles. And on the mth side, counted from the unloaded one in both directions, is laid any $(1+e_m)$-partitioned $(2+a_m)$-gon, once in every interval between an agonal and a diagonal axis.

The equations to be satisfied are

$$\left.\begin{aligned} r=n \quad &+4h\left(a_1+a_2+..+a_{\frac{n-\alpha h}{\alpha h}}\right)+2h\left(2\tfrac{\alpha_n}{\alpha h}+1\right) \\ k=n-2h&+4h\left(e_1+e_2+..+e_{\frac{n-\alpha h}{\alpha h}}\right)+2h\tfrac{\varepsilon_n}{\alpha h}, \end{aligned}\right\} \quad \ldots \quad (A''.)$$

$$a_m \geqq 0,\ e_m \not> a_m - 1;$$

for reasons sufficiently given in art. XXV. These cannot exist unless

$$r-n-2h\left(2\tfrac{\alpha_n}{\alpha h}+1\right) \text{ and } k-n-2h\left(\tfrac{\varepsilon_n}{\alpha h}-1\right)$$

are divisible by $4h$. Every r-gon constructed by loading the sides of the n-gon as above described will be $(1+k)$-partitioned and reversible about h clear agonal and h loaded diagonal axes.

Leaving now undisturbed the $\left(3+2\frac{\alpha_n}{\alpha h}\right)$-gons, we can change the arrangements of diagonals in the $(2+a_1)$-gon, $(2+a_2)$-gon, &c. in

$$D(2+a_1, e_1).D(2+a_2, e_2)..D\left(2+a_{\frac{n-\alpha h}{\alpha h}}, e_{\frac{n-\alpha h}{\alpha h}}\right)=\Delta_{\frac{n-\alpha h}{\alpha h}}$$

different ways, without altering at all our solution of equations (A''.). And we can combine each of these arrangements of diagonals with every $\left(1+\frac{\varepsilon_n}{\alpha h}\right)$-partitioned $\left(3+2\frac{\alpha_n}{\alpha h}\right)$-gon in our list of monogonally reversibles, of which there are

$$M=\Sigma_m R^{m\cdot mo}\left(3+2\tfrac{\alpha_n}{\alpha h}, \tfrac{\varepsilon_n}{\alpha h}\right);$$

so that the entire number of $(1+k)$-partitioned r-gons, having not fewer than h clear agonal and h loaded diagonal axes, constructible from a single solution of (A''.), is the product $M\Delta_{\frac{n-\alpha h}{\alpha h}}$; and a similar product being formed from every solution of (A''.), counting every change of value or order of $a_1 e_1$, $a_2 e_2$, &c. as a solution, we obtain the result

$$\Sigma.M.\Delta_{\frac{n-\alpha h}{\alpha h}}=\Sigma_i R_{\alpha\varepsilon}^{nh\cdot ogdi}(r, k)_n,$$

the number of products on the left under Σ being that of those solutions; or

$$R_{\alpha\varepsilon}^{nh\cdot ogdi}(r, k)_n=\Sigma.M.\Delta_{\frac{n-\alpha h}{\alpha h}}-\Sigma_i.R_{\alpha\varepsilon}^{nh(i+1)\cdot ogdi}(r, k)_n; \ (i>0).$$

XXXIII. In equations (A″.) $\frac{n-4h}{4h}$ may have any value $\geqq 0$. When $n=4h$, we have

$$r-n=\frac{n}{2}\cdot(2\alpha_1+1),$$

$$k-\frac{n}{2}=\frac{n}{2}\varepsilon_1;$$

$$3+2\alpha_1=\frac{2r}{n};\quad \varepsilon_1=\frac{2k-n}{n};$$

hence $$R_{ac}^{n.agdi}(r,\,k)_n=0;$$

and as $\Delta_0=1$,

$$R_{ac}^{\frac{n}{2}agdi}(r,\,k)_n=\Sigma_m R^{m.mo}\left(\frac{2r}{n},\ \frac{2k-n}{n}\right);$$

which of course $=0$ if the fractions are irreducible, and also (IX.), if r is divisible by n.

From this we obtain, putting the subindex $\frac{n-4h}{4h}=1$, or $2h=\frac{n}{4}$,

$$R_{ac}^{\frac{n}{4}agdi}(r,\,k)_n=\Sigma M\Delta_1-\Sigma_m R^{m.mo}\left(\frac{2r}{n},\ \frac{2k-n}{n}\right),$$

and thus $R_{ac}^{2hagdi}(r,\,k)$ can be found in succession for every whole value of $\frac{n}{4h}$ from $2h=\frac{n}{2}$ to $2h=2$, which gives

$$r-n-2h\left(2\alpha_{\frac{n}{4h}}+1\right)\ \text{and}\ k-n-2h\left(\varepsilon_{\frac{n}{4h}}-1\right)$$

divisible by $4h$.

We proceed with our partitions of the octagon, by finding $R_{ac}^{2hagdi}(8,\,k)_n$.

$R_{ac}^{2hagdi}(8,\,k)_6=0$, because the subindex $\frac{6-4h}{4h}$ is not integer.

If $n=4$, $\frac{4-4h}{4h}=0$ and $h=1$; when (A″.) becomes $8=4+2(2\alpha_1+1)$, which is absurd;

$\therefore\ R_{ac}^{2hagdi}(8,\,k)=0$, for any nucleus.

XXXIV. Problem e. *To find* $R_0^{2hagdi}(r,\,k)_n$, *the number of* (1+k)*-partitioned* r*-gons having an* n*-gonal nucleus and* h *diagonal and* h *agonal axes of reversion, of which none are clear.*

The construction is like the preceding, with the exception that $2h$ equidistant sides, viz. the

$$n\text{th},\ \frac{n}{2h}\text{th},\ \frac{2n}{2h}\text{th},\ \ldots\ \frac{2h-1}{2h}n\text{th},$$

are all loaded with the same $(1+\varepsilon_0)$-partitioned $(4+2\alpha_0)$-gon, having at least one agonal axis of reversion, ε_0 and α_0 being any numbers in our register of agonally reversibles, $\varepsilon_{\frac{n}{4h}}$ and $\alpha_{\frac{n}{4h}}$ being found as in XXXII.

The equations to be satisfied are,

$$\left.\begin{aligned} r&=n+2h.(2+2\alpha_0)+4h\left(a_1+a_2+\ldots+a_{\frac{n}{4h}}\right)+2h\left(2\alpha_{\frac{n}{4h}}+1\right)\\ k&=n+2h.\varepsilon_0\qquad\quad +4h\left(e_1+e_2+\ldots+e_{\frac{n}{4h}}\right)+2h.\varepsilon_{\frac{n}{4h}}\end{aligned}\right\}\quad\ldots\quad(A'''.)$$

where $a_m\geqq 0$, and $e_m\geqslant a_m-1$.

An r-gon so constructed from any one solution has k diagonals, and is reversible about at least h agonal and h diagonal axes, all loaded.

We can, as before, make the product

$$\mathrm{D}(2+a_1, e_1)\mathrm{D}(2+a_2, e_2)\ldots\mathrm{D}\left(2+a_{\frac{n-ik}{ik}}, e_{\frac{n-ik}{ik}}\right)=\Delta_{\frac{n-ik}{ik}}$$

with the numbers a_1, a_2, &c. in our solution of (A$'''$.), which is the number of variations of diagonals that can be made in the $(2+a_1)$-gon, $(2+a_2)$-gon, &c., while the $4h$ reversible polygons upon the axes are undisturbed. We can also disturb these polygons alone, varying the $2h$ agonally reversibles, in

$$\Sigma_m\left(2\mathrm{R}^{m\text{-}ag}(4+2\alpha_0, \varepsilon_0)+\mathrm{R}^{2m\text{-}agdi}(4+2\alpha_0, \varepsilon_0\right)=\mathrm{H}$$

different ways, as in XXIX., and the $2h$ monogonally reversibles in

$$\Sigma_m\mathrm{R}^{m\text{-}mo}\left(3+\alpha_{\frac{n}{ik}}, \varepsilon_{\frac{n}{ik}}\right)=\mathrm{M}$$

different ways, as in XXXII., wherefore the entire number of r-gons $(1+k)$-partitioned and having h equidistant agonal and as many diagonal axes of reversion all loaded, that can be formed by one solution of (A$'''$.), is the product

$$\mathrm{H}.\mathrm{M}.\Delta_{\frac{n-ik}{ik}},$$

all the numbers $\alpha_0 \varepsilon_0$, $a_1 e_1$, &c. appearing in the product being those found in that one solution. The sum of these products, one for every possible solution of equation (A$'''$.), all orders and values of the quantities $\alpha\varepsilon$, ae, &c. counting as solutions, is the total number of $(1+k)$-partitioned r-gons, built on the n-gonal nucleus, to have h agonal and h diagonal axes of reversion none of them clear, *i. e.*

$$\Sigma_i\mathrm{R}_0^{2hi\cdot agdi}(r, k)_n=\Sigma\mathrm{H}.\mathrm{M}.\Delta_{\frac{n-ik}{ik}};$$

the number of products under Σ on the right being that of the solutions of (A$'''$.).

Wherefore $\quad \mathrm{R}_0^{2h\cdot agdi}(r, k)_n=\Sigma\mathrm{H}.\mathrm{M}.\Delta_{\frac{n-ik}{ik}}-\Sigma_i\mathrm{R}_0^{2k(i+1)agdi}(r, k)_n\ (i>0);$

for whatever be the value of i, there must be h equidistant agonal and as many diagonal axes bisecting the intervals between them.

XXXV. The highest value of h in (A$'''$.) gives $n-2h=0$, $2h=\frac{n}{2}$. The equations become in this case

$$\frac{2r-5n}{2n}=\alpha_0+\alpha_1,$$

$$\frac{2k-2n}{n}=\varepsilon_0+\varepsilon_1,$$

which have of course no solution if the fractions are irreducible; and $\Sigma\Delta_0=1$: wherefore $\mathrm{R}_0^{n\text{-}agdi}(r, k)_n=0$,

$$\mathrm{R}_0^{\frac{n}{2}\text{-}agdi}(r, k)_n=\Sigma\left\{\Sigma_m\left(2\mathrm{R}^{m\text{-}ag}(4+2\alpha_0, \varepsilon_0)+\mathrm{R}^{2m\text{-}agdi}(4+2\alpha_0, \varepsilon_0)\right)\Sigma_m\mathrm{R}^{m\text{-}mo}(3+\alpha_1, \varepsilon_1)\right\}=\Sigma\mathrm{HM}\Delta_0,$$

a sum of products easily obtained from our register of reversibles, a different product for every solution of the two equations just written in α_0, α_1, ε_0, ε_1.

We next put $\frac{n-4h}{4h}=1$, or $2h=\frac{n}{4}$, and find

$$R_0^{\frac{n}{4}\cdot agdi}(r, k)_n=\Sigma HM\Delta_1-R_0^{\frac{n}{4}\cdot agdi}(r, k)_n,$$

H, M and Δ_1 being properly determined from the numbers that satisfy

$$r=n+\tfrac{n}{4}(2+2\alpha_0)+\tfrac{n}{2}\cdot a_1+\tfrac{n}{4}(2\alpha_2+1),$$

$$k=n+\tfrac{n}{4}\cdot\varepsilon_0 \qquad +\tfrac{n}{2}\cdot e_1+\tfrac{n}{4}\cdot\varepsilon_2.$$

In like manner we can obtain $R_0^{2h\cdot agdi}(r, k)_n$ for every value of $2h$ that gives, in equations (A^m.),

$$n,\ r-6h, \text{ and } k-n-2h\left(\varepsilon_0+\varepsilon_{\frac{n}{2h}}\right),$$

each divisible by $4h$; α_0, ε_0, $\alpha_{\frac{n}{2h}}$ and $\varepsilon_{\frac{n}{2h}}$ being numbers in our register of reversibles.

It is easily seen that

$$R_0^{2h\cdot agdi}(8, k)_6=0=R_0^{2h\cdot agdi}(8, k)_4,$$

whence by what precedes we obtain, for $k>0$,

$$R^{2h\cdot agdi}(8, k)=0,$$

or the octagon has no partition ago-diagonally reversible, except itself.

Thus we have determined the four numbers

$$R_e^{2h\cdot agdi}(r, k)_n+R_{ae}^{2h\cdot agdi}(r, k)_n+R_{ae}^{2h\cdot agdi}(r, k)_n+R_0^{2h\cdot agdi}(r, k)_n=R^{2h\cdot agdi}(r, k)_n.$$

XXXVI. The next step is to obtain

$$R^{h\cdot ag}(r, k)_n=R_e^{h\cdot ag}(r, k)_n+R_{ae}^{h\cdot ag}(r, k)_n+R_0^{h\cdot ag}(r, k)_n.$$

Problem f. *To find* $R_e^{h\cdot ag}(r, k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons built on the* n*-gonal nucleus, to be reversible about* h *agonal axes, all clear.*

Whether n be odd or even, $2h$ sides of the n-gon, viz.—

$$\text{the } n\text{th, } \tfrac{n}{h}\text{th, } \tfrac{2n}{h}\text{th}\ldots\tfrac{h-1}{h}n\text{th sides,}$$

$$\text{and the } \tfrac{n}{2h}\text{th, } \tfrac{3n}{2h}\text{th}\ldots\tfrac{2h-1}{2h}n\text{th sides,}$$

which are equidistant from pairs of the preceding h, are unloaded, and the sides between the nth and $\frac{n}{2h}$th are loaded, the mth with a $(1+e_m)$-partitioned $(2+a_m)$-gon, &c., the configuration of this interval being reversed in the two adjoining intervals of $\frac{n-2h}{2h}$ sides each, so as to make a figure reversible about all the axes. And the equations following will be true:—

$$\left.\begin{aligned} r&=n+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right),\ a_m\geqq 0,\\ k&=n-2h+2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right),\ e_m\geqq a_m-1.\end{aligned}\right\}\quad\ldots\ldots\quad(\text{B.})$$

The above construction from every solution of this equation gives an r-gon k-partitioned end reversible about at least h clear agonal axes bisecting $2h$ equidistant sides, both of the r-gon and n-gon (Theorems B, N). And with this solution, *i. e.* without changing the weight or order of the imposed loads, we can, by disturbing the arrangement of the e_m diagonals of the $(2+a_m)$-gon in any way whatever, produce a new configuration about the axis bisecting the nth side; and the entire number of configurations so brought into view by such disturbances will be

$$\mathrm{D}(2+a_1,\ e_1)\mathrm{D}(2+a_2,\ e_2)\ldots\mathrm{D}\left(2+a_{\frac{n-2h}{2h}},\ e_{\frac{n-2h}{2h}}\right)=\Delta_{\frac{n-2h}{2h}};$$

and a like expression for the number of configurations generable from every solution of equation (B.), *i. e.* for every change in the weight or order of imposed loads, being added to the above, we obtain $\Sigma\Delta_{\frac{n-2h}{2h}}$ different configurations about the axis through the nth side, the number of terms under Σ being that of the solutions. Among these will be found once, and once only, every aspect of any $(1+k)$-partitioned r-gon about a clear agonal axis, which has $2h$ equidistant sides in common with the n-gon, bisected by agonal axes of reversion. Now two of these configurations are found on every one of $\mathrm{R}^{hiag}_{c}(r,\ k)_n$, for each has $i\times 2h$ such equidistant sides of the n-gon; one will be found on each of $\mathrm{R}^{2hi\cdot ag}_{\frac{1}{2}c}(r,\ k)_n$ for the same reason, and one will be found on each of $\mathrm{R}^{2hiagdi}_{ac}(r,\ k)_n$ and on each of $\mathrm{R}^{2hiagdi}_{c}(r,\ k)_n$ for the same reason (Theorem C); that is $(i>0)$,

$$\Sigma\Delta_{\frac{n-2h}{2h}}=\Sigma_i\{2\mathrm{R}^{hiag}_{c}(r,\ k)_n+\mathrm{R}^{2hiag}_{\frac{1}{2}c}(r,\ k)_n+\mathrm{R}^{2hiagdi}_{ac}(r,\ k)_n+\mathrm{R}^{2hiagdi}_{c}(r,\ k)_n\},\ \text{and}$$

$$\mathrm{R}^{h\cdot ag}_{c}(r,\ k)_n=\tfrac{1}{2}\left\{\Sigma\Delta_{\frac{n-2h}{2h}}-\Sigma_i\left(2\mathrm{R}^{h(i+1)ag}_{c}(r,\ k)_n+\mathrm{R}^{2hiag}_{\frac{1}{2}c}(r,\ k)_n+\mathrm{R}^{2hiagdi}_{ac}(r,\ k)_n+\mathrm{R}^{2hi\cdot agdi}_{c}(r,\ k_n)\right)\right\}.$$

XXXVII. Before we proceed with the discussion of this formula, it is necessary,

Problem g. *To find* $\mathrm{R}^{hag}_{\frac{1}{2}c}(\mathrm{r,\ k})_n$, *the number of* $(1+\mathrm{k})$*-partitioned* r*-gons having* $\frac{\mathrm{h}}{2}$ *clear and* $\frac{\mathrm{h}}{2}$ *loaded agonal axes, and built on the* n*-gonal nucleus.*

These r-gons have all on h sides of the n-gon, the

$$n\text{th},\ \tfrac{n}{h}\text{th},\ \tfrac{2n}{h}\text{th},\ \ldots\ \tfrac{h-1}{h}\,n\text{th sides},$$

a $(1+\varepsilon_0)$-partitioned $(4+2\alpha_0)$-gon having an agonal axis, and nothing on the

$$\tfrac{n}{2h}\text{th},\ \tfrac{3n}{2h}\text{th}\ldots\tfrac{2h-1}{2h}\,n\text{th sides}.$$

The equations to be satisfied are

$$\left.\begin{aligned} r&=n+h(2+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right),\\ k&=n-h+h\varepsilon_0\quad+2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right);\end{aligned}\right\}\quad\ldots\ldots\quad(\text{B}'.)$$

α_0 and ε_0 being any number in our list of reversibles about an agonal axis.

The configurations producible about the loaded agonal axis through the nth side of the n-gon, from any one solution of (B′.), are the product $H\Delta_{\frac{n-2h}{2h}}$, of

$$H=\Sigma_m\{2R^{maq}(4+2\alpha_0,\ \varepsilon_0)+R^{2maqdi}(4+2\alpha_0,\ \varepsilon_0)\},$$

and
$$\Delta_{\frac{n-2h}{2h}}=D(2+a_1,\ e_1).D(2+a_2,\ e_2)\ldots.D\left(2+a_{\frac{n-2h}{2h}},\ e_{\frac{n-2h}{2h}}\right);$$

and the sum of these products, one for every solution of (B′.), all values and orders of the quantities counting among the solutions, is the sum of the configurations bisected by the loaded agonal axes of all $(1+k)$-partitioned r-gons built on the n-gonal nucleus, to have $\frac{1}{2}h$ loaded agonal axes, and $\frac{1}{2}h$ clear agonal axes bisecting the equal intervals between the loaded ones.

Now the whole number of these r-gons is that of these configurations, since no such r-gon has more than one; and these r-gons are in number exactly

$$\Sigma_i R_{\frac{1}{2}c}^{(2i+1)h.aq}(r,\ k)_n=\Sigma H.\Delta_{\frac{n-2h}{2h}},\ (i\overline{>}0);$$

for all these have $(2i+1)\times\frac{1}{2}h$ loaded agonal axes, and among these are $\frac{h}{2}$ equidistant ones, in the intervals of which lie an odd number $(2i+1)$ of clear agonal axes. But $R_{\frac{1}{2}c}^{2ihaq}(r,\ k)_n$ is not in the number of these r-gons, because in the intervals of any $\frac{1}{2}h$ equidistant loaded axes of these, lies an even number of clear axes. Wherefore, $(i>0)$

$$R_{\frac{1}{2}c}^{haq}(r,\ k)_n=\Sigma\left(H\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i R_{\frac{1}{2}c}^{(2i+1)haq}(r,\ k)_n.$$

XXXVIII. The highest value of h in (B′.) gives $n-2h=0$; which reduces them to

$$\frac{2r-2n}{n}=2+2\alpha_0,\ \frac{2k-n}{n}=\varepsilon_0,\ \Delta_0=1;$$

wherefore
$$R_{\frac{1}{2}c}^{naq}(r,\ k)_n=0$$
$$R_{\frac{1}{2}c}^{\frac{n}{2}aq}(r,\ k)_n=\Sigma_m\left\{2R^{maq}\left(\frac{2r}{n},\ \frac{2k-n}{n}\right)+R^{2maqdi}\left(\frac{2r}{n},\ \frac{2k-n}{n}\right)\right\}.$$

We can next put $\frac{n-2h}{2h}=1$ in equations (B′.) and find $R_{\frac{1}{2}c}^{\frac{n}{4}aq}(r,\ k)_n$, and thus every number $R_{\frac{1}{2}c}^{haq}(r,\ k)_n$, for any even value of h, so that

$$n,\ r,\ \text{and}\ k-h(\varepsilon_0-1)$$

shall be multiples of $2h$, ε_0 being some number of diagonals on our list of agonally reversibles.

In XXXVI., equations (B.), the highest value of h gives $\frac{n-2h}{2h}=1$, or $h=\frac{n}{4}$, when those equations become

$$\frac{2r-2n}{n}=a_1,\quad \frac{2k-n}{n}=e_1,$$

so that r is divisible by n: $D(2+a_1,\ e_1)=\Sigma\Delta_1$.

Hence
$$R_{\frac{1}{2}c}^{\frac{n}{2}aqdi}(r,\ k)_n=0;$$

and since $R_c^{\frac{n}{2}\,agdi}(r, k)_n = 0$, by XXVIII.,

and $R_{ac}^{\frac{n}{2}\,agdi}(r, k)_n = 0$, by XXXIII.,

when r is divisible by n,

$$R_c^{\frac{n}{4}\,ag}(r, k)_n = \tfrac{1}{2}\{D(2+a_1, e_1) - R_{bc}^{\frac{n}{2}\,ag}(r, k)_n\}$$
$$= \tfrac{1}{2}\left\{D\left(\frac{2r}{n}, \frac{2k-n}{n}\right) - \Sigma_m\left\{2R^{m\,ag}\left(\frac{2r}{n}, \frac{2k-n}{n}\right) + R^{2m\,agdi}\left(\frac{2r}{n}, \frac{2k-n}{n}\right)\right\}\right\}.$$

If now we put in (B.) $\frac{n-2h}{2h} = 2$, we can obtain $R_c^{\frac{n}{3}\,ag}(r, k)_n$, and every number $R_c^{h\,ag}(r, k)_n$ in which n, r, and k are multiples of $2h$, including the case of $h=1$.

We shall now find $R_c^{h\,ag}(8, k)_n$ and $R_{bc}^{h\,ag}(8, k)_n$.

For $n=6$ in equations (B.), we can only put $\frac{6-2h}{2h} = 2$, $h=1$, and $k=2$; then

$$r = 8 = 6 \quad +2.1(1+0) = 6+2.1(0+1)$$
$$k = 2 = 6-2+2.1(0-1) = 6+2.1(-1+0)$$
$$\Delta_3 = 2D(3, 0) = 2$$
$$R_c^{ag}(8, 2)_6 = \tfrac{1}{2}2 = 1.$$

If $n=4$, $\frac{4-2h}{2n} =$ integer gives only $(r>n)$ $h = \frac{n}{4}$, and $\frac{2k-4}{4}$ being integer in the formula for $R_c^{\frac{n}{4}\,ag}$, gives $k=2$, or else $k=4$. Also, by the same formula,

$$R_c^{ag}(8, 2)_4 = \tfrac{1}{2}\{D(4, 0) - R^{4\,agdi}(4, 0)\} = \tfrac{1}{2}(1-1) = 0$$
$$R_c^{ag}(8, 4)_4 = \tfrac{1}{2}\left\{D(4, 1) - 2\left(R^{ag}(4, 1) + R^{agdi}(4, 1)\right)\right\} = \tfrac{1}{2}(2-0) = 1;$$

for the only entries in our register for $r=4$ are

$$R^{4\,agdi}(4, 0) = 1 \text{ and } R^{adi}(4, 1) = 1.$$

When $n=6$ in equations (B'.), $\frac{n-2h}{2h}$ is not integer for an even value of h; but when $n=4$, $h=2=\frac{n}{2}$, and by the formula for $R_{bc}^{\frac{n}{2}\,agdi}(r, k)_n$,

$$R_{bc}^{2\,ag}(8, 2)_4 = R^{4\,agdi}(4, 0) = 1;$$

for $k=2$ is the only value that finds $\frac{2k-n}{n}$ integer in our register.

XXXIX. Problem h. *To find* $R_c^{h\,ag}(r, k)_n$, *the number of* (1+k)-*partitioned* r-*gons having* h *agonal axes only of reversion, all loaded axes.*

In any one of these there are $2h$ equidistant sides, the nth, the $\frac{n}{2h}$th, the $\frac{2n}{2h}$th, &c. of the n-gon loaded, half with a $(1+\epsilon_0)$-partitioned $(4+2\alpha_0)$-gon, and the rest, alternating with these, loaded with a $\left(1+\epsilon_{\frac{n}{2h}}\right)$-partitioned $\left(4+2\alpha_{\frac{n}{2h}}\right)$-gon, all having at least one agonal axis of reversion, which is also one in the n-gon. The numbers α_0, ϵ_0, as well as $\alpha_{\frac{n}{2h}}$, $\epsilon_{\frac{n}{2h}}$,

which may be the same or not with the former, are any that we choose from our register of agonally reversibles. The intervals are loaded as before, so as to satisfy

$$\left.\begin{aligned} r &= n+h(2+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right)+h\left(2+2\alpha_{\frac{n}{2h}}\right) \\ k &= n+h\epsilon_0 \qquad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right)+h\epsilon_{\frac{n}{2h}}; \end{aligned}\right\} \quad \ldots \quad (\mathrm{B''}.)$$

where $$a_m \geqq 0, \quad e_m \geqslant a_m - 1.$$

Having chosen any solution of these equations, and posited the polygons, we can, without disturbing the first-named $2h$ polygons on the axes, make

$$\mathrm{D}(2+a_1, e_1).\mathrm{D}(2+a_2, e_2)\ldots\mathrm{D}\left(2+a_{\frac{n-2h}{2h}}, e_{\frac{n-2h}{2h}}\right)=\Delta_{\frac{n-2h}{2h}}$$

variations of the $e_1, e_2, \ldots$ diagonals, and produce about the axis through the nth side as many configurations. And with each of these variations we can combine any $(4+2\alpha_0)$-gon in our register which has ϵ_0 diagonals and an agonal axis, and also any $\left(4+2\alpha_{\frac{n}{2h}}\right)$-gon having $\epsilon_{\frac{n}{2h}}$ diagonals and an agonal axis, and in any posture giving h agonal axes of the r-gon. That is, we can, without altering our solution of (B''.), produce about the axis through the nth side $\mathrm{H}.\mathrm{H}'.\Delta_{\frac{n-2h}{2h}}$ different configurations reversible about h agonal axes all loaded as described, where

$$\mathrm{H}=\Sigma_m\{2.\mathrm{R}^{m\text{-}ag}(4+2\alpha_0, \epsilon_0)+\mathrm{R}^{m\text{-}agdi}(4+2\alpha_0, \epsilon_0)\}$$

$$\mathrm{H}'=\Sigma_m\left\{2.\mathrm{R}^{m\text{-}ag}\left(4+2\alpha_{\frac{n}{2h}}, \epsilon_{\frac{n}{2h}}\right)+\mathrm{R}^{m\text{-}agdi}\left(4+2\alpha_{\frac{n}{2h}}, \epsilon_{\frac{n}{2h}}\right)\right\}.$$

If then $\Sigma\mathrm{H}\mathrm{H}'\Delta$ denote the sum of these products made from every solution of (B''.), obtained by every change of value or order in the numbers *aa se*, this will denote the entire number of such configurations.

Now two of these are found in each of $\mathrm{R}_0^{hi\cdot ag}(r, k)_n$;

one of them, namely, about the loaded axes, in each of $\mathrm{R}_{hi}^{2hi\cdot ag}(r, k)_n$,

and these are among the configurations obtained with $\alpha_0=\alpha_{\frac{n}{2h}}$ and $\epsilon_0=\epsilon_{\frac{n}{2h}}$;

also one of them is in each of $\mathrm{R}_0^{2hi\cdot agdi}(r, k)_n$,

and one of them in each of $\mathrm{R}_{ai}^{2hi\cdot agdi}(r, k)_n$;

for both these have h equidistant agonal axes loaded, of course alike $\left(\alpha_0=\alpha_{\frac{n}{2h}} \text{ and } \epsilon_0=\epsilon_{\frac{n}{2h}}\right)$. Nor is there anything to prevent the $2hi$ terminations of axes in some of $\mathrm{R}_0^{hiag}(r, k)_n$ being loaded all alike; for this cannot hinder the configurations about alternate axes from being different. Therefore

$$\Sigma\mathrm{H}.\mathrm{H}'.\Delta_{\frac{n-2h}{2h}}=\Sigma_i\left(2\mathrm{R}_0^{hi\cdot ag}(r, k)_n+\mathrm{R}_{hi}^{2hi\cdot ag}(r, k)_n+\mathrm{R}_0^{2hi\cdot agdi}(r, k)_n+\mathrm{R}_{ai}^{2hi\cdot agdi}(r, k)_n\right), \text{ or}$$

$$\mathrm{R}_0^{hiag}(r, k)_n=\tfrac{1}{2}\left\{\Sigma\mathrm{H}.\mathrm{H}'.\Delta_{\frac{n-2h}{2h}}-\Sigma_i\left(2\mathrm{R}_0^{h(i+1)ag}(r, k)_n+\mathrm{R}_{hi}^{2hiag}(r, k)_n+\mathrm{R}_0^{2hi\cdot agdi}(r, k)_n+\mathrm{R}_{ai}^{2hi\cdot agdi}(r, k)_n\right)\right\}.$$

XL. The highest value of h in (B″.) comes from $n-2h=0$, the case when there are no intervals to load. Then,

$$\frac{r-3n}{n}=\alpha_0+\alpha_1$$

$$\frac{k-n}{n}=\epsilon_0+\epsilon_1$$

$\Delta_0=1=\Sigma\Delta_0$; so that

$$R^{n\cdot sf}_{\ 0}(r,k)_n=0,$$

and since $R^{n\cdot sf}_{\ 0}(r,k)_n=0$ (by XXIV.)$=R^{n\cdot sf}_{\ \frac{n}{h}}(r,k)_n$ by XXVII.,

$$R^{\frac{n}{2}\cdot sf}_{\ 0}(r,k)_n=\tfrac{1}{2}\{\Sigma H.H'-R^{n\cdot sf}_{\ \frac{n}{2}}(r,k)_n\},$$

H and H′ being given functions of $\alpha_0\epsilon_0$ and $\alpha_1\epsilon_1$, expressed in the preceding article, of course both $=0$, if r and k are not multiples of n.

We next put $n-2h=2h$ in (B″.), and readily deduce $R^{\frac{n}{4}\cdot sf}_{\ 0}(r,k)_n$; and thus in order we obtain $R^{h\cdot sf}_{\ 0}(r,k)_n$ for every value of $h>1$, which makes

$$n,\ r,\text{ and } k-h\left(\epsilon_0+\epsilon_{\frac{n}{2h}}\right) \text{ divisible by } 2h;$$

conditions necessary for the solution of equations (B″.). We cannot find $R^{sf}_{\ 0}(r,k)_n$ by this formula ($h=1$), for there is no such division reducible to a nucleus (K, XIII.). We are to proceed with our partitions of the octagon. In equation (B″.) $r-n\not<4$, $\therefore n\not>4$. When $n=4$, $\frac{4-2h}{2h}=1$ or $=0$, giving $h=1$ or $h=2$, the latter of which is inadmissible, since $8-4\not<2h.2$. Therefore

$$r=8=4+1.2+2.0+1.2;\text{ or } \alpha_0=\alpha_{\frac{n}{2h}}=\alpha_1=0,$$
$$k=2=4+2.-1,$$

and D(2, −1) is to be considered unity $=\Delta_1$;

$$R^{sf}_{\ 0}(8,2)_4=\tfrac{1}{2}\{R^{4\cdot sf}(4,0).R^{4\cdot sf}(4,0)-R^{2\cdot sf}_{\ 0}(8,2)_4\}$$
$$=\tfrac{1}{2}(1.1-1)=0.$$

Thus we have found (XXXVI. XXXVII. XXXIX.),

$$R^{h\cdot sf}(r,k)_n=R^{h\cdot sf}_{\ c}(r,k)_n+R^{h\cdot sf}_{\ 0}(r,k)_n+R^{h\cdot sf}_{\ \infty}(r,k)_n.$$

XLI. We have to investigate next, formulæ for

$$R^{h\cdot sf}_{\ 0}(r,k)_n+R^{h\cdot sf}_{\ \infty}(r,k)_n+R^{h\cdot sf}_{\ c}(r,k)_n=R^{h\cdot sf}(r,k)_n.$$

Problem i. *To find* $R^{h\cdot sf}_{\ 0}(r,k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons having diagonal axes only of reversion, all loaded, and built on an* n*-gonal nucleus.*

Each r-gon of this number has a $(1+\epsilon_0)$-partitioned $(3+2\alpha_0)$-gon laid on h equidistant sides, the nth, the $\frac{n}{h}$th, the $\frac{2n}{h}$th, &c. of the r-gon, and a $\left(1+\epsilon_{\frac{n}{2h}}\right)$-partitioned $\left(3+2\alpha_{\frac{n}{2h}}\right)$-gon laid on h sides each equidistant from two of the other h, all being polygons reversible

about at least one monogonal axis, which is an agonal axis of the n-gon (Theorem N), bisecting the side on which the polygon is placed. The intervals of $\frac{n-2h}{2h}$ sides being loaded as in the previous constructions, the equations of condition are

$$\left.\begin{aligned} r=n+h(1+2\alpha_0)+2h\left(\alpha_1+\alpha_2+\ldots+\alpha_{\frac{n-2h}{2h}}\right)+h\left(1+2\alpha_{\frac{n}{2h}}\right) \\ \mathrm{R}=n+h.\varepsilon_0 \qquad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right)+h\varepsilon_{\frac{n}{2h}} \\ \alpha_m \geqq 0, \quad e_m \geqslant a_m-1, \end{aligned}\right\} \quad \ldots \quad (\mathrm{C.})$$

and both α, ε are chosen from our register of monogonally reversibles. Leaving undisturbed the $2h$ polygons on the axes, we have in our power, from any one solution of equations (C.),

$$\mathrm{D}(2+a_1, e_1)\mathrm{D}(2+a_2, e_2)\ldots\mathrm{D}\left(2+a_{\frac{n-2h}{2h}}, e_{\frac{n-2h}{2h}}\right)=\Delta_{\frac{n-2h}{2h}}$$

variations of arrangement of the e_1, e_2, &c. diagonals in the intervals, producing as many configurations about the axis through the nth side of the n-gon, and all reversible about at least h monogonal axes. With any one of these variations we can combine any polygon of the number

$$\mathrm{M}=\Sigma_m \mathrm{R}^{m.\,mo}(3+2\alpha_0, \varepsilon_0),$$

and any one of

$$\mathrm{M}'=\Sigma_m \mathrm{R}^{m.\,mo}\left(3+2\alpha_{\frac{n}{2h}}, \varepsilon_{\frac{n}{2h}}\right);$$

so that we obtain $\mathrm{MM}'\Delta_{\frac{n-2h}{2h}}$ configurations from this one solution of (C.).

If we form this product for every possible solution, in every order of values of α a, ε e, &c., we shall produce about the axis through the nth side of the n-gon,

$$\Sigma\mathrm{MM}'\Delta_{\frac{n-2h}{2h}}$$

different configurations.

Now two of these are seen in each of $\mathrm{R}_0^{hi.\,di}(r, k)_n$,

one of them in each of $\mathrm{R}_{\frac{1}{2}e}^{2hi.\,di}(r, k)_n$,

one in each of $\mathrm{R}_0^{2hi.\,ogdi}(r, k)_n$,

and one in each of $\mathrm{R}_{\infty}^{2hi.\,ogdi}(r, k)_n$;

for every one of the last three classes has $h\times i$ terminations of agonal axes of the n-gon loaded with monogonally reversible polygons. Wherefore

$$\Sigma\mathrm{M}.\mathrm{M}'.\Delta_{\frac{n-2h}{2h}}=\Sigma_i\{2\mathrm{R}_0^{hi.\,di}(r, k)_n+\mathrm{R}_{\frac{1}{2}e}^{2hi.\,di}(r, k)_n+\mathrm{R}_0^{2hi.\,ogdi}(r, k)_n+\mathrm{R}_{\infty}^{2hi.\,ogdi}(r, k)_n\},$$

and $(i>0)$,

$$\mathrm{R}_0^{hi}(r, k)_n=\tfrac{1}{2}\left\{\Sigma\mathrm{MM}'\Delta_{\frac{n-2h}{2h}}-\Sigma_i\left(2\mathrm{R}_0^{h(i+1)\,di}(r, k)_n+\mathrm{R}_{\frac{1}{2}e}^{2hi\,di}(r, k)_n+\mathrm{R}_0^{2hi\,ogdi}(r, k)_n+\mathrm{R}_{\infty}^{2hi.\,ogdi}(r, k)_n\right)\right\}.$$

This result is useless until $\mathrm{R}_{\frac{1}{2}e}^{hi}(r, k)_n$ is known.

XLII. Problem k. *To find* $\mathrm{R}_{\frac{1}{2}e}^{hi}(r, k)_n$, *the number of* (1+k)*-partitioned* r*-gons built on an* n*-gonal nucleus, to have* $\frac{1}{2}$h *loaded and* $\frac{1}{2}$h *clear axes only, all diagonal axes.*

A clear diagonal axis of the r-gon is a diagonal of the nucleus, evidently. These

r-gons will have only n sides of the n-gon loaded with $(1+\epsilon_0)$-partitioned $(3+2\alpha_0)$-gons monogonally reversible, namely the nth, the $\frac{n}{h}$th, the $\frac{2n}{h}$th, &c. sides. And half the interval between the nth and $\frac{n}{h}$th, namely $\frac{n-h}{2h}$ sides, will be loaded, the mth with a $(1+\epsilon_m)$-partitioned $(2+\alpha_m)$-gon, which will be placed also on the $\left(\frac{n}{h}-m\right)$th side, so as to form a figure reversible about the diagonal bisecting that interval.

Our equations of condition are now

$$\left.\begin{aligned} r &= n+h(1+2\alpha_0)+2h\left(a_1+a_2+..+a_{\frac{n-h}{2h}}\right)\\ k &= n+h\epsilon_0 \qquad\quad +2h\left(e_1+e_2+..+e_{\frac{n-h}{2h}}\right)\\ a_m &\overline{>} 0,\ e_m \geqslant a_m-1, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (C.)$$

where α_0 and ϵ_0 are any numbers we select from our list of monogonally reversibles.

Reasoning as in the previous constructions, we deduce that the entire sum, made from every solution of $(C.)$, of products $M\Delta$ of

$$M=\Sigma_m R^{m\cdot m}(3+2\alpha_0,\ \epsilon_0), \text{ and}$$

$$\Delta_{\frac{n-h}{2h}}=D(2+a_1,\ e_1).D(2+a_2,\ e_2)\ldots D\left(2+a_{\frac{n-h}{2h}},\ e_{\frac{n-h}{2h}}\right),$$

is the complete number of configurations about a loaded axis, of r-gons having $\frac{h}{2}$ equidistant loaded, and bisecting the intervals between these, $\frac{h}{2}$ clear, diagonal axes of reversion.

Now one of these configurations about a loaded axis is seen in each of $R_{\frac{h}{2}}^{(2i+1)h\cdot di}(r,\ k)_n$; for it has in the intervals of any equidistant $\frac{h}{2}$ loaded axis $2i+1$ clear ones, one of them bisecting each interval. But none of the class $R_{\frac{h}{2}}^{2ih\cdot di}(r,\ k)_n$ has that configuration, because in the equal intervals of $\frac{h}{2}$ loaded axes it has an even number of clear ones, and none bisecting an interval. And no ago-diagonally reversible can have this configuration, because the diagonal axes are in these all clear or all loaded.

Wherefore $$\Sigma.M\Delta_{\frac{n-h}{2h}}=\Sigma_i R_{\frac{h}{2}}^{(2i+1)h\cdot di}(r,\ k)_n,$$

and $$R_{\frac{h}{2}}^{hdi}(r,\ k)_n=\Sigma M.\Delta_{\frac{n-h}{2h}}-\Sigma_i R_{\frac{h}{2}}^{(2i+3)h\cdot di}(r,\ k)_n;\ (i\overline{>}0),$$

where $(2i+3)h$ is a value h' that makes $\frac{n-h'}{2h'}$ integer.

XLIII. The greatest value of h in $(C.)$ is $h=n$, when there is no interval between the axes. Then,

$$\frac{r-2n}{n}=2\alpha_0,\ \frac{k-n}{n}=\epsilon_0,$$

$$\Delta_0=1=\Sigma\Delta,\quad M=\Sigma M,$$

and $$R_{\frac{n}{2}}^{di}(r,\ k)_n=\Sigma_m R^{m\cdot m}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right),$$

which $=0$, if r and k are not multiples of n.

Next putting $\frac{n-h}{2h}=1$, we find $R_{\frac{1}{2}c}^{\frac{n}{2}\cdot\frac{di}{2}}(r,k)_n$, and so on for every even value of $h>0$ that makes $n-h,\ r-n-h(1+2\alpha_0)$, and $k-n-hs_0$, all divisible by $2h$.

The greatest value of h in equation (C.), XLI., is $h=\frac{1}{2}n$. Those equations become

$$\frac{r-2n}{n}=\alpha_0+\alpha_1$$

$$\frac{2k-2n}{n}=s_0+s_1;$$

and $$\Delta_0=1=\Sigma\Delta_0,$$

wherefore, $$R_0^{n\cdot di}(r,k)=0,$$

and because (XXXIII., XXXV.)

$$R_{\infty}^{n\cdot\alpha p di}(r,k)_n=0=R_0^{n\cdot\alpha p di}(r,k)_n,$$

$$R_0^{\frac{1}{2}n\cdot di}(r,k)_n=\tfrac{1}{2}\Sigma M.M'-\tfrac{1}{2}R_{\frac{1}{2}c}^{n\cdot di}(r,k)_n,$$

or $$R_0^{\frac{1}{2}n\cdot di}(r,k)_n=\tfrac{1}{2}\Sigma.\left\{\Sigma_m\left(R^{m\cdot m}(3+2\alpha_0,s_0).\Sigma_m R^{m\cdot m}(3+2\alpha_1,s_1)\right)\right\}-\tfrac{1}{2}\Sigma_m\left(R^{m\cdot m}\left(\frac{r+n}{n},\frac{k-n}{n}\right)\right);$$

the number of products MM' under the first Σ being that of the solutions of the above equations in $\alpha_0\,\alpha_1$, $s_0\,s_1$, every order of values counting for a solution. From this $R_0^{\frac{1}{2}n\cdot di}(r,k)_n$ can be found for every value of $h>1$ in equation (C.), XLI., that makes n, r, and $k-h\left(s_0+s_{\frac{n}{2h}}\right)$ divisible by $2h$, conditions imposed by those equations. But we cannot attempt by these formulæ to find $R_0^{di}(r,k)_n$, $(h=1)$, because (XIII.) this subclass reduces to no nucleus.

XLIV. We proceed with our partitions of the 8-gon, to find $R_{\frac{1}{2}c}^{\frac{1}{2}\cdot di}(8,k)_n$ and $R_0^{\frac{1}{2}\cdot di}(8,k)_n$.

In equations (C.), $n=6$ or $n=4$, for $r=8$; and $\frac{6-h}{2h}=$integer gives only $h=2$. Hence

$$r=8=6+2.(1+0)+4.0,$$

$$k=2=6+2.0+4.-1,$$

the only value of k, because $s_0\not>1+2\alpha_0-1$. Wherefore

$$R_{\frac{1}{2}c}^{2di}(8,2)_6=R^{m\cdot 0}(3,0)\Delta_1=1.$$

Next, if $n=4$, $\frac{4-h}{2h}=$integer, $=0$, gives $h=4$,

$$r=8=4+4(1+0)$$

$$k=4=4+4.0,$$

the only value of k;

$$R_{\frac{1}{2}c}^{4di}(8,4)_4=R^{m\cdot 0}(3,0)\Delta_0=1.$$

Next, in equations (C.), $r=8$ gives $n=6$, or $n=4$, of which the first, putting $\frac{6-2h}{2h}=$integer, gives only $h=1$, the value above forbidden. But $\frac{4-2h}{2h}=$integer $=0$, gives $h=2$; whence $k=4$, and by the formula for $R_0^{\frac{1}{2}n\cdot di}(r,k)_n$ we obtain $(\alpha_0=s_0=0)$,

$$R_0^{2di}(8,4)_4=\tfrac{1}{2}\{R(3,0).R(3,0)-R_{\frac{1}{2}c}^{4di}(8,4)_4\}=\tfrac{1}{2}\{1-1\}=0.$$

XLV. Problem l. *To find* $R_c^{h\cdot di}(r, k)_n$, *the number of* (1+k)*-partitioned* r*-gons built on the* n*-gonal nucleus, which have* h *clear diagonal axes of reversion, and none others.*

In these all the axes are diagonal axes of the nucleus, and terminate at $2h$ equidistant angles of it. The interval of $\frac{n}{2h}$ sides between any two, as the 1st, 2nd, ... $\frac{n}{2h}$th sides of the n-gon, is loaded, the mth side with a $(1+e_m)$-partitioned $(2+a_m)$-gon, as in the previous constructions, so that all shall be reversible about every axis. We have to solve

$$\left.\begin{aligned} r&=n+2h\left(a_1+a_2+\ldots a_{\frac{n}{2h}}\right)\\ k&=n+2h\left(e_1+e_2+\ldots e_{\frac{n}{2h}}\right)\end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (C'.)$$

$$a_m \geqq 0,\ e_m \geqq a_m - 1.$$

The sum of products, one for every solution, all orders of the values counted among solutions,

$$\Sigma\Delta_{\frac{n}{2h}} = \Sigma D(2+a_1,\ e_1)D(2+a_2,\ e_2)\ldots D\left(2+a_{\frac{n}{2h}},\ e_{\frac{n}{2h}}\right),$$

is the entire number of configurations about a clear diagonal axis, if r-gons have h equidistant clear diagonal axes, and built on this nucleus.

Now of these configurations

two are seen on each of $R_c^{hi\cdot di}(r, k)_n$,

one is seen on each of $R_{\frac{1}{2}c}^{2h\cdot di}(r, k)_n$,

namely, that about the clear axes in these,

one is seen on each of $R_c^{2h\cdot eqdi}(r, k)_n$,

and one on each of $R_{\frac{1}{2}c}^{2h\cdot eqdi}(r, k)_n$,

namely, that about the diagonal axes in these. Wherefore $(i>0)$,

$$\Sigma\Delta_{\frac{n}{2h}} = \Sigma_i\{2R_c^{hi\cdot di}(r, k)_n + R_{\frac{1}{2}c}^{2hi\cdot di}(r, k)_n + R_c^{n\cdot eqdi}(r, k)_n + R_{\frac{1}{2}c}^{n\cdot eqdi}(r, k)_n\}, \text{ and}$$

$$2\cdot R_c^{h\cdot di}(r, k)_n = \Sigma\Delta_{\frac{n}{2h}} - \Sigma_i\{2R_c^{(i+1)h\cdot di}(r, k)_n + R_{\frac{1}{2}c}^{2hi\cdot di}(r, k)_n + R_c^{n\cdot eqdi}(r, k)_n + R_{\frac{1}{2}c}^{n\cdot eqdi}(r, k)_n\}.$$

As $r>n$ in $(C'.)$, the greatest value of h gives $\frac{n}{2h}=1$. Those equations then become

$$\frac{r-n}{n}=a_1,\ \frac{k-n}{n}=e_1$$

$$\Delta_1 = D\left(\frac{r+n}{n},\ \frac{k-n}{n}\right) = \Sigma\Delta_1;$$

hence $R_c^{di}(r, k)_n = 0,$

and since $R_c^{n\cdot eqdi}(r, k) = 0$ (XVIII.),

$$R_{\frac{1}{2}c}^{hn\cdot di}(r, k)_n = \frac{1}{2}\left\{D\left(\frac{r+n}{n},\ \frac{k-n}{n}\right) - R_{\frac{1}{2}c}^{n\cdot di}(r, k)_n - R_{\frac{1}{2}c}^{n\cdot eqdi}(r, k)_n\right\}$$

a given number, by XLIII. and XXX.

And we can proceed to find $R_c^{h\cdot di}(r, k)_n$, for every value of h down to $h=1$, which gives, in $(C'.)$, r, n, and k, divisible by $2h$.

Thus we have completely determined

$$R^{h\cdot a}(r,\,k)_n = R^{h\cdot a}_e(r,\,k)_n + R^{h\cdot a}_{\bar{e}}(r,\,k)_n + R^{h\cdot a}_{\bar{v}}(r,\,k)_n.$$

XLVI. When $r=8$ in equation (C''.), $n=6$, or $n=4$. If $n=6$, $\frac{n}{2h}=3$ and $h=1$ are the only values making $\frac{8}{2h}$ integer. We have then the three solutions

$$r=8=6+2(1+0+0)=6+2(0+1+0)=6+2(0+0+1),$$
$$k=2=6+2(0-1-1)=6+2(-1+0-1)=6+2(-1-1+0);$$
$$\Sigma\Delta_3=3D(3,\,0)=3, \text{ and we obtain}$$
$$R^a_e(8,\,2)_4=\tfrac{1}{3}\{3-R^{a\cdot a}_{\bar{v}}(8,\,2)_4\}=1.$$

If $n=4$, $h=2$ or $h=1$; for the first of which we obtain by the formula for $R^{h\cdot ai}_e(r,\,k)_n$,

$$R^{a_2}_e(8,\,4)_4=\tfrac{1}{3}\{D(3,\,0)-R^{a_2}_{\bar{v}}(8,\,4)_4\}=\tfrac{1}{3}(1-1)=0,$$

$k=4$ being the only possible value.

If $h=1$, we may have $k=4$ or $k=2$; for $k=4$, we find the three solutions $\left(\frac{n}{2h}=2\right)$,

$$8=4+2(2+0)=4+2(0+2)=4+2(1+1)$$
$$4=4+2(1-1)=4+2(-1+1)=4+2(0+0);$$
$$\Sigma\Delta_3=2D(4,\,1)+D(3,\,0)=4+1=5,$$

whence $\quad R^a_e(8,\,4)_4=\tfrac{1}{3}\{5-R^{a_2}_{\bar{v}}(8,\,4)_4\}=2.$

For $k=2$, there are two solutions,

$$8=4+2(2+0)=4+2(0+2),$$
$$2=4+2(0-1)=4+2(-1+0);$$

giving $\quad \Sigma\Delta_3=2D(4,\,0)=2,$

and $\quad R^a_e(8,\,2)_4=\tfrac{1}{3}(2-0)=1;$

for there is nothing in our list under $R(8,\,2)_4$ to subtract in the formula for $2R^{h\cdot ai}_e(r,\,k)_n$.

XLVII. Our next step is to determine

$$R^{h\cdot m_0}(r,\,k)n = R^{h\cdot m_0}_e(r,\,k)_n + R^{h\cdot m_0}_{\bar{e}}(r,\,k)_n.$$

Problem m. *To find* $R^{h\cdot m_0}(r,\,k)_n$, *the number of* (1+k)*-partitioned* r*-gons, built on the* n*-gonal nucleus to have* h *clear monogonal axes of reversion, and none besides.*

The h clear axes (Theorem O) are monogonal axes of the n-gon, bisecting h equidistant sides of it; between which lie h vertices of the n-gon upon those h axes; and n is odd. The interval of $\frac{n-h}{2h}$ sides between such a vertex and bisected side is loaded the mth from the vertex with a $(1+e_m)$-partitioned $(2+a_m)$-gon, so that all is reversible about the h axes. The equations following are satisfied in the construction:

$$\left.\begin{aligned} r&=n+\quad 2h\left(a_1+a_2+..+a_{\frac{n-h}{2h}}\right)\\ k&=n-h+2h\left(e_1+e_2+..+e_{\frac{n-h}{2h}}\right)\end{aligned}\right\};\quad (D.)$$

$$(a_m\geqq 0),\ (e_m\geqq a_m-1).$$

For every solution of these equations we can make

$$D(2+a_1,\ e_1)D(2+a_2,\ e_2)\times\ldots D\left(2+a_{\frac{n-h}{2h}},\ e_{\frac{n-h}{2h}}\right)=\Delta_{\frac{n-h}{2h}}$$

different arrangements of the e_1, e_2, &c. diagonals, giving as many distinct configurations about the monogonal axis through the nth side of the n-gon. And, as before, we obtain $\Sigma\Delta_{\frac{n-h}{2h}}$ different configurations in all by adding together as many such products as there are solutions of equations (D.), every order of the quantities counted as a solution; which sum is the entire number of configurations of $(1+k)$-partitioned r-gons built on this nucleus to have h clear monogonal equidistant axes of reversion.

Now one of these configurations is seen on each r-gon of the number $R_c^{(i+1)h,\,mo}(r,\ k)_n$; for each has h equidistant monogonal axes, and on none besides, because if one monogonal axis is clear, all are clear; otherwise there would be more than two axial configurations, and the axes are odd in number (Theorem F). Wherefore

$$\Sigma\Delta_{\frac{n-h}{2h}}=\Sigma_i R_c^{(2i+1)h,\,mo}(r,\ k)_n,$$

and $(i\geqq 0)$

$$R_c^{h,\,mo}(r,\ k)_n=\Sigma\Delta_{\frac{n-h}{2h}}-\Sigma_i R_c^{(2i+2)h,\,mo}(r,\ k)_n.$$

XLVIII. Since $r>n$ in equations (D.), $n>h$. Let $n-h=2h$; then $h=\frac{n}{3}$ is the highest value, and

$$r=n+\frac{2n}{3}a_1,\qquad 2+a_1=\frac{3r+n}{2n},$$

$$k=\frac{2n}{3}+\frac{2n}{3}e_1,\qquad e_1=\frac{3k-2n}{2n};$$

$$\Delta_1=\Sigma\Delta_1=D\left(\frac{3r+n}{2n},\ \frac{3k-2n}{2n}\right),$$

and

$$R_c^{n,\,mo}(r,\ k)_n=0,$$

$$R_c^{\frac{n}{3},\,mo}(r,\ k)_n=D\left(\frac{3r+n}{2n},\ \frac{3k-2n}{2n}\right),$$

which of course $=0$ if the fractions are irreducible.

From this we can proceed to find $R_c^{h,\,mo}(r,\ k)_n$, for every odd value of h down to $h=1$ which makes $n-h$, $r-n$, and k, divisible by $2h$.

The octagon has no monogonal axis of reversion (Theorem E).

XLIX. Problem n. *To find* $R_e^{h,\,mo}(r,\ k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons built on the* n*-gonal nucleus to have* h *only monogonal axes of reversion, all loaded.*

The axes will be either all monogonal or all agonal axes of the n-gon (Theorem N, O), for every axis of the r-gon carries the same two configurations (Theorem C). Let them first be all monogonal, *i. e.* let n be odd; this requires that each be loaded at one end

with an agonally reversible (about one or more axes) $(1+\epsilon_0)$-partitioned $(4+2\alpha_0)$-gon. The intervals of $\frac{n-h}{2h}$ sides will be loaded as in the last preceding construction, to satisfy

$$\left.\begin{aligned} r&=n+h(2+2\alpha_0)+2h\left(a_1+a_2+..+a_{\frac{n-h}{2h}}\right),\\ k&=n+h\epsilon_0 \qquad\quad +2h\left(e_1+e_2+..+e_{\frac{n-h}{2h}}\right);\end{aligned}\right\} \quad \ldots\ldots \text{(D′.)}$$

$$(a_m \overline{\geqq} 0,\quad e_m \neq a_m-1),\ (n=2m+1),$$

$(4+2\alpha_0)$ and ϵ_0 being numbers selected at pleasure from our register of agonally or ago-diagonally reversibles. Putting, as in XIX.,

$$\Sigma_m\{2R^{m\cdot ag}(4+2\alpha_0,\ \epsilon_0)+R^{2m\cdot agd}(4+2\alpha_0,\ \epsilon_0)\}=H,$$

and forming the product $\Delta_{\frac{n-h}{2h}}$ from a solution of (D′.) which contains those numbers α_0, ϵ_0, we deduce, as before, that $\Sigma H.\Delta_{\frac{n-h}{2h}}$, comprising as many products under Σ as those solutions of (D′.), all orders of the quantities a_i e_i, &c. being counted as solutions, is the total number of axial configurations of $(1+k)$-partitioned r-gons having h loaded monogonal axes and built on the n-gonal nucleus when n is odd.

And as one of these configurations is seen on each r-gon of the number $R_0^{(2i+1)h\cdot mo}(r,k)_n$, and only one, read from the loaded end of the axis, we obtain

$$\Sigma\left(H.\Delta_{\frac{n-h}{2h}}\right)=\Sigma_i R_0^{(2i+1)h\cdot mo}(r,\ k)_n,$$

and

$$R_0^{h\cdot mo}(r,\ k)_n=\Sigma H.\Delta_{\frac{n-h}{2h}}-\Sigma_i R_0^{(2i+3)h\cdot mo}(r,\ k)_n,$$

where n is odd, and $i \overline{\geqq} 0$.

L. Now let the h axes be agonal axes of the n-gon; *i. e.* let n be even.

One end of every axis will carry (Theorem O) the same $(3+2\alpha_0)$-gon $(1+\epsilon_0)$-partitioned and reversible about at least one monogonal axis. The other ends will carry the same $\left(2+2\alpha_{\frac{n}{2h}}\right)$-gon, $\left(1+\epsilon_{\frac{n}{2h}}\right)$-partitioned and reversible about at least one agonal axis, which will be no load if $\alpha_{\frac{n}{2h}}=0$. The interval of $\frac{n-2h}{2h}$ sides between these reversibles will be loaded, on the mth side from the first, with a $(1+e_m)$-partitioned $(2+a_m)$-gon; and the equations to be satisfied are,

$$\left.\begin{aligned} r&=n+h(1+2\alpha_0)+2h\left(a_1+a_2+..+a_{\frac{n-2h}{2h}}\right)+h\alpha_{\frac{n}{2h}},\\ k&=n+h\epsilon_0 \qquad\quad +2h\left(e_1+e_2+..+e_{\frac{n-2h}{2h}}\right)+h\epsilon_{\frac{n}{2h}};\end{aligned}\right\} \quad \ldots\ldots \text{(D″.)}$$

where $a_m \overline{\geqq} 0$, $e_m \neq a_m-1$, and α_0, ϵ_0, $\alpha_{\frac{n}{2h}}$, $\epsilon_{\frac{n}{2h}}$ are chosen at pleasure, the first pair from our register of monogonally reversibles, and the second from our agonally or ago-diagonally reversibles.

Taking now any one solution of these equations, the product $H''M\Delta$ of

$$H''=\Sigma_m\left\{2R^{m\cdot ag}\left(2+\alpha_{\frac{n}{2h}},\ \epsilon_{\frac{n}{2h}}\right)+R^{2m\cdot agd}\left(2+\alpha_{\frac{n}{2h}},\ \epsilon_{\frac{n}{2h}}\right)\right\},$$

(which is to be considered unity when $\alpha_{\frac{n}{2h}}=0$, the load being then simply the edge of the n-gon),

$$M=\Sigma_m R^{m\cdot m^0}(3+\alpha_0,\ \epsilon_0),$$

$$\Delta_{\frac{n-2h}{2h}}=D(2+a_1,\ e_1).D(2+a_2,\ e_2)\times..D\left(2+\alpha_{\frac{n-2h}{2h}},\ e_{\frac{n-2h}{2h}}\right),$$

is the number of configurations that we can produce, about the axis through the nth side of the n-gon, read from that side, of $(1+k)$-partitioned r-gons built on this nucleus from this one solution, to have h loaded monogonal axes. And the sum of these products belonging to every solution, all orders and values counting as solutions, is the exact number of all such configurations possible. Now one, and only one, of these is seen in each r-gon of the number $R_0^{(2i+1)h\cdot m^0}(r,\ k)_n$, for n even. That is,

$$\Sigma\left(H''.M.\Delta_{\frac{n-2h}{2h}}\right)=\Sigma_i R_0^{(2i+1)h\cdot m^0}(r,\ k)_n,$$

$$R_0^{h\cdot m^0}(r,\ k)_n=\Sigma\left(H''.M.\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i R_0^{(2i+2)h\cdot m^0}(r,\ k)_n,$$

where n is even, and $i\geqq 0$; and $H''=1$ whenever $\alpha_{\frac{n}{2h}}=0$.

LI. In equation (D′.), the highest value of h is $h=n$. Then those equations become

$$\frac{r-n}{n}=2+2\alpha_0,\ \frac{k-n}{n}=\epsilon_0$$

$$H=\Sigma(H\Delta_0)=\Sigma_m\left\{2R^{m\cdot q}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right)+R^{2m\cdot q\cdot 0}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right)\right\}=R_0^{n\cdot m^0}(r,\ k)_n,$$

since $\Delta_0=1$, where n is odd; but $R_0^{n\cdot m^0}(r,\ k)_n=0$ of course, if n is even.

From this we can proceed to find $R_0^{h\cdot m^0}(r,\ k)_n$, for all odd values of $h>1$, which make, in equations (D′.),

$$n-h,\ r-n-h(2+2\alpha_0)\text{ and }k-n-h\epsilon_0,$$

divisible by $2h$; α_0 and ϵ_0 being selected from our register of agonally and ago-diagonally reversibles.

In equations (D″.), the greatest value of h gives $n-2h=0$, or $h=\frac{1}{2}n$. Those equations become

$$\frac{2r-3n}{2n}=\alpha_0+\alpha_1,$$

$$\frac{2k-2n}{n}=\epsilon_0+\epsilon_1,$$

$$\Delta_0=\Sigma\Delta_0=1,$$

whence, if $\frac{1}{2}n$ be an odd number (for h is always odd),

$$R_0^{\frac{n}{2}\cdot m^0}(r,\ k)_n=\Sigma H''.M,$$

$\left(\text{but this is }=0\text{ if }\frac{n}{2}\text{ be even}\right)$, H'' and M being defined in the preceding article, and the number of products $H''.M$ being equal to the number of solutions of the above reduced

form of equations (D″.), made with numbers α_0, ϵ_0, $\alpha_{\frac{n}{2h}}$, $\epsilon_{\frac{n}{2h}}$, selected from our register, as above stated (L.).

From this value of $R_0^{\frac{n}{2}mo}(r, k)_n$, we can proceed to find, when n is even, $R_0^{hmo}(r, k)_n$ for all values of $h>1$, which make

$$n, r-h\left(1+2\alpha_0+\alpha_{\frac{n}{2h}}\right), \text{ and } k-h\left(\epsilon_0+\epsilon_{\frac{n}{2h}}\right),$$

divisible by $2h$. But we cannot find $R_0^{mo}(r, k)_n$ by these formulæ, there being no such division reducible to a nucleus (VIII.).

Thus we have found (XLIX., L.) $R_0^{hmo}(r, k)_n$ for any value of $n>2$.

LII. All the numbers R of article XX. have now been determined, for every value of h when the axes are not all loaded, and for every value of $h>1$ when they are all loaded, the nucleus being n-gonal, and $n\not<3$. We have next to enumerate the doubly and singly reversibles which are built on the 2-gonal nucleus, that is, upon a line, which is of course a drawn axis of reversion of the r-gon, and also the singly reversibles which have an e-scored axis. These have no proper nucleus (Theorem K).

These r-gons now to be discussed fall into the classes following,

$$R^{2agdi}(r, k)_2,\ R^{2di}(r, k)_2,\ R^{di}(r, k)_2,\ R_0^{ag}(r, k),\ R_0^{di}(r, k),\ R_0^{mo}(r, k);$$

of which the first two have a drawn diagonal axis standing as a perpendicular score upon an undrawn axis (Theorem G); the third has a single drawn diagonal axis (Theorem G); and the fourth, fifth and sixth have a single undrawn and scored axis of reversion (Theorem K). The subscript 2 in the first three symbols shows that the nucleus is a drawn line, which may be considered as a 2-gon reversible about a diagonal and an agonal axis.

Problem o. *To find* $R^{2agdi}(r, k)_2$, *the* (1+k)*-partitioned* r*-gons built on a nucleus-line to have a diagonal and an agonal axis of reversion.*

If the drawn diagonal be erased, the figure will be still reversible about the same two axes undrawn, for (Theorem B, VI.) the erasure has not disturbed the symmetry about either axis. But the erasure may have restored the symmetry about some n-gonal nucleus of which that erased line is a clear diagonal axis (Theorem R, XVIII.). And every r-gon which has a clear diagonal axis perpendicular to an agonal one, that is, every one of $R_0^{(4h+2)agdi}(r, k-1)_n$ and of $R_{dc}^{(4h+2)agdi}(r, k-1)_n$, whatever h or n may be, if $n>2$ (*Cor.* 1, VII.), will give us one and only one (Theorem G) of $R^{2agdi}(r, k)_2$, by the drawing of any diagonal axis of reversion; for all these axes bisect the same configuration. Wherefore

$$R^{2agdi}(r, k)_2=\Sigma_h\Sigma_n\{R_0^{(4h+2)agdi}(r, k-1)_n+R_{dc}^{(4h+2)agdi}(r, k-1)_n\},$$

including all values of h, and of $n>2$ in our register; n of course being even, else it could have no diameter.

As we have nothing in our register under $R^{(4h+2)agdi}(8, k)_n$, $R^{2agdi}(8, k)_2=0$, whatever k may be.

LIII. Problem p. *To find* $R^{2di}(r, k)_2$, *the number of* (1+k)*-partitioned* r*-gons built on a line-nucleus, to have two diagonal axes of reversion.*

If the drawn diagonal axis be erased, the figure still remains reversible about the same axis, as in the preceding article, and we may have restored the symmetry about some n-gonal nucleus of which that erased line is a clear diagonal axis. Every r-gon of the number

$$R_e^{2h,di}(r, k-1)_n$$

will give us two of $R^{2di}(r, k)_n$, for we can draw a diagonal axis bisecting either of its configurations; and every r-gon of the number

$$(R_{de}^{2h,di}+R_e^{(2h+q),di}+R_{de}^{(2h+q),di})(r, k-1)_n,$$

which have all a clear diagonal axis at right angles to a diagonal axis (VI. VII. *Cor.* 1), will give us one of $R^{2di}(r, k)_n$, by drawing any diagonal axis of reversion, which can be drawn without meeting diagonals; *i. e.*

$$R^{2di}(r, k)_n=\Sigma_h\Sigma_n\{2R_e^{2h,di}(r, k-1)_n+R_{de}^{2h,di}(r, k-1)_n+R_e^{(2h+q),di}(r, k-1)_n+R_{de}^{(2h+q),di}(r, k-1)_n\};$$

comprehending all the values of $h>0$ and $n>2$ in our register of reversibles.

Hence follows (XLIV.),

$$R^{2di}(8, 5)_3=R_{de}^{4di}(8, 4)_4 =1,$$
$$R^{2di}(8, 3)_3=R_{de}^{4di}(8, 2)_6 =1,$$
$$R^{2di}(8, 1)_3=R_e^{4q,di}(8, 0)_3=1.$$

LIV. Problem q. *To find* $R^{di}(r, k)_n$, *the number of* (1+k)*-partitioned singly reversible* r*-gons having a drawn diagonal axis.*

If this axis be erased, the figure F remains reversible about the same diagonal axis, and the erasure may have restored the symmetry about some n-gonal nucleus of which that erased line is a clear diagonal axis, one of an odd number of axes of reversion in F; for if this number were even, there would then be an axis (VI.) perpendicular to the erased one, about which the symmetry would not be disturbed by the erasure, so that the figure before the erasure must have been at least doubly reversible, contrary to hypothesis.

Now every one of the number $R_e^{(2h+1),di}(r, k-1)_n$ will give one (Theorems B, G), and only one, of $R^{di}(r, k)_n$, by drawing of any of its axes of reversion, because all these carry the same configuration. Consequently

$$R^{di}(r, k)_n=\Sigma_h\Sigma_n R_e^{(2h+1),di}(r, k-1)_n,$$

comprehending all values of h including $h=0$, and all of n, in our register of k-partitions. Hence we obtain (XLVI.),

$$R^{di}(8, 5)_3=R_e^{di}(8, 4)_4=2,$$
$$R^{di}(8, 3)_3=R_e^{di}(8, 2)_6+R_e^{di}(8, 2)_4=2.$$

LV. Problem r. *To find* $R_e^{me}(r, k)$, *the number of* (1+k)*-partitioned* r*-gons reversible only about a single scored monogonal axis.*

The number of ways in which e diagonals can be drawn at right angles across an axis

of one of the r-gons of the division $R_c^{(2h+1)me}(r, k-e)_n$ is $\left(\frac{n-3}{2}\right)^{e-1}.\overline{e+1}^{-1}$; for any e vertices may be chosen out of $\frac{n-3}{2}$ on one side of the monogonal axis. The whole number of operations described in Theorems P and S, by which a $(1+k)$-partitioned monogonal can arise, is the above operation performed on every $(1+k-e)$-partitioned monogonal in our register for every value of e, n, and h. The results, by Theorems E, P, and S, are all monogonally reversibles. And by Theorem Q we see that every $(1+k)$-partitioned $(2h+3)$-ly reversible about loaded monogonal axes will be among these results, whatever be its nucleus; and every singly reversible with a loaded monogonal axis will be among them; for if this be cleared, we have before us one of our subjects of operation just referred to in our register. Wherefore

$$\Sigma_h\Sigma_e\Sigma_n\left(\frac{n-3}{2}\right)^{e-1}.\overline{e+1}^{-1}.R_c^{(2h+1)me}(r, k-e)_n=\Sigma_h R_0^{(2h+1)me}(r, k),$$

and $$R_0^{me}(r, k)=\Sigma_h\Sigma_e\Sigma_n\left(\frac{n-3}{2}\right)^{e-1}.\overline{e+1}^{-1}.R_c^{(2h+1)me}(r, k-e)_n-\Sigma_h R_0^{(2h+3)me}(r, k),$$

where the omission of the subscript n shows that all nuclei are included. This is a given number; and as no monogonally reversible can be obtained by scoring any but a monogonal axis, it is the whole number $R_0^{me}(r, k)$.

LVI. Problem s. *To find* $R_0^{se}(r, k)$, *the number of* $(1+k)$*-partitioned* r*-gons singly reversible about a scored agonal axis.*

$R_0^{se}(r, k)$ and $R_0^{de}(r, k)$ are obtained by the operations of Theorems P, R, S, by scoring both $2h$-ly and $(2h+1)$-ly reversibles. We write

$$R_0^{se}(r, k)=R_0^{se}(r, k)'+R_0^{se}(r, k)'',$$

the first of the right member denoting the number of those r-gons obtained by scoring one of an odd number, and the second, those by one of an even number, of axes.

The number $R_0^{se}(r, k)'$ is obtained by Theorems P, Q, S, exactly as $R_0^{me}(r, k)$ was obtained, except that the number of angles of the nucleus from which on one side of the clear axis e scores can be drawn, is $\frac{n-4}{2}$ in the first, instead of $\frac{n-3}{2}$ in the second. That is,

$$R_0^{se}(r, k)'=\Sigma_e\Sigma_n\left(\frac{n-4}{2}\right)^{e-1}.\overline{e+1}^{-1}.R_c^{(2h+1)se}(r, k-e)_n-\Sigma_h R_0^{(2h+3)se}(r, k),$$

where again the omission of the subscript n shows that constructions upon all nuclei are included.

LVII. We have next to determine $R_0^{se}(r, k)''$, obtained by scoring a clear axis out of an even number of axes (Theorem R).

Whatever e may be, the constructions of Theorem R are enumerated by subtracting the symmetrical arrangements of the e scores from the whole number of arrangements, and dividing the remainder by two; for every unsymmetrical one occurs twice in this remainder, once as read from each end of the scored axis, which readings are the two configurations of the singly reversible about the axis (Theorem C).

Let e first be even, in constructing $R^{\prime\prime}_{n}(r, k)''$. The entire number of e-plets eligible from the $\frac{n-4}{2}$ numerals on one side of the axis is

$$\left(\frac{n-4}{2}\right)^{e-1}.\overline{e+1}^{-1},$$

and that of the $\frac{e}{2}$-plets eligible alike on opposite sides of the centre of the axis is

$$\left(\frac{n-4}{4}\right)^{\frac{e}{2}-1}\times\overline{\frac{e}{2}+1}^{-1},$$

when $n=4m$, and, when $n=4m+2$, it is

$$\left(\frac{n-6}{4}\right)^{\frac{e}{2}-1}\times\overline{\frac{e}{2}+1};$$

wherefore, using the circulators 4_n and 4_{n-3} and 2_e,

$$\frac{2_e}{2}\left\{\left(\frac{n-4}{2}\right)^{e-1}.\overline{e+1}^{-1}-\left(\frac{n-4}{4}\right)^{\frac{e}{2}-1}.\overline{\frac{e}{2}+1}.4_n-\left(\frac{n-6}{4}\right)^{\frac{e}{2}-1}.\overline{\frac{e}{2}+1}.4_{n-3}\right\}. \quad (a.)$$

is the number out of $R^{\prime\prime}_{n}(r, k)''$, when e is even, that can be made from any $(1+k-e)$-partitioned r-gon having an n-gonal nucleus, $2h$ clear axes, and clear agonal ones, by scoring any one of these with e parallels at right angles to it.

But when e is odd, a diameter of the n-gon must be one of the scores if they be symmetrical about the centre, which requires $n=4m+2$. Hence, if $n=4m$, every odd e scores that can be drawn gives an unsymmetrical arrangement, so that there are thus

$$4_n.2_{e-1}.\tfrac{1}{2}\left(\frac{n-4}{2}\right)^{e-1}.\overline{e+1}^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b.)$$

out of $R^{\prime\prime}_{n}(r, k)''$ obtained by so scoring any clear agonal axis of a $2h$-ly reversible.

If $n=4m+2$, and e is odd, we are to subtract from the whole number of ways of drawing e scores that of the ways of doing it symmetrically, which, when we have drawn a diameter for one, is that of the ways of drawing $\frac{1}{2}(e-1)$ scores on one side of it. This difference, divided by two, is

$$4_{n-3}.2_{e-1}.\tfrac{1}{2}\left(\left(\frac{n-4}{2}\right)^{e-1}.\overline{e+1}^{-1}-\left(\frac{n-6}{4}\right)^{\frac{e-1}{2}-1}.\overline{\frac{e+1}{2}}^{-1}\right), \quad . \quad . \quad . \quad . \quad (c.)$$

which is the number of ways of drawing an odd number e of scores across the agonal axis of a $(4m+2)$-gonal nucleus.

Uniting the expressions $(a.)$, $(b.)$, $(c.)$, we find

$$\tfrac{1}{2}\left\{\left(\frac{n-4}{2}\right)^{e-1}.\overline{e+1}^{-1}-2_e\left(4_n.\left(\frac{n-4}{4}\right)^{\frac{e}{2}-1}.\overline{\frac{e}{2}+1}^{-1}+4_{n-3}.\left(\frac{n-6}{4}\right)^{\frac{e}{2}-1}.\overline{\frac{e}{2}+1}^{-1}\right)\right.$$
$$\left.-2_{e-1}\left(4_{n-3}.\left(\frac{n-6}{4}\right)^{\frac{e-1}{2}-1}.\overline{\frac{e+1}{2}}^{-1}\right)\right\}$$

for the correct number out of $R^{\prime\prime}_{n}(r, k)''$ that can be made by drawing e diagonals of the

n-gonal nucleus in any $2h$-ly reversible $(1+k-e)$-partitioned r-gon, across a clear agonal axis.

When the axes are all clear and agonal, we can score across either configuration, *i. e.* upon either of two adjacent axes, and thus obtain twice this number towards $R_v^{ag}(r, k)''$; but if only half are clear and agonal, we have only one configuration to score, and can obtain this number only. Hence we see that the product under $\Sigma_h\Sigma_e\Sigma_n$ of the number above written, into

$$\Sigma_h\Sigma_e\Sigma_n\{2R_c^{2h\cdot ag}(r, k-e)_n+R_{\frac{1}{2}c}^{2h\cdot ag}(r, k-e)_n+R_c^{2h\cdot agdi}(r, k-e)_n+R_{ac}^{2hagdi}(r, k-e)_n\},$$

is the entire sum $R_v^{ag}(r, k)''$, of r-gons in $R_v^{ag}(r, k)$ obtainable by scoring one of an even number of axes, by the construction of Theorem R.

LVIII. Adding, then, to this entire sum the portion of $R_v^{ag}(r, k)$ obtained by scoring one of an odd number of axes (LVI.), we express the complete result thus:—

$$R_v^{ag}(r, k)=\Sigma_h\Sigma_e\Sigma_n\Big[R_c^{(2h+1)ag}(r, k-e)_n\cdot\Big(\frac{n-4}{2}\Big)^{n-1}\cdot\overline{e+1}^{-1}-R_v^{(2h+3)ag}(r, k)$$

$$+\tfrac{1}{2}\{2R^{2hag}(r, k-e)_n+R_{\frac{1}{2}c}^{2hag}(r, k-e)_n+R_c^{2h\cdot agdi}(r, k-e)_n+R_{ac}^{2h\cdot agdi}(r, k-e)_n\}$$

$$\times\Big\{\Big(\frac{n-4}{2}\Big)^{n-1}\cdot\overline{e+1}^{-1}-2_e\Big(4_n\cdot\Big(\frac{n-4}{4}\Big)^{\frac{e}{2}-1}\cdot\overline{\frac{e}{2}+1}^{-1}+4_{n-2}\cdot\Big(\frac{n-6}{4}\Big)^{\frac{e}{2}-1}\overline{\frac{e}{2}+1}^{-1}\Big)$$

$$-2_{e-1}\Big(4_{n-2}\cdot\Big(\frac{n-6}{4}\Big)^{\frac{e-1}{2}-1}\cdot\overline{\frac{e+1}{2}}^{-1}\Big)\Big\}\Big],$$

comprising all the values of e, h and n on our register of $(1+k-e)$-partitioned reversibles specified in the right member: $(n>2,\ e>0,\ h\geqq 0)$.

When $k=e$, the r-gon scored is the simple r-gon, or $n=r$.

The only numbers of those in the right member of this expression that our list contains for $r=8$, are

$$R_c^{2agdi}(8, 0)_8=1,\ R_c^{ag}(8, 2)_6=1,\ R_c^{ag}(8, 4)_4=1,\ R_{\frac{1}{2}c}^{2ag}(8, 2)_4=1$$

(XXVIII. XXXVIII.), of which the two last disappear by the multiplier $n-4$, since e always >0. Therefore

$$R_v^{ag}(8, 1)=\tfrac{1}{2}\Big(R_c^{2agdi}(8, 0)_8\times\frac{8-4}{2.1}\Big)=1$$

$$R_v^{ag}(8, 2)=\tfrac{1}{2}\Big\{R_c^{2\cdot agdi}(8, 0)_8\times\Big((\tfrac{4}{3})^{8-1}\cdot\overline{3}^{-1}-\frac{4}{4.1}\Big)\Big\}=0$$

$$R_v^{ag}(8, 3)=R_c^{ag}(8, 2)_6\times\Big(\frac{6-4}{2}\cdot 1\Big)=1$$

$$R_v^{ag}(8, 4)=R_c^{ag}(8, 2)_6\times(\tfrac{3}{2})^{8-1}\cdot\overline{3}^{-1}=0.$$

LIX. Problem t. *To find* $R_v^{di}(r, k)$, *the number of* $(1+k)$*-partitioned* r*-gons singly reversible about a scored diagonal axis.*

We write again in the sense of art. LVI.,

$$R_v^{di}(r, k)=R_v^{di}(r, k)'+R_v^{di}(r, k)''.$$

The number of angles of the nucleus from which, on one side of a clear diagonal axis, e scores can be drawn, is $\frac{1}{2}(n-2)$. Wherefore, by the reasoning of LV.,

$$R^{a}_{r}(r,\ k)'=\Sigma_{e}\Sigma_{m}\left(\frac{n-2}{2}\right)^{e-1}.\overline{e+1}^{-1}.R^{(2h+1)m}_{e}(r,\ k-e)_{m}-\Sigma_{s}R^{(2h+2)s}_{e}(r,\ k).$$

LX. When the scored axis is a diagonal one out of an even number of axes, the constructions $R^{a}_{r}(r,\ k)''$ of Theorem R are enumerated by a subtraction and division like those of LVII. Let e first be even.

The entire number of e-plets eligible from the $\frac{n-2}{2}$ numerals on one side of this diagonal axis, is $\left(\frac{n-2}{2}\right)^{e-1}\times\overline{e+1}^{-1}$; and that of $\frac{e}{2}$-plets eligible alike on opposite sides of the centre of the axis is $\left(\frac{n-2}{4}\right)^{\frac{e}{2}-1}\times\overline{\frac{e}{2}+1}^{-1}$, when $n=4m+2$, and $\left(\frac{n-4}{4}\right)^{\frac{e}{2}-1}.\overline{\frac{e}{2}+1}^{-1}$, when $n=4m$: wherefore

$$\tfrac{1}{2}\left\{\left(\frac{n-2}{2}\right)^{e-1}.\overline{e+1}^{-1}-\left(\frac{n-2}{4}\right)^{\frac{e}{2}-1}.\overline{\frac{e}{2}+1}^{-1}.4_{m-s}-\left(\frac{n-4}{4}\right)^{\frac{e}{2}-1}.\overline{\frac{e}{2}+1}^{-1}.4_{m}\right\}.2_{s}\ .\quad(a.)$$

is the number out of $R^{a}_{r}(r,\ k)''$, when e is even, that can be made by e-scoring a clear diagonal axis of a $2h$-ly reversible $(1+k-e)$-partitioned r-gon.

But when e is odd, a diameter of the r-gon must be one of the e scores, if all is symmetrical about the centre, which requires $r=4m$. Then when $n=4m+2$, every odd e scores that can be drawn gives an unsymmetrical configuration, so that there are thus

$$4_{m-s}.2_{e-1}.\left\{\frac{1}{2}.\left(\frac{n-2}{2}\right)^{e-1}.\overline{e+1}^{-1}\right\},\text{ out of }R^{a}_{r}(r,\ k)'',\quad.\quad.\quad.\quad.\quad.\quad(b.)$$

obtained by so scoring a clear diagonal axis of a $2h$-ly reversible. And when $n=4m$, we have merely to subtract from this entire number of ways as before, the number of symmetrical ways, which, after drawing a diameter, is that of drawing $\frac{1}{2}(e-1)$ scores on one side of the centre, and then to divide by two, as before, thus:—

$$\tfrac{1}{2}.2_{e-1}.4_{m}\left(\left(\frac{n-2}{2}\right)^{e-1}.\overline{e+1}^{-1}-\left(\frac{n-4}{4}\right)^{\frac{e-1}{2}-1}.\overline{\frac{e}{2}+1}^{-1}\right).\quad.\quad.\quad.\quad.\quad.\quad(c.)$$

Then uniting the expressions $(a.)$, $(b.)$, $(c.)$, we find

$$\tfrac{1}{2}\left\{\left(\frac{n-2}{2}\right)^{e-1}.\overline{e+1}^{-1}-2_{e}\left(4_{m-s}.\left(\frac{n-2}{4}\right)^{\frac{e}{2}-1}.\overline{\frac{e}{2}+1}^{-1}+4_{m}.\left(\frac{n-4}{4}\right)^{\frac{e}{2}-1}.\overline{\frac{e}{2}+1}^{-1}\right)\right.$$
$$\left.-2_{e-1}\left(4_{m}.\left(\frac{n-4}{4}\right)^{\frac{e-1}{2}-1}.\overline{\frac{e+1}{2}}^{-1}\right)\right\},$$

for the exact number out of $R^{a}_{r}(r,\ k)''$ that can be made from any one $2h$-ly reversible $(1+k-e)$-partitioned r-gon, by drawing e scores upon a clear diagonal axis.

When the axes of the r-gon are all clear and diagonal, we obtain twice this number, for we can score an axis carrying either of two configurations; but we obtain it once

only, when only half the axes are clear and diagonal. That is, if we multiply this number, under $\Sigma_s\Sigma_n$, by

$$\Sigma_s\Sigma_r\Sigma_n\{2R_v^{2s,di}(r,\,k-e)_n+R_{v'}^{2s\cdot di}(r,\,k-e)_n+R_v^{2s\cdot ogdi}(r,\,k-e)_n+R_{v'}^{2s\cdot ogdi}(r,\,k-e)_n\},$$

we have the full number $R_v^{di}(r,\,k)''$, out of $R_v^{di}(r,\,k)$, that are made by scoring one of an even number of axes.

LXI. Adding the expression in LIX. to this product, we obtain

$$R_v^{di}(r,\,k)=\Sigma_s\Sigma_r\Sigma_n\Big[R_v^{(2s+1)\cdot di}(r,\,k-e)_n\cdot\Big(\frac{n-2}{2}\Big)^{d-1}\cdot\overline{|e+1}^{-1}-R_v^{(2s+2)\cdot di}(r,\,k)$$

$$+\tfrac{1}{2}\{2R_v^{2s,di}(r,\,k-e)_n+R_{v'}^{2s,di}(r,\,k-e)_n+R_v^{2s\cdot ogdi}(r,\,k-e)_n+R_{v'}^{2s\cdot ogdi}(r,\,k-e)_n\}$$

$$\times\Big\{\Big(\frac{n-2}{2}\Big)^{d-1}\cdot\overline{|e+1}^{-1}-2_r\Big(4_{n-2}\cdot\Big(\frac{n-2}{4}\Big)^{\frac{e}{2}|-1}\cdot\overline{\Big|\frac{e}{2}+1}^{-1}+4_n\Big(\frac{n-4}{4}\Big)^{\frac{e}{2}|-1}\cdot\overline{\Big|\frac{e}{2}+1}^{-1}\Big)$$

$$-2_{r-1}\Big(4_n\cdot\Big(\frac{n-4}{4}\Big)^{\frac{e-1}{2}|-1}\cdot\overline{\Big|\frac{e+1}{2}}^{-1}\Big)\Big\}\Big],$$

comprising all the values of e, h, and n, in our register of $(1+k-e)$-partitioned r-gons here specified ($h\equiv 0$, $e>0$, $n>2$).

Thus we have determined the six classes of doubly and singly reversibles of art. LII., which are not reducible to a polygonal nucleus, and can register them in order as well as the twelve classes of reversibles of art. XX., which are so reducible.

The only numbers in the right member of the above equation contained in our list are

$$R_v^{s\cdot ogdi}(8,\,0)_8=1,\ R_{v'}^{di}(8,\,2)_8=1,\ R_v^{di}(8,\,2)_4=1,\ R_v^{di}(8,\,4)_4=2,\ R_{v'}^{di}(8,\,4)_4=1,$$

(XLIV. and XLVI.). Wherefore

$$R_v^{di}(8,1)=\tfrac{1}{2}R_v^{s\cdot ogdi}(8,\,0)_8\Big(\frac{8-2}{2}\cdot 1-\Big(\frac{8-4}{4}\Big)^0\Big) \qquad =1,$$

$$R_v^{di}(8,\,2)=\tfrac{1}{2}R_v^{s\cdot ogdi}(8,\,0)_8\Big\{\frac{3.2}{1.2}-\frac{8-4}{4}\Big\} \qquad =1,$$

$$R_v^{di}(8,\,3)=R_v^{di}(8,\,2)_8\cdot\frac{6-2}{2}+R_v^{di}(8,\,2)_4\cdot\frac{4-2}{2}+\tfrac{1}{2}\Big\{R_{v'}^{di}(8,\,2)_8\cdot\frac{6-2}{2}\Big\} \qquad =4,$$

$$R_v^{di}(8,\,4)=R_v^{di}(8,\,2)_8\cdot\Big(\frac{6-2}{2}\Big)^{|-1}\cdot\frac{1}{1.2}+\tfrac{1}{2}\Big\{R_{v'}^{di}(8,\,2)_8\Big(\Big(\frac{6-2}{2}\Big)^{|-1}\cdot\frac{1}{1.2}-\frac{6-2}{4}\Big)\Big\}=1.1+0=1,$$

$$R_v^{di}(8,\,5)=R_v^{di}(8,\,4)_4\cdot 1+\tfrac{1}{2}\cdot R_{v'}^{di}(8,\,4)_4(1-0^s) \qquad =2.$$

This completes our register of reversible partitions of the octagon. Collecting them from XXVIII., XXXVIII., XLIV., XLVI., LIII., LIV., LVIII., we add to those above written,

$$R_v^{s\cdot ogdi}(8,\,0)_8=1,\ R_v^{sg}(8,\,2)_8=1,\ R_v^{sg}(8,\,4)_4=1,\ R_{v'}^{sg}(8,\,2)_4=1,$$
$$R_{v'}^{di}(8,\,2)_8=1,\ R_{v'}^{di}(8,\,4)_4=1,$$
$$R_v^{di}(8,\,2)_8=1,\ R_v^{di}(8,\,4)_4=2,\ R_v^{di}(8,\,2)_4=1,$$
$$R^{sdi}(8,\,5)_8=1,\ R^{sdi}(8,\,3)_8=1,\ R^{sdi}(8,\,1)_8=1,$$
$$R^{di}(8,\,5)_8=2,\ R^{di}(8,\,3)_8=2,$$
$$R_v^{sg}(8,\,1)=1,\ R_v^{sg}(8,\,3)=1.$$

That is,

$R(8, 1)=2$, $R(8, 2)=4$, $R(8, 3)=7$, $R(8, 4)=4$, $R(8, 5)=4$, $R^1(8, 1)=1$, $R^1(8, 2)=2$
$R^1(8, 3)=1$, $R^1(8, 5)=1$, $R^1(8, 4)=1$, $R^1(8, 0)=1$.

The singly reversibles $R(r, k)$ form seven classes,

$$R_v^{ev}(r, k)_n,\ R_v^{ds}(r, k)_n,\ R_v^{ms}(r, k)_n,\ R^{ds}(r, k)_n,\ R_0^{ev}(r, k),\ R_0^{ds}(r, k),\ R_0^{ms}(r, k).$$

LXII. We have now to investigate the number of irreversible $(1+k)$-partitions of the r-gon. And when we have determined $I^{m+1}(r, k)_n$, $m \geqq 0$, the equations of art. XXIII. will enable us to complete the tedious solution of our problem. It is necessary to demonstrate the theorems following.

THEOREM V. *Every* h*-ly irreversible partition of an* r*-gon, if* $h>2$, *and every doubly irreversible partition in which a diameter of the* r*-gon is not drawn, is regularly built on a polygonal nucleus.*

For in each of the h sequences of configuration (art. III.) in the circuit of the r-gon, we see f marginal faces. These being all erased, we see an r'-gon ($r'<r$) with fh fewer diagonals, and still h sequences, for those of the r-gon have been treated alike. If this r'-gon is reversibly partitioned, it is not singly reversible, because it has repeated sequences (V.); therefore the theorem is proved by XII.; for there is no drawn diameter by hypothesis. If the r'-gon be still irreversible, it has h' sequences ($h' \leqq h$), from which marginal faces can be removed as before; and thus we shall finally reduce the figure, either to a reversibly-partitioned r''-gon, or to one having no marginal faces, *i. e.* to an n-gonal nucleus. And by reversing the process of reduction, the r-gon can be regularly constructed on that nucleus. Q. E. D.

LXIII. THEOREM W. *If the nucleus line of a doubly irreversible* r*-gon built on it be erased, the figure becomes* 2h*-ly reversible or else* 2h*-ly irreversible, and has no drawn diameter.*

For it has still two irreversible sequences, occupying each half the circuit of the r-gon, since the two sequences terminated by the nucleus-line have been treated alike by the erasure; wherefore it can neither be oddly reversible nor oddly irreversible (*Obs.* 1, 3, IV.; and 5, VII.). And evidently the r-gon cannot have a drawn diameter meeting its nucleus line.

Cor. The r-gon, after the erasure, is one of those reducible to a polygonal nucleus.

THEOREM X. *If in any* 2m*-ly reversible, or* 2m*-ly irreversible,* k*-partition of an* r*-gon, a diameter be drawn which is not an axis of reversion, and which meets no diagonal, the figure becomes a doubly irreversible* $(1+k)$*-partition of the* r*-gon, built on that drawn line.*

For such a drawn line disturbs the symmetry about every axis of a reversible, because it is (VI.) perpendicular to none of them; therefore the result is not reversible. And every $2m$-ly reversible or $2m$-ly irreversible has an irreversible sequence occupying half its circuit (III, V.), beginning at any angle not on an axis of reversion; therefore the result is not singly irreversible. And as there is no other sequence terminated by or exhibiting a drawn diameter, since two cannot be drawn, the result has no sequence

more than twice repeated, and is doubly irreversible. And evidently the figure is built on the drawn line, by applying to it the same polygon on both sides, so as not to give a reversible result.

LXIV. Problem u. *To find* $\mathrm{I}^{h}(r, k)_{n}$, $(h>1, n>2)$, *the number of* h*-ly irreversible* $(1+k)$*-partitions of the* r*-gon, which are built on the* n*-gonal nucleus.*

The construction differs from that of reversibles, in that the $(1+e_m)$-partitioned $(2+a_m)$-gon which is laid on the mth side of the n-gon in the first interval of $\frac{n}{h}$ sides, is laid also on the mth side of every interval, so that the series of loads imposed is h times repeated, and not reverted. The result will be an r-gon in whose circuit a sequence occupying $\frac{n}{h}$ sides of the nucleus is h times repeated.

The equations to be satisfied are

$$\left.\begin{aligned} r&=n+h\left(a_1+a_2+\ldots+a_{\frac{n}{h}}\right)\\ k&=n+h\left(e_1+e_2+\ldots+e_{\frac{n}{h}}\right)\end{aligned}\right\}, \qquad \ldots\ldots\ldots \text{(E.)}$$

$$a_m \overline{>} 0, \quad e_m \not> a_m - 1;$$

exactly as in our previous constructions.

We take any one solution, and lay on the polygons, and then make every possible change in the arrangement of the diagonals of the imposed polygons; this gives us

$$\mathrm{D}(2+a_1, e_1)\mathrm{D}(2+a_2, e_2)\ldots\mathrm{D}\left(2+a_{\frac{n}{h}}, e_{\frac{n}{h}}\right)=\Delta_{\frac{n}{h}}$$

different sequences read in the r-gon upon the first $\frac{n}{h}$ sides of its nucleus, which are all h times repeated in the same order in the circuit of the r-gon. We then take every other solution of (E.) that can arise from changing either the values or order of the numbers $a_1, a_2 \ldots, e_1, e_2 \ldots$; and forming a similar product from each solution, products which will some of them be equal to each other, we shall have enumerated in the sum $\Sigma\Delta_{\frac{n}{h}}$, every possible sequence that can be read on $\frac{n}{h}$ successive sides of the n-gonal nucleus of any $(1+k)$-partitioned r-gon, which has that sequence h times repeated in its circuit. And no two of these sequences can be alike, because they are either made from different solutions of (E.), or have a different arrangement of diagonals in the $\frac{n}{h}$ imposed polygons.

Now every hi-ly irreversible $(1+k)$-partition of the r-gon has $\frac{2n}{hi}$, and no more, of these sequences; for it has a different i-ple sequence of $\frac{n}{h}$ vertices commencing at every one of $\frac{n}{hi}$ successive vertices, and no more, reading always in one direction, because the sequence beginning at the $\left(\frac{n}{hi}+1\right)$th vertex is identical with that commencing at the first; and this i-ple sequence of $\frac{n}{h}$ vertices is h times repeated in the circuit, because a simple sequence

is hi times repeated. And as the hi-ly irreversible has the same sequences reversed on its other face, giving $\frac{n}{hi}$ different aspects which we have all constructed as read from the first side of the n-gon in one direction, we have had this same hi-ly irreversible $2n:hi$ times before us among our results.

Again, every hi-ly reversible $(1+k)$-partitioned r-gon has been $\frac{n}{hi}$ times brought before us in our $\Sigma\Delta_{\frac{n}{i}}$ constructions. For this has a different i-ple sequence reversible or irreversible of $\frac{n}{h}$ vertices of the n-gonal nucleus commencing at any one of $\frac{n}{hi}$ successive vertices, and we never see a sequence repeated in it till we have read over an interval equal to that between two alternate axes, that is an interval of $\frac{n}{hi}$ sides of the n-gon (art. V.). This i-ple sequence is h times repeated in the circuit of the r-gon; so that we have counted this hi-ly reversible $\frac{n}{hi}$ times among those constructed from equation (E.). Wherefore

$$\Sigma\Delta_{\frac{n}{i}}=\Sigma_i\left\{\frac{2n}{hi}\mathrm{I}^{hi}(r,k)_n+\frac{n}{hi}\mathrm{R}^{hi}(r,k)_n\right\}, \text{ and}$$

$$\mathrm{I}^h(r,k)_n=\frac{h}{2n}\left(\Sigma\Delta_{\frac{n}{i}}-\Sigma_i\left(\frac{2n}{h(i+1)}\mathrm{I}^{(i+1)h}(r,k)_n+\frac{n}{hi}\mathrm{R}^{hi}(r,k)_n\right)\right), (i>0).$$

LXV. The highest value of h in (E.) comes from putting the subindex $\frac{n}{h}=1$, or $h=n$. This gives

$$\mathrm{I}^n(r,k)_n=\tfrac{1}{2}\left\{\mathrm{D}\left(\frac{r+n}{n},\frac{k-n}{n}\right)-\mathrm{R}^n(r,k)_n\right\},$$

a given number, since $\mathrm{R}^n(r,k)_n$ is found in our register.

Next putting $\frac{n}{h}=2$, or whatever next factor n may have, we can obtain $\mathrm{I}^h(r,k)_n$ for every value of $h>1$ that is a divisor of the three numbers r, k and n.

Since $h>1$, $n>2$, r, k and n are none of them prime; the only values of $\mathrm{I}^h(8,k)_n$ that give them all divisible by h, and $k\not>8-3$, are

$$\mathrm{I}^2(8,2)_8,\ \mathrm{I}^2(8,2)_4,\ \mathrm{I}^2(8,4)_4, \text{ and } \mathrm{I}^4(8,4)_4.$$

By the last written formula,

$$\mathrm{I}^4(8,4)_4=\tfrac{1}{2}\{\mathrm{D}(3,0)-\mathrm{R}^4(8,4)_4\}=\tfrac{1}{2}(1-1)=0,$$

for $\mathrm{I}^2(8,2)_8$ we have three solutions, $\left(\frac{n}{h}=3\right)$

$$8=6+2(1+0+0)=\&c.$$
$$2=6+2(0-1-1)=\&c.$$

giving $\Sigma\Delta_3=3\mathrm{D}(3,0)=3$, and as $\mathrm{R}^2(8,2)_8=1$,

$$\mathrm{I}^2(8,2)_8=\tfrac{2}{8}\{3-\tfrac{8}{2}\mathrm{R}^2(8,2)_8\}=\tfrac{1}{4}(3-3)=0.$$

For $I^3(8, 2)_4$ we have $\left(\frac{n}{h}=2\right)$,

$$8=4+2.(1+1)=4+2(2+0)=4+2(0+2)$$
$$2=\qquad\qquad 4+2(0-1)=4+2(-1+0)$$
$$\Sigma\Delta_3=2D(4, 0)=2, \text{ and } R^3(8, 2)_4=1,$$
$$\therefore\ I^3(8, 2)_4=\tfrac{2}{3}\left\{2-\tfrac{4}{2.1}.1\right\}=0.$$

For $I^3(8, 4)_4$ we have $\left(\frac{n}{h}=2\right)$,

$$8=4+2(1+1)=4+2(0+2)=4+2(2+0)$$
$$4=4+2(0+0)=4+2(-1+1)=4+2(1-1)$$
$$\Sigma\Delta_3=D(3, 0)D(3, 0)+2D(4, 1)=1+4=5,$$

and $R^4(8, 4)_4=1$, whence

$$I^3(8, 4)_4=\tfrac{2}{3}\left\{5-\tfrac{4}{2.2}.1\right\}=1,$$

which is the only form of $I^4(8, k)_n$, for $h>1$, $n>2$.

LXVI. Problem w. *To find* $I^3(r, k)_n$, *the doubly irreversible* $(1+k)$*-partitions of the r-gon, which have a nucleus line.*

It is proved in Theorems W and X, that these are all obtained by drawing a diameter δ which is not an axis of reversion in one of the r-gons $I^{2m}(r, k-1)_n$ and $R^{2m}(r, k-1)_n$; and that any such diameter drawn in any one of these gives us one of $I^3(r, k)_n$. And it is evident that δ is a diameter of the nucleus about which all is symmetrical (*Cor.* LXIII.).

When no diameter of the nucleus is a diagonal axis of reversion of the r-gon, every diameter that can be drawn is such a line δ. Now in $I^{2h}(r, k-1)_n$ you may draw δ from any one θ of $\frac{n}{2h}$ vertices of the nucleus, and from no more, because the configuration about θ is repeated $2h$ times, about a vertex in each of the $2h$ repeated sequences. In a reversible partition no diameter of the nucleus is a diagonal axis of reversion, unless it be a clear axis. Therefore, in

$$R_0^{2h\cdot 2qdi}(r, k-1)_n+R_{ac}^{2h2qdi}(r, k-1)_n+R^{2h2q}(r, k-1)_n,$$

whether the axes be clear or not in the last one, you can draw δ from any one of $\frac{n}{4h}$ vertices of the r-gon; for none of the partitions in these classes have a clear diagonal axis, and each vertex ϵ in one of the $4h$ intervals between two adjacent axes begins a different sequence, which does not occur reverted till you come to ϵ', equidistant with ϵ from the axis between them. Wherefore

$$I^3(r, k)_3=\Sigma_h\Sigma_n\left\{\tfrac{n}{2h}I^{2h}(r, k-1)_n+\tfrac{n}{4h}\left(R_0^{2h2qdi}+R_{ac}^{2h2qdi}+R^{2h2q}\right)(r, k-1)_n+\right.$$

After the + must come, first,

$$\tfrac{n-2h}{4h}(\mathrm{R}^{2h\alpha\varepsilon\delta i}_{\delta\varepsilon}+\mathrm{R}^{2h\alpha\varepsilon\delta i}_{\varepsilon}+\mathrm{R}^{2h\delta i}_{\delta\varepsilon})(r,\,k-1)_n,$$

because δ cannot be drawn from any of the $2h$ terminations of clear diagonal axes, but may be drawn to begin a certain sequence $4h$ times from other vertices, once in the interval between the terminations of every adjacent pair of axes; and secondly,

$$+\tfrac{n-4h}{4h}\mathrm{R}^{2h\delta i}_{\varepsilon}(r,\,k-1)_n$$

must be added, because, omitting the $4h$ terminations of clear diagonal axes, you can draw δ to begin a certain sequence $4h$ times from the remaining vertices of the nucleus. Therefore, finally,

$$\mathrm{I}^h(r,\,k)_n=\Sigma_h\Sigma_n\Big\{\tfrac{n}{2h}\,\mathrm{I}^h(r,\,k-1)_n+\tfrac{n}{4h}\big(\mathrm{R}^{2h\alpha\varepsilon\delta i}_{\delta}(r,\,k-1)_n+\mathrm{R}^{2h\alpha\varepsilon\delta i}_{\alpha\varepsilon}(r,\,k-1)_n+\mathrm{R}^{2h\alpha\varepsilon}(r,\,k-1)_n\big)$$
$$+\tfrac{n-2h}{4h}\big(\mathrm{R}^{2h\alpha\varepsilon\delta i}_{\alpha\varepsilon}(r,\,k-1)_n+\mathrm{R}^{2h\alpha\varepsilon\delta i}_{\varepsilon}(r,\,k-1)_n+\mathrm{R}^{2h\delta i}_{\varepsilon}(r,\,k-1)_n\big)+\tfrac{n-4h}{4h}\mathrm{R}^{2h\delta i}_{\varepsilon}(r,\,k-1)_n\Big\};$$

where every value of $h>0$ and $n>2$ is included, that is found in our register, filled up by the formulæ already given.

LXVII. Hence, since we have found

$$\mathrm{R}^{8\alpha\varepsilon\delta i}(8,\,0)_8=\mathrm{R}^{2\alpha\varepsilon}_{4\varepsilon}(8,\,2)_4=\mathrm{R}^{2\delta i}_{4\varepsilon}(8,\,2)_8=\mathrm{R}^{4\delta i}_{4\varepsilon}(8,\,4)_4=\mathrm{I}^2(8,\,4)_4=1,$$
$$\mathrm{I}^2(8,\,1)_8\quad=0.\,\mathrm{R}^{8\alpha\varepsilon\delta i}(8,\,0)_8=0,$$
$$\mathrm{I}^2(8,\,3)_8\quad=\tfrac{4}{4}\,\mathrm{R}^{2\alpha\varepsilon}(8,\,2)_4+\tfrac{6-2}{4}\mathrm{R}^{2\delta i}_{4\varepsilon}(8,\,2)_8=2,$$
$$\mathrm{I}^2(8,\,5)_8\quad=\tfrac{4-4}{8}\,\mathrm{R}^{4\delta i}_{4\varepsilon}(8,\,4)_4+\tfrac{4}{4}.\,\mathrm{I}^2(8,\,4)_4=2.$$

LXVIII. Problem x. *To find* I(r, k), *the number of singly irreversible* (1+k)*-partitions of the* r*-gon.*

By the formula of XXIII. we have

$$\mathrm{I}(r,\,k)=\tfrac{1}{2r}\Big\{\mathrm{D}(r,\,k)-\Sigma_m\Big(\tfrac{r}{m}.\,\mathrm{R}^m(r,\,k)+\tfrac{2r}{m+1}.\,\mathrm{I}^{m+1}(r,\,k)\Big)\Big\},$$

which completes the solution of our problem of the (1+k)-partitions of the polygon, or of the polyace; for these are analytically and numerically the same. Hence follows,

$\mathrm{I}(8,\,0)=\frac{1}{16}\{\mathrm{D}(8,\,0)-1.\,\mathrm{R}^8(8,\,0)\}$ $= 0,$

$\mathrm{I}(8,\,1)=\frac{1}{16}\{\mathrm{D}(8,\,1)-\frac{8}{8}.\,\mathrm{R}^8(8,\,1)-8\mathrm{R}(8,\,1)\}$ $=\frac{1}{16}\{20-4-16\}$ $= 0,$

$\mathrm{I}(8,\,2)=\frac{1}{16}\{\mathrm{D}(8,\,2)-\frac{8}{8}\mathrm{R}^8(8,\,2)-8.\,\mathrm{R}(8,\,2)\}$ $=\frac{1}{16}\{120-4\cdot2-8\cdot4\}$ $= 5,$

$\mathrm{I}(8,\,3)=\frac{1}{16}\{\mathrm{D}(8,\,3)-\frac{8}{8}\mathrm{R}^8(8,\,3)-8\mathrm{R}(8,\,3)-\frac{16}{8}\mathrm{I}^2(8,\,3)\}=\frac{1}{16}\{300-4\cdot1-8\cdot7-8\cdot2\}=14,$

$\mathrm{I}(8,\,4)=\frac{1}{16}\{\mathrm{D}(8,\,4)-\frac{8}{4}\mathrm{R}^4(8,\,4)-8.\,\mathrm{R}(8,\,4)-\frac{16}{8}\mathrm{I}^2(8,\,4)\}=\frac{1}{16}\{330-2\cdot1-8\cdot4-8\cdot1\}=18,$

$\mathrm{I}(8,\,5)=\frac{1}{16}\{\mathrm{D}(8,\,5)-\frac{8}{8}\mathrm{R}^8(8,\,5)-8.\,\mathrm{R}(8,\,5)-\frac{16}{8}\mathrm{I}^2(8,\,5)\}=\frac{1}{16}\{132-4\cdot1-8\cdot4-8\cdot2\}=5,$

which, along with the results in LXI., LXV., LXVII., complete the enumeration of the partitions of the octagon.

LXIX. It may be useful to collect into one view all our formulæ for $R^n(r, k)$ and $I^n(r, k)$.

Problem b. To find $R_c^{2h\cdot agdi}(r, k)_n$ $(n>2)$, we have in XXIV., XXVIII.,

$$\left.\begin{aligned} r&=n \quad +4h\left(a_1+a_2+..+a_{\frac{n-2h}{4h}}\right)\\ k&=n-2h+4h\left(e_1+e_2+..+e_{\frac{n-2h}{4h}}\right)\end{aligned}\right\}, \qquad (A.)$$

$$(a_m \geqq 0, \quad e_m \not> a_m-1,$$

which is always to be understood in what follows), for the equations of condition. Here $r-n$ and k are both multiples of $4h$.

$$D(2+a_1, e_1)\times..\times D\left(2+a_{\frac{n-2h}{4h}}, e_{\frac{n-2h}{4h}}\right)=\Delta_{\frac{n-2h}{4h}},$$

$$R_c^{2h\cdot agdi}(r, k)_n=\Sigma\Delta_{\frac{n-2h}{4h}}-\Sigma_i R_c^{2(i+1)h\cdot agdi}(r, k)_n, \ (i>0);$$

$$R_c^{n\cdot agdi}(r, k)=0=R_c^{\frac{n}{2}agdi}(r, k)_n;$$

$$R_c^{\frac{n}{3}agdi}(r, k)_n=D\left(\frac{3r+n}{2n}, \frac{3k-2n}{2n}\right).$$

Problem c. To find $R_{dc}^{2h\cdot agdi}(r, k)_n$ (XXIX., XXX.),

$$\left.\begin{aligned} r&=n+2h(2+2\alpha_0)+4h\left(a_1+a_2+..+a_{\frac{n-2h}{4h}}\right)\\ k&=n+2h\varepsilon_0 \quad +4h\left(e_1+e_2+..+e_{\frac{n-2h}{4h}}\right)\end{aligned}\right\}. \qquad (A'.)$$

$$H=\Sigma_m\{2R^{m\cdot ag}(4+2\alpha_0, \varepsilon_0)+R^{2m\cdot agdi}(4+2\alpha_0, \varepsilon_0)\},$$

$$R_{dc}^{2h\cdot agdi}(r, k)_n=\Sigma\left(H.\Delta_{\frac{n-2h}{4h}}\right)-\Sigma_i R_{dc}^{2h(i+1)\cdot agdi}(r, k)_n, \ (i>0);$$

$$R_{dc}^{n\cdot agdi}(r, k)_n=\Sigma_m\left\{2R^{m\cdot ag}\left(\frac{r+n}{n}, \frac{k-n}{n}\right)+R^{2m\cdot agdi}\left(\frac{r+n}{n}, \frac{k-n}{n}\right)\right\},$$

$$R_{dc}^{\frac{n}{2}agdi}(r, k)_n=0,$$

$$R_{dc}^{\frac{n}{3}agdi}(r, k)_n=\Sigma(H.\Delta_1)-R_{dc}^{n\cdot agdi}(r, k)_n.$$

Problem d. To find $R_{ac}^{2h\cdot agdi}(r, k)_n$ (XXXII., XXXIII.),

$$\left.\begin{aligned} r&=n \quad +4h\left(a_1+a_2+..+a_{\frac{n-4h}{4h}}\right)+2h\left(2\alpha_{\frac{n}{4h}}+1\right)\\ k&=n-2h+4h\left(e_1+e_2+..+e_{\frac{n-4h}{4h}}\right)+2h.\varepsilon_{\frac{n}{4h}}\end{aligned}\right\}; \qquad (A''.)$$

$$M=\Sigma_m R^{m\cdot mo}\left(3+2\alpha_{\frac{n}{4h}}, \varepsilon_{\frac{n}{4h}}\right),$$

$$R_{ac}^{2h\cdot agdi}(r, k)_n=\Sigma\left(M\Delta_{\frac{n-4h}{4h}}\right)-\Sigma_i R_{ac}^{2h(i+1)agdi}(r, k)_n, \ (i>0),$$

$$R_{ac}^{n\cdot agdi}(r, k)_n=0,$$

$$R_{ac}^{\frac{n}{2}agdi}(r, k)_n=\Sigma_m R^{m\cdot mo}\left(\frac{2r}{n}, \frac{2k-n}{n}\right),$$

$$R_{ac}^{\frac{n}{4}agdi}(r, k)_n=\Sigma M\Delta_1-\Sigma_m R^{m\cdot mo}\left(\frac{2r}{n}, \frac{2k-n}{n}\right).$$

Problem e. To find $R_q^{2hsqdi}(r, k)_n$ (XXXIV., XXXV.),

$$\left.\begin{aligned} r&=n+2h(2+2\alpha_0)+4h\left(a_1+a_2+..+a_{\frac{n-2h}{2h}}\right)+2h\left(2\alpha_{\frac{n}{2h}}+1\right)\\ k&=n+2hs_0\qquad\quad +4h\left(e_1+e_2+..+e_{\frac{n-2h}{2h}}\right)+2hs_{\frac{n}{2h}}\end{aligned}\right\}, \quad . \quad . \quad . \quad (A''')$$

$$H=\Sigma_m\left(2R^{m\cdot sq}(4+2\alpha_0, s_0)+R^{2msqdi}(4+2\alpha_0, s_0)\right),$$

$$M=\Sigma_m R^{m\cdot ms}\left(3+2\alpha_{\frac{n}{2h}}, s_{\frac{n}{2h}}\right),$$

$$R_q^{2hsqdi}(r, k)_n=\Sigma\left(H.M.\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i R_q^{2(i+1)h\cdot sqdi}(r, k)_n, \ (i>0),$$

$$R_q^{sqdi}(r, k)_n=0,$$

$$R_q^{hm\cdot sqdi}(r, k)_n=\Sigma(H.M),$$

$$R_q^{hm\cdot sqdi}(r, k)_n=\Sigma(HM\Delta_1)-R_q^{hm\cdot sqdi}(r, k)_n,$$

Problem f. To find $R_c^{h\cdot sq}(r, k)_n$ (XXXVI., XXXVIII.),

$$\left.\begin{aligned} r&=n+2h\qquad\left(a_1+a_2+..+a_{\frac{n-2h}{2h}}\right)\\ k&=n-2h+2h\left(e_1+e_2+..+e_{\frac{n-2h}{2h}}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B.)$$

$$R_c^{h\cdot sq}(r, k)_n=\tfrac{1}{2}\left\{\Sigma\Delta_{\frac{n-2h}{2h}}(i>0)-\Sigma_i\left(2R_c^{h(i+1)\cdot sq}(r, k)_n+R_{qc}^{2hi\cdot sq}(r, k)_n+R_{ac}^{2hisqdi}(r, k)_n+R_q^{2hisqdi}(r, k)_n\right)\right\},$$

$$R_c^{sq}(r, k)_n=R_c^{hsq}(r, k)_n=0,$$

$$R_c^{hsq}(r, k)_n=\tfrac{1}{2}\left\{D\left(\frac{2r}{n}, \frac{2k-n}{n}\right)-\Sigma_m\left\{\left(2R^{m\cdot sq}+R^{2msqdi}\right)\left(\frac{2r}{n}, \frac{2k-n}{n}\right)\right\}\right.$$

Problem g. To find $R_{qc}^{hsq}(r, k)_n$ (XXXVII., XXXVIII.),

$$\left.\begin{aligned} r&=n+h(2+2\alpha_0)+2h\left(a_1+a_2+..+a_{\frac{n-2h}{2h}}\right)\\ k&=n-h-hs_0\quad\ +2h\left(e_1+e_2+..+e_{\frac{n-2h}{2h}}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B'.)$$

$$H=\Sigma_m\{2R^{msq}(4+2\alpha_0, s_0)+R^{2msqdi}(4+2\alpha_0, s_0)\}$$

$$R_{qc}^{hsq}(r, k)_n=\Sigma\left(H.\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i R_{qc}^{2(i+1)h\cdot sq}(r, k)_n, \ (i>0),$$

$$R_{qc}^{sqq}(r, k)_n=0,$$

$$R_{qc}^{hc\cdot sq}(r, k)_n=\Sigma_m\left\{\left(2R^{msq}+R^{2msqdi}\right)\left(\frac{2r}{n}, \frac{2k-n}{n}\right)\right\}$$

Problem h. To find $R_q^{hsq}(r, k)_n$ (XXXIX. XL.),

$$\left.\begin{aligned} r&=n+h(2+2\alpha_0)+2h\left(a_1+a_2+..+a_{\frac{n-2h}{2h}}\right)+h.\left(2+2\alpha_{\frac{n}{2h}}\right),\\ k&=n+hs_0\qquad\quad +2h\left(e_1+e_2+..+e_{\frac{n-2h}{2h}}\right)+hs_{\frac{n}{2h}};\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (B''.)$$

$$H=\Sigma_m\{2R^{msq}(4+2\alpha_0, s_0)+R^{2msqdi}(4+2\alpha_0, s_0)\},$$

$$H'=\Sigma_m\left\{2R^{msq}\left(4+2\alpha_{\frac{n}{2h}}, s_{\frac{n}{2h}}\right)+R^{2msqdi}\left(4+2\alpha_{\frac{n}{2h}}, s_{\frac{n}{2h}}\right)\right\};$$

$$R_q^{hsq}(r, k)_n=\tfrac{1}{2}\left\{\Sigma\left(H.H'.\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i\left(2R_q^{h(i+1)sq}(r, k)_n+R_{qc}^{2hisq}(r, k)_n+R_q^{2hisqdi}(r, k)_n+R_{qc}^{2hisqdi}(r, k)_n\right)\right\},$$

where $i>0$:

$$R_0^{n,q}(r,k)_n=0,$$

$$R_0^{h,n,q}(r,k)_n=\tfrac{1}{2}\{\Sigma(H.H')-R_{hr}^{n,q,di}(r,k)_n\}.$$

Problem i. To find $R_0^{h,di}(r,k)_n$ (XLI., XLIII.),

$$\left.\begin{aligned} r&=n+h(1+2a_0)+2h\left(a_1+a_2+..+a_{\frac{n-2h}{2h}}\right)+h\left(1+2a_{\frac{n}{2h}}\right),\\ k&=n+hs_0\qquad\quad +2h\left(e_1+e_2+..+e_{\frac{n-2h}{2h}}\right)+hs_{\frac{n}{2h}}; \end{aligned}\right\}\quad\ldots\ldots\quad(C.)$$

here r and n are both even.

$$M=\Sigma_m R^{m\cdot,m_0}(3+2a_0,\ s_0),$$

$$M'=\Sigma_m R^{m\cdot,m_0}(3+2a_{\frac{n}{2h}},\ s_{\frac{n}{2h}}),$$

$$R_0^{h,di}(r,k)_n=\tfrac{1}{2}\left\{\Sigma\left(MM'\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i(2R_0^{h(i+1),di}+R_{hr}^{2hi}+R_0^{2hi,q,di}+R_{hr}^{2hi,q,di})(r,k)_n\right\},\ (i>0).$$

$$R_0^{n,di}(r,k)_n=0$$

$$R_0^{h,di}(r,k)_n=\tfrac{1}{2}\Sigma\left\{\Sigma_m\left(R^{m\cdot,m_0}(3+2a_0,\ s_0)\right).\Sigma_m\left(R^{m\cdot,m_0}(3+2a_1,\ s_1)\right)\right\}-\tfrac{1}{2}\Sigma_m R^{m\cdot,m_0}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right),$$

where $\frac{r-2n}{n}=a_0+a_1$, and $\frac{2k-2n}{n}=s_0+s_1$.

Problem k. To find $R_{hr}^{h,di}(r,k)_n$ (XLII., XLIII.);

here h is even, consequently n is even.

$$r=n+h(1+2a_0)+2h\left(a_1+a_2+..+a_{\frac{n-h}{2h}}\right),$$

$$k=n+hs_0\qquad\quad 2h\left(e_1+e_2+..+e_{\frac{n-h}{2h}}\right);$$

and r is divisible by $2h$.

$$M=\Sigma_m.R^{m,m_0}(3+2a_0,\ s_0)$$

$$R_{hr}^{h,di}(r,k)_n=\Sigma\left(M\Delta_{\frac{n-h}{2h}}\right)-\Sigma_i R_{hr}^{(i+1)h,di}(r,k)_n,\ i\geqq 0.$$

$$R_{hr}^{n,di}(r,k)_n=\Sigma_m R^{m\cdot,m_0}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right).$$

Problem l. To find $R_c^{h,di}(r,k)_n$ (XLV., XLVI.),

$$\left.\begin{aligned} r&=n+2h\left(a_1+a_2+..+a_{\frac{n}{2h}}\right),\\ k&=n+2h\left(e_1+e_2+..+e_{\frac{n}{2h}}\right); \end{aligned}\right\}\quad\ldots\ldots\quad(C.)$$

here r, k and n are all even and divisible by $2h$.

$$R_c^{h,di}(r,k)_n=\tfrac{1}{2}\Sigma\Delta_{\frac{n}{2h}}-\tfrac{1}{2}\Sigma_i\{(2R_c^{(i+1)h,di}+R_{hr}^{2hi,di}+R_c^{2hi,\cdot,q,di}+R_{hr}^{2hi,q,di})(r,k)_n\};\ (i>0),$$

$$R_c^{n,di}(r,k)_n=0$$

$$R_c^{h,di}(r,k)_n=\tfrac{1}{2}\left\{D\left(\frac{r+n}{n},\ \frac{k-n}{n}\right)-R_{hr}^{n,di}(r,k)_n-R_{hr}^{n,q,di}(r,k)_n\right\}.$$

Problem m. To find $R_r^{kmo}(r, k)_n$ (XLVII., XLVIII.);
here n, h and r are odd numbers.

$$\left.\begin{aligned} r &= n \quad +2h\left(a_1+a_2+\ldots+a_{\frac{n-h}{2h}}\right), \\ k &= n-h+2h\left(e_1+e_2+\ldots+e_{\frac{n-h}{2h}}\right), \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(D.)}$$

$$R_r^{k\cdot mo}(r, k)_n = \Sigma\Delta_{\frac{n-h}{2h}} - \Sigma_i R_r^{(2i+3)mo}(r, k)_n, \ (i \overline{\geq} 0).$$

$$R_r^{kmo}(r, k)_n = 0$$

$$R_r^{k\cdot mo}(r, k)_n = D\left(\frac{3r+n}{2n}, \frac{3k-2n}{2n}\right).$$

Problem n. To find $R_0^{k\cdot mo}(r, k)_n$ (XLIX., L.), for n odd and n even,

$$\left.\begin{aligned} r &= n+h(2+2a_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-h}{2h}}\right), \\ k &= n+hs_0 \quad\quad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right), \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(D'.)}$$

$$H = \Sigma_m\{2R^{mo}(4+2a_0, s_0)+R^{2mo;di}(4+2a_0, s_0)\}$$

$$R_0^{kmo}(r, k)_n = \Sigma H.\Delta_{\frac{n-h}{2h}} - \Sigma_i.R_0^{k(2i+3)mo}(r, k)_n, \ (i \overline{\geq} 0),$$

when n is odd. And, when n is even,

$$r = n+h(1+2a_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right)+ha_{\frac{n}{2h}}$$

$$k = n+hs_0 \quad\quad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right)+hs_{\frac{n}{2h}};$$

$$H'' = \Sigma_m\left(2R^{mo;r}+R^{2mo;di}\left(2+a_{\frac{n}{2h}}, e_{\frac{n}{2h}}\right)\right)$$

$$M = \Sigma_m R^{mo;r}(3+a_0, s_0);$$

$$R_0^{k\cdot mo}(r, k)_n = \Sigma\left(H''M.\Delta_{\frac{n-2h}{2h}}\right) - \Sigma_i R_0^{(2i+3)k\cdot mo}(r, k)_n, \ (i \overline{\geq} 0),$$

$$R_0^{\cdot mo}(r, k)_n = \Sigma_m\left\{(2R^{mo;r}+R^{2mo;di})\left(\frac{r+n}{n}, \frac{k-n}{n}\right)\right\} \ (n \text{ odd});$$

$$R_0^{mo;r}(r, k)_n = 0, \text{ for } n \text{ even};$$

$$R_0^{kmo;r}(r, k)_n = \Sigma H''M, \text{ if } \frac{n}{2} \text{ be odd, and } = 0, \text{ if } \frac{n}{2} \text{ be even.}$$

Problem o. To find $R^{2mo;di}(r, k)_2$ (LII.),

$$R^{2mo;di}(r, k)_2 = \Sigma_h\Sigma_m\{R_r^{(4h+3)mo;di}(r, k-1)_n + R_{2r}^{(4h+3)mo;di}(r, k-1)_n\}.$$

Here r and $k-1$ are both even, by Problems b, c, as well as n.

Problem p. To find $R^{2di}(r, k)_2$ (LIII.),

$$R^{2di}(r, k)_2 = \Sigma_h\Sigma_m\{(2R_r^{2hdi}+R_{2r}^{2hdi}+R_r^{4hmo;di}+R_{2r}^{4hmo;di})(r, k-1)_n\}.$$

Here r and $k-1$ are both even (Problems b, c, k, l).

Problem q. To find $R^{di}(r, k)_2$ (LIV.),

$$R^{di}(r, k)_2 = \Sigma_h\Sigma_m R_r^{(2h+1)di}(r, k-1)_n.$$

Here r and $k-1$ are both even, by Problem l.

Problem r. To find $R_0^{mo}(r, k)$ (LV.),

$$R_0^{mo}(r, k) = \Sigma_h \Sigma_e \Sigma_n \left(\frac{n-3}{2}\right)^{e-1} \cdot \overline{e+1}^{-1} \cdot R_e^{(2h+1)mo}(r, k-e)_n - \Sigma_h R_0^{(2h+2)mo}(r, k).$$

Problem s. To find $R_0^{ag}(r, k)$ (LVI., LVII.),

$$\begin{aligned} R_0^{ag}(r, k) = \Sigma_h \Sigma_e \Sigma_n \Bigg[& R_e^{(2h+1)ag}(r, k-e)_n \cdot \left(\frac{n-4}{2}\right)^{e-1} \cdot \overline{e+1}^{-1} - R_0^{(2h+2)ag}(r, k) \\ & + \tfrac{1}{2}\{2R^{2hag}(r, k-e)_n + R_{\frac{1}{2}e}^{2hag}(r, k-e)_n + R_e^{2h \cdot ag di}(r, k-e)_n + R_{\frac{1}{2}e}^{2h \cdot ag di}(r, k-e)_n\} \\ & \times \left\{ \left(\frac{n-4}{2}\right)^{e-1} \cdot \overline{e+1}^{-1} - 2_e \left(4_n \left(\frac{n-4}{4}\right)^{\frac{e}{2}\left|-1\right.} \cdot \overline{\frac{e}{2}+1}^{-1} + 4_{n-2^r} \left(\frac{n-6}{4}\right)^{\frac{e}{2}\left|-1\right.} \cdot \overline{\frac{e}{2}+1}^{-1} \right) \right. \\ & \left. - 2_{e-1} \left(4_n \left(\frac{n-6}{4}\right)^{\frac{e-1}{2}\left|-1\right.} \cdot \overline{\frac{e+1}{2}}^{-1} \right) \right\} \Bigg]. \end{aligned}$$

Problem t. To find $R_0^{di}(r, k)$ (LXI.),

$$\begin{aligned} R_0^{di}(r, k) = \Sigma_h \Sigma_e \Sigma_n \Bigg[& R_e^{(2h+1)di}(r, k-e)_n \cdot \left(\frac{n-2}{2}\right)^{e-1} \cdot \overline{e+1}^{-1} - R_0^{(2h+2)di}(r, k) \\ & + \tfrac{1}{2}\{2R_e^{2hdi}(r, k-e)_n + R_{\frac{1}{2}e}^{2hdi}(r, k-e)_n + R_e^{2h \cdot agdi}(r, k-e)_n + R_{\frac{1}{2}e}^{2h \cdot agdi}(r, k-e)_n\} \\ & \times \left\{ \left(\frac{n-2}{2}\right)^{e-1} \cdot \overline{e+1}^{-1} - 2_e \left(4_{n-2} \left(\frac{n-2}{4}\right)^{\frac{e}{2}\left|-1\right.} \cdot \overline{\frac{e}{2}+1}^{-1} + 4_n \left(\frac{n-4}{4}\right)^{\frac{e}{2}\left|-1\right.} \cdot \overline{\frac{e}{2}+1}^{-1} \right) \right. \\ & \left. - 2_{e-1} \left(4_n \left(\frac{n-4}{4}\right)^{\frac{e-1}{2}\left|-1\right.} \cdot \overline{\frac{e+1}{2}}^{-1} \right) \right\} \Bigg]. \end{aligned}$$

Problem u. To find $I^h(r, k)_n$, $h>1$, $n>2$ (LXIV.),

$$r = n + h\left(a_1 + a_2 + \ldots + a_{\frac{n}{h}}\right),$$
$$k = n + h\left(e_1 + e_2 + \ldots + e_{\frac{n}{h}}\right).$$

Here h divides r, n, and k.

$$I^h(r, k)_n = \frac{h}{2n}\left\{ \Sigma \Delta_{\frac{n}{h}} - \Sigma_i \left(\frac{2n}{h(i+1)} I^{(i+1)h}(r, k)_n + \frac{n}{hi} R^{hi}(r, k)_n \right) \right\} \quad (i>0),$$

$$I^n(r, k)_n = \tfrac{1}{2}\left\{ D\left(\frac{r+n}{n}, \frac{k-n}{n}\right) - R^n(r, k)_n \right\}.$$

Problem w. To find $I^2(r, k)_2$ (LXVI.),

$$\begin{aligned} I^2(r, k)_2 = \Sigma_h \Sigma_n \Bigg\{ & \frac{n}{2h} I^{2h}(r, k-1)_n + \frac{n}{4h} \cdot \left(R_0^{2h \cdot agdi}(r, k-1)_n + R_{\frac{1}{2}e}^{2h \cdot agdi}(r, k-1)_n + R^{2h \cdot ag}(r, k-1)_n \right) \\ & + \frac{n-2h}{4h} \left(R_{\frac{1}{2}e}^{2h \cdot agdi}(r, k-1)_n + R_e^{2h \cdot agdi}(r, k-1)_n + R_{\frac{1}{2}e}^{2hdi}(r, k-1)_n \right. \\ & \left. + \frac{n-4h}{4h} R_e^{2hdi}(r, k-1)_n \right) \Bigg\}. \end{aligned}$$

Here r and $k-1$ are both even.

Problem x. To find $I(r, k)$ (LXVIII.),

$$I(r, k)=\frac{1}{2r}\left\{D(r, k)-\Sigma_m\left(\frac{r}{m}R^m(r, k)+\frac{2r}{m+1}I^{m+1}(r, k)\right)\right\}.$$

LXX. As a further illustration of these formulæ and their use, and for a comparison of methods, it will be worth our while to deduce the seven partitions of the 9-gon, and the eight partitions of the 10-gon, which are obtained by way of example in the memoir referred to, in the first article.

First, to find $R^1(9, 6)$ and $I^1(9, 6)$.

As the 9-gon is divided into triangles, there can be only two nuclei, a triangle or a line. We ask, then, what are $R^1(9, 6)_3$ and $R^1(9, 6)_2$?

By Theorem E, the axes are monogonal only, and $R^{1mo}(9, 6)_3$ is what we are seeking. By Problem m,

$$R^{2mo}_v(9, 6)_3=0,$$

$$R^{1\cdot mo}_v(9, 6)_3=D(5, 2)=5;$$

and by Problem n,

$$R^{2mo}_0(9, 6)_3=(2R^{mo\,sy}+R^{2mo\,sy\,di})(4, 1)=0,$$

$R^{2di}(4, 1)_2=1$ being the only entry in our register under (4, 1).

For $R^{mo}_0(9, 6)_3$, we have, by equations (D′.), $\frac{n-h}{2h}=1=h$.

$$9=3+2+2a_0+2a_1,$$

$$6=3+\quad\ \epsilon_0+2e_1,$$

$$4+2a_0=8-2a_1, \qquad \Delta_1=D(2+a_1, e_1),$$

$$\epsilon_0=3-2e_1,$$

$$R^{mo}_0(9, 6)_3=\Sigma.\{\Sigma(2R^{mo\,sy}+R^{2mo\,sy\,di})(8-2a_1, 3-2e_1)\}.D(2+a_1, e_1).$$

The only values of e_1 are 0 or 1; if $e_1=1$, $a_1\lessdot 2$, if $e_1=0$, $a_1\lessdot 1$. And as we have nothing in our register under $(R^{mo\,sy}$ or $R^{mo\,sy\,di})((4, 1)$ or $(6, 3))$,

$$R^{mo}_0(9, 6)_3=0,$$

and (Problems o, p, q) $\quad R(9, 6)_3=0;$

wherefore $\quad R(9, 6)=R^{mo}_v(9, 6)_3=5.$

Next to find $I^1(9, 6)_3$; by Problem u,

$$I^1(9, 6)_3=\tfrac{1}{3}\{D(4, 1)-R^1(9, 6)_3\}=\tfrac{1}{3}(2-0)=1,$$

the subtracted term having just been proved to be zero. And as 9 is odd (Problem w),

$I^2(9, 6)_3=0$, wherefore (Problem x),

$$I(9, 6)=\tfrac{1}{18}\{D(9, 6)-9R(9, 6)-\tfrac{1}{3}\cdot 9I^1(9, 6)=\tfrac{1}{18}\{429-9\cdot 5-6\cdot 1\}=21.$$

This agrees with the results at p. 409 of my former memoir, saving an error of transcription in the seventh line from the bottom, in which $+1.ii(8, 2, 0)$ is omitted from the value of $I(10, 2)$ as it stands above (line 14). If the indulgent reader will kindly correct the sixth and seventh lines thus,

$$I(10, 2)=2.\tfrac{1}{2}\{ii(8, 2, 0).ii(4, 1, 0))+1.ii(8, 2, 0)$$
$$=2.\tfrac{1}{2}\{4-2\}+4=6,$$

he will find the sum

$$I(10, 2)+I(10, 3)+I(10, 4)=6+13+2=21,$$
$$I^2(10, 3)=1,$$

and
$$R(10, 2)+R(10, 4)=5,$$

the 7-partitions of the nonagon, in that notation, as we have just found them: vide (I) of this memoir.

LXXI. We shall for a final example determine the 8-partitions of the decagon. There is no nucleus but 3 and 2; we want then

$$R^{\lambda}(10, 7)_3,\ R^{\lambda}(10, 7)_2,\ R^{a}(10, 7)_2,\ R_{\epsilon}(10, 7),\ I^{\lambda}(10, 7)_2\ I^{\lambda}(10, 7)_2.$$

By Theorems M, E (XIV., IX), no axis can be agonal, or monogonal; we seek then $R^{\lambda di}(10, 7)_2$, $R^{\lambda di}(10, 7)_2$, $R^{di}(10, 7)_2$, and $R^{di}_{\epsilon}(10, 7)$.

$R^{\lambda di}_{\epsilon}(10, 7)_2=0$ by Problem i, because n is odd; and $R^{\lambda di}_{\epsilon'}(10, 7)_2=0$ for the same reason, as is also $R^{\lambda di}_{\epsilon''}(10, 7)_2$. Therefore $R^{\lambda di}(10, 7)_2=0$.

By Problem p,

$$R^{\lambda di}(10, 7)_2=\Sigma_{\lambda}\Sigma_{n}\{2R^{2\lambda di}_{\epsilon}+R^{\lambda\lambda di}_{\epsilon'}+R^{\lambda\lambda\epsilon di}_{\epsilon''}+R^{\lambda\lambda\epsilon di}_{\epsilon'''})(10, 6)_n\}=0;\ \text{for}$$

$R^{2\lambda di}_{\epsilon}$ $(10, 6)_n=0$, because 10 is not a multiple of $4h$ (Problem l);

$R^{\lambda\lambda di}_{\epsilon'}$ $(10, 6)_n=0$, for the same reason (Problem k);

$R^{\lambda\lambda\epsilon di}_{\epsilon''}(10, 6)_n=0$, because 10 is no multiple of $8h$ (Problem b);

$R^{\lambda\lambda\epsilon di}_{\epsilon'''}(10, 6)_n=0$, because $10-n$, $(n>2)$ is no multiple of $8h$ (Problem c).

R^{di} $(10, 7)_2=\Sigma_{\lambda}\Sigma_{n}R^{(2\lambda+1)di}_{\epsilon}(10, 6)_n=R^{di}_{\epsilon}(10, 6)_n$, because 10 is a multiple of $4h+2$ (Problems l, q);

$$R^{di}_{0}\quad (10, 7)=\Sigma_{\lambda}\Sigma_{\epsilon}\Sigma_{n}\left[R^{(2\lambda+1)di}_{\epsilon}(10, 7-e)_n\left\{\left(\frac{n-2}{2}\right)^{e-1}.\overline{e+1}^{-1}\right\}^2+\&c.\right]\text{(Problem t),}$$

all the remaining part of the expression vanishing; for we shall presently see that $n=4$ only in both these equations; and $R^{(2\lambda+3)di}_{0}(10, 7)$ has been proved above to be $=0$, as have $R^{2\lambda di}_{\epsilon}(10, 7-e)_n$ and $R^{2\lambda di}_{\epsilon'}(r, 7-e)_n$ also; and $R^{2\lambda\epsilon di}_{\epsilon''}(10, 7-e)_n=0$, because $7-e$ and $10-n$ are not multiples of $4h$, unless $h=1$, $n=6$; $(n>2)$, which makes equations (A.)

$$\left(\frac{n-2h}{4h}=1\right),\qquad 10=6+4.1,\ \text{and}\ 7-e=6-2+4.-1,$$

$\therefore e=7$; but $R^{3\epsilon di}(10, 0)_6$ has no existence, for if there are no diagonals the nucleus is the 10-gon itself. $R^{3\epsilon di}_{\epsilon''}(10, 7-e)_n=0$; for this can only be $R^{3\epsilon di}_{\epsilon''}(10, 7-e)_6$, for a reason just given; but (Problem c) $7-e-6$ is even, therefore $e=1$, and as $h=1$, $=\frac{n-2a}{4h}$, equations (A'.) become

$$10=6+.2(2+0)+4.0,\ \text{and}\ k=6=6+2.2+4.-1,$$

i. e. in H, $\alpha_0=0$, $\epsilon_0=2$, or a 4-gon has two diagonals not crossing, which is absurd.

Therefore $R_e^{(2h+1)di}(10, 7-e)_n$ is what we have to determine for every value of h, e, and n.

Now in $R_e^{hdi}(10, 7-e)_n$ (Problem l), when h is odd, 10, $7-e$ and n are not all divisible by $2h$ if $n>6$; for $n<10$. And in $R_e^{hdi}(10, 7-e)_6$, h is either 1 or 3. For $h=3$ the formula for R^{hdi} gives $R^{hdi}(10, 7-e)_6=0$, because $\frac{10+6}{6}$ is not integer; and for $h=1$, $\frac{n}{2h}=3$,

$$10=6+2.(a_1+a_2+a_3)$$
$$7-e=6+2.(e_1+e_2+e_3);$$

the solutions are

$$\frac{10-6}{2}=2=1+1+0, \text{ or } =2+0+0$$
$$1-e=2(0+0-1), \quad \text{or } =2(1-1-1), \text{ or } =2(0-1-1),$$

the second line of which is absurd, if $e<3$. But for $e>2$, the multiplier $\left(\frac{6-2}{2}\right)^{3-1}=0$. Therefore n is not 6.

Let then $n=4$, and seek $R^{hdi}(10, 7-e)_4$, for h odd. If $\frac{4}{2h}=1$, the formula for R^{hdi} gives $R_e^{hdi}(10, 7-e)_4=0$, because $\frac{10+4}{4}$ is not integer. Then $\frac{4}{2h}=2$, $h=1$, and we have for $e=1$,

$$10=4+2.(2+1)=4+2(3+0)$$
$$6=4+2\ (1+0)=4+2(2-1),$$

giving four solutions; and for $e=3$, the multiplier $\left(\frac{4-2}{2}\right)^{4-1}$ vanishes. These four are then the only solutions, and

$$\Sigma(\Delta_4)=2D(4, 1)D(3, 0)+2D(5, 2)=2(2+5)=14;$$

whence $$R_e^n(10, 6)_4=\tfrac{1}{8}\{14-\Sigma_i(2R_e^{(i+1)di}+R_{\frac{n}{2}}^{di}+R_e^{n\cdot iqdi}+R_{\frac{n}{2}}^{\frac{n}{2}iqdi})(10, 6)_4\}$$

where $(i>0)$.

Now

$R_e^{(i+1)di}(10, 6)_4=0$ (Problem l), because 10 and 6 and 4 have no common measure $2(i+1)$.

$R_{\frac{n}{2}}^{di}\ (10, 6)_4=0$ (Problem k), because 10 is not divisible by 4.

$R_e^{n\cdot iqdi}(10, 6)_4=0$ (Problem b), because 10 is no multiple of $4i$.

$R_{\frac{n}{2}}^{\frac{n}{2}iqdi}\ (10, 6)_4=0$ (Problem c), because $10-4$ is no multiple of $4i$.

Therefore $$R_e^n(10, 6)_4=\tfrac{1}{8}14=7=R_e^n(10, 7),$$

and $$R(10, 7)=R^n(10, 7)_2+R_e^n(10, 7)=2R_e^n(10, 6)_4=14.$$

LXXII. We have next to determine $I^n(10, 7)$, $h>1$.

The formulæ for I^n (Problem u) give

$$I^3(10, 7)_3=0, \text{ because } \frac{10+3}{3} \text{ is no integer.}$$

And (Problem w)

$$I^s(10, 7)_s = \Sigma_s \Sigma_n \left\{ \frac{h}{2^i h} I^{is}(10, 6)_n + \frac{n}{4^i h} R^{2hsgdi}_{ac}(10, 6)_n \right\},$$

the quantities here omitted from the formula being

$R^{2hsgdi}_{b}(10, 6)_n = 0$, because 10 is not divisible by $4h$ (Problem e).

$R^{2hsg}\ (10, 6)_n = 0$, for the same reason (Problems f, g, h).

$R^{2hsgdi}_{ac}(10, 6)_n = 0$, as it has been proved in the preceding article.

$R^{2hsgdi}_{c}(10, 6)_n = 0$, because 10 is not divisible by $4h$ (Problem b).

$R^{2hdi}_{bc} = R^{2hdi}_{c} = 0$, for the same reason (Problems k, l).

Now the equations of Problem u for $I^{is}(10, 6)_n$ are those of Problem l for $R^s_n(10, 6)_n$, and we have proved in the preceding article that $h=1$ and $n=4$ are the only values affording a solution; which gives

$$\Sigma\left(\Delta_{\frac{s}{2h}}\right) = \Sigma(\Delta 2) = 14.$$

Wherefore (Problem w) ($i > 0$),

$$I^s(10, 6)_4 = \frac{s}{8}\left\{14 - \Sigma_i \left(\frac{8}{2(i+1)} I^{(i+1)s}(10, 6)_4 + \frac{4}{2^i} R^{is}(10, 6)_4 \right)\right\}.$$

The formula for $I^n(r, k)_n$ shows that $I^4(10, 6)_4 = 0$, for $\frac{10+4}{4}$ is not integer, and we have just proved that $R^{is}(10, 6)_n = 0$ under every form except $R^{2hsgdi}_{ac}(10, 6)_n$, which remains to be determined.

In Problem d we see that $r - n - 2h$ is divisible by $4h$; wherefore $n=4$ and $h=1$. Therefore, by the formula for R^{ts}_{ac},

$$R^{2sgdi}_{ac}(10, 6)_4 = \Sigma_m R^{m \cdot ms}(5, 2) = R^{ms}(5, 2) = 1,$$

as is easily verified. Consequently

$$I^s(10, 6)_4 = \frac{s}{8}\{14 - \tfrac{4}{2} R^{2sgdi}(10, 6)_4\} = \frac{14-2}{4} = 3,$$

and $$I^s(10, 7) = I^s(10, 7)_s = \tfrac{4}{2} I^s(10, 6)_4 + \tfrac{4}{4} . R^{2sgdi}(10, 6)_4 = 6 + 1 = 7,$$

whence, finally (Problem x),

$$I(10, 7) = \tfrac{1}{3^0}\{D(10, 7) - \tfrac{1^0}{1} R(10, 7) - \tfrac{2^0}{2} I^2(10, 7)\} = \tfrac{1}{3^0}(1430 - 10.14 - 10.7) = 61.$$

And this enumeration of the 8-partitions of the 10-gon agrees with that at p. 410 of my memoir mentioned in the first article.

It is evident that this last example would have caused far less trouble, if we had had our register of partitions filled up for inspection up to the 7-partitions of the 10-gon.

There is little difficulty, as I have verified to some extent, in framing, by the aid of the results here given, algebraical expressions, containing circulating functions for $R^k(r, k)$ and $I^k(r, k)$, in terms of r and k only, and thence by addition, complete expressions of the $(1+k)$-partitions of the r-gon. But the subject has been pursued far enough for one communication.

LXXIII. The notation above employed for the partitions of the r-gon is applicable, with hardly any change, to express those of the r-ace.

An h-ly reversible $(1+k)$-partition of the r-ace has h axial planes of reversion, which are *achorial*, *diachorial*, or *monochorial*, according as they cut none, two, or one only of the faces ($\chi\omega\rho\acute{\iota}\alpha$) about the r-ace. The k partitioning lines may conveniently be called *diapipeds*, or shorter, *diapeds*, being each in two planes about the r-ace, as a diagonal is in two summits of the r-gon.

Putting Ac, Di, and Mo, for achorial, diachorial and monochorial, we have the following account to give, N standing for the nucleus n-ace, of the $(1+k)$-partitions of the r-ace:

$$\mathrm{R}^{h\cdot Ac\cdot Di}(r,\,k)_N = \mathrm{R}^{h\cdot ac\cdot di}(r,\,k)_n,$$
$$\mathrm{R}^{h\cdot Ac}\quad(r,\,k)_N = \mathrm{R}^{h\cdot ac}\quad(r,\,k)_n,$$
$$\mathrm{R}^{h\cdot Di}\quad(r,\,k)_N = \mathrm{R}^{h\cdot di}\quad(r,\,k)_n,$$
$$\mathrm{R}^{h\cdot Mo}\quad(r,\,k)_N = \mathrm{R}^{h\cdot mo}\quad(r,\,k)_n,$$
$$\mathrm{I}^{h}\qquad(r,\,k)_N = \mathrm{I}^{h}\qquad(r,\,k)_n.$$

XIII. *A Memoir upon Caustics.* *By* Arthur Cayley, *Esq.*

Received May 1,—Read May 8, 1856.

The following memoir contains little or nothing that can be considered new in principle; the object of it is to collect together the principal results relating to caustics *in plano*, the reflecting or refracting curve being a right line or a circle, and to discuss with more care than appears to have been hitherto bestowed upon the subject, some of the more remarkable cases. The memoir contains in particular researches relating to the caustic by refraction of a circle for parallel rays, the caustic by reflexion of a circle for rays proceeding from a point, and the caustic by refraction of a circle for rays proceeding from a point; the result in the last case is not worked out, but it is shown how the equation in rectangular coordinates is to be obtained by equating to zero the discriminant of a rational and integral function of the sixth degree. The memoir treats also of the secondary caustic or orthogonal trajectory of the reflected or refracted rays in the general case of a reflecting or refracting circle and rays proceeding from a point; the curve in question, or rather *a* secondary caustic, is, as is well known, the Oval of Descartes or 'Cartesian': the equation is discussed by a method which gives rise to some forms of the curve which appear to have escaped the notice of geometers. By considering the caustic as the evolute of the secondary caustic, it is shown that the caustic, in the general case of a reflecting or refracting circle and rays proceeding from a point, is a curve of the sixth class only. The concluding part of the memoir treats of the curve which, when the incident rays are parallel, must be taken for the secondary caustic in the place of the Cartesian, which, for the particular case in question, passes off to infinity. In the course of the memoir, I reproduce a theorem first given, I believe, by me in the Philosophical Magazine, viz. that there are six different systems of a radiant point and refracting circle which give rise to identically the same caustic. The memoir is divided into sections, each of which is to a considerable extent intelligible by itself, and the subject of each section is for the most part explained by the introductory paragraph or paragraphs.

I.

Consider a ray of light reflected or refracted at a curve, and suppose that ξ, η are the coordinates of a point Q on the incident ray, α, β the coordinates of the point G of incidence upon the reflecting or refracting curve, a, b the coordinates of a point N upon the normal at the point of incidence, x, y the coordinates of a point q on the reflected or refracted ray.

Write for shortness,

$$(b-\beta)(\xi-\alpha)-(a-\alpha)(\eta-\beta)=\nabla \mathrm{QGN}$$
$$(a-\alpha)(\xi-\alpha)+(b-\beta)(\eta-\beta)=\square \mathrm{QGN},$$

then $\nabla \mathrm{QGN}$ is equal to twice the area of the triangle QGN, and if ξ, η instead of being the coordinates of a point Q on the incident ray were current coordinates, the equation $\nabla \mathrm{QGN}=0$ would be the equation of the line through the points G and N, *i. e.* of the normal at the point of incidence; and in like manner the equation $\square \mathrm{QGN}=0$ would be the equation of the line through G perpendicular to the line through the points G and N, *i. e.* of the tangent at the point of incidence.

We have

$$\overline{\mathrm{NG}}^2=(a-\alpha)^2+(b-\beta)^2$$
$$\overline{\mathrm{QG}}^2=(\xi-\alpha)^2+(\eta-\beta)^2,$$

and therefore identically,

$$\overline{\mathrm{NG}}^2.\overline{\mathrm{QG}}^2=\overline{\nabla \mathrm{QGN}}^2+\overline{\square \mathrm{QGN}}^2.$$

Suppose for a moment that φ is the angle of incidence and φ' the angle of reflexion or refraction; and let μ be the index of refraction (in the case of reflexion $\mu=-1$), then writing

$$(b-\beta)(x-\alpha)-(a-\alpha)(y-\beta)=\nabla q\mathrm{GN}$$
$$(a-\alpha)(x-\alpha)+(b-\beta)(y-\beta)=\square q\mathrm{GN},$$

and

$$\overline{q\mathrm{G}}^2=(x-\alpha)^2+(y-\beta)^2,$$

we have

$$\sin\varphi=\frac{\nabla \mathrm{QGN}}{\mathrm{NG.GQ}},\ \sin\varphi'=\frac{\nabla q\mathrm{GN}}{\mathrm{NG}.\mathrm{G}q};$$

and substituting these values in the equation

$$\sin^2\varphi-\mu^2\sin^2\varphi'=0,$$

we obtain

$$\overline{q\mathrm{G}}^2\ \overline{\nabla \mathrm{QGN}}^2-\mu^2\ \overline{\mathrm{QG}}^2\ \overline{\nabla q\mathrm{GN}}^2=0,$$

an equation which is rational of the second order in x, y, the coordinates of a point q on the refracted ray; this equation must therefore contain, as a factor, the equation of the refracted ray; the other factor gives the equation of a line equally inclined to, but on the opposite side of the normal; this line (which of course has no physical existence) may be termed *the false refracted ray*. The caustic is *geometrically* the envelope of *the pair of rays*, and for finding the equation of the caustic it is obviously convenient to take the equation of the two rays conjointly in the form under which such equation has just been found, without attempting to break the equation up into its linear factors.

It is however interesting to see how the resolution of the equation may be effected; for this purpose multiply the equation by $\overline{\mathrm{NG}}^2$, then reducing by means of a previous formula, the equation becomes

$$\left(\overline{\nabla q\mathrm{GN}}^2+\overline{\square q\mathrm{GN}}^2\right)\ \overline{\nabla \mathrm{QGN}}^2-\mu^2\left(\overline{\nabla \mathrm{QGN}}^2+\overline{\square \mathrm{QGN}}^2\right)\ \overline{\nabla q\mathrm{GN}}^2=0.$$

which is equivalent to

$$\nabla qGN^2(\mu^2 \square QGN^2 + (\mu^2-1)\ \nabla QGN^2) - \square qGN^2\ \nabla QGN^2 = 0,$$

and the factors are

$$\nabla qGN \sqrt{\mu^2 \square QGN^2 + (\mu^2-1)\ \nabla QGN^2} \mp \square qGN . \nabla QGN = 0;$$

it is in fact easy to see that these equations represent lines passing through the point G and inclined to GN at angles $\pm\phi'$, where ϕ' is given by the equations

$$\sin\phi = \mu \sin\phi'$$

$$\tan\phi = \frac{\nabla QGN}{\square QGN},$$

and there is no difficulty in distinguishing in any particular case between the refracted ray and the false refracted ray.

In the case of reflexion $\mu = -1$, and the equations become

$$\nabla qGN . \square QGN \mp \square qGN . \nabla QGN = 0;$$

the equation

$$\nabla qGN . \square QGN - \square qGN . \nabla QGN = 0$$

is obviously that of the incident ray, which is what the false refracted ray becomes in the case of reflexion; and the equation

$$\nabla qGN . \square QGN + \square qGN . \nabla QGN = 0$$

is that of the reflected ray.

II.

But instead of investigating the nature of the caustic itself, we may begin by finding the secondary caustic or orthogonal trajectory of the refracted rays, *i. e.* a curve having the caustic for its evolute; suppose that the incident rays are all of them normal to a certain curve, and let Q be a point upon this curve, and considering the ray through the point Q, let G be the point of incidence upon the refracting curve; then if the point G be made the centre of a circle the radius of which is $\mu^{-1}.GQ$, the envelope of the circles will be the secondary caustic. It should be noticed, that if the incident rays proceed from a point, the most simple course is to take such point for the point Q. The remark, however, does not apply to the case where the incident rays are parallel; the point Q must here be considered as the point in which the incident ray is intersected by some line at right angles to the rays, and there is not in general any one line which can be selected in preference to another. But if the refracting curve be a circle, then the line perpendicular to the incident rays may be taken to be a diameter of the circle. To translate the construction into analysis, let ξ, η be the coordinates of the point Q, and α, β the coordinates of the point G, then ξ, η, α, β are in effect functions of a single arbitrary parameter; and if we write

$$\overline{GQ}^2 = (\xi-\alpha)^2 + (\eta-\beta)^2$$

$$\overline{Gq}^2 = (x-\alpha)^2 + (y-\beta)^2,$$

then the equation

$$\mu^2\overline{Gq}^2 - \overline{GQ}^2 = 0,$$

where x, y are to be considered as current coordinates, and which involves of course the arbitrary parameter, is the equation of the circle, and the envelope is obtained in the usual manner. This is the well-known theory of GERGONNE and QUETELET.

III.

There is however a simpler construction of the secondary caustic in the case of the reflexion of rays proceeding from a point. Suppose, as before, that Q is the radiant point, and let G be the point of incidence. On the tangent at G to the reflecting curve, let fall a perpendicular from Q, and produce it to an equal distance on the other side of the tangent; then if q be the extremity of the line so produced, it is clear that q is a point on the reflected ray Gq, and it is easy to see that the locus of q is the secondary caustic. Produce now QG to a point Q′ such that GQ′=QG, it is clear that the locus of Q′ will be a curve similar to and similarly situated with and twice the magnitude of the reflecting curve, and that the two curves have the point Q for a centre of similitude. And the tangent at Q′ passes through the point q, i. e. q is the foot of the perpendicular let fall from Q upon the tangent at Q′; we have therefore the theorem due to DANDELIN, viz.

If rays proceeding from a point Q are reflected at a curve, then the secondary caustic is the locus of the feet of the perpendiculars let fall from the point Q upon the tangents of a curve similar to and similarly situated with and twice the magnitude of the reflecting curve, and such that the two curves have the point Q for a centre of similitude.

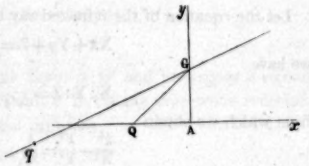
IV.

If rays proceeding from a point Q are reflected at a line, the reflected rays will proceed from a point q situate on the perpendicular let fall from Q, and at an equal distance on the other side of the reflecting line. The point q may be spoken of as the image of Q; it is clear that if Q be considered as a variable point, then the locus of the image q will be a curve equal and similar but oppositely situated to the curve, the locus of Q, and which may be spoken of as the image of such curve. Hence it at once follows, that if the incidental rays are tangent, or normal, or indeed in any other manner related to a curve, then the reflected rays will be tangent or normal, or related in a corresponding manner to a curve the image of the first-mentioned curve. The theory of the combined reflexions and refractions of a pencil of rays transmitted through a plate or prism, is, by the property in question, rendered very simple. Suppose, for instance, that a pencil of rays is refracted at the first surface of a plate or prism, and after undergoing any number of internal reflexions, finally emerges after a second refraction at the first or second surface; in order to find the caustic enveloped by the rays after the first refraction, it is only necessary to form the successive images of this caustic corresponding to the different reflexions, and finally to determine the caustic for refraction in the case where the

incident rays are the tangents of the caustic which is the last of the series of images; the problem is not in effect different from that of finding the caustic for refraction in the case where the incident rays are the tangents to the caustic after the first refraction, but the line at which the second refraction takes place is arbitrarily situate with respect to the caustic. Thus *e. g.* suppose the incident rays proceed from a point, the caustic after the first refraction is, it will be shown in the sequel, the evolute of a conic; for the complete theory of the combined reflexions and refractions of the pencil by a plate or prism, it is only necessary to find the caustic by refraction, where the incident rays are the normals of a conic, and the refracting line is arbitrarily situate with respect to the conic.

V.

Suppose that rays proceeding from a point Q are refracted at a line; and take the refracting line for the axis of y, the axis of x passing through the radiant point Q, and take the distance QA for unity. Suppose that the index of refraction μ is put equal to $\frac{1}{k}$. Then if φ be the angle of incidence and φ' the angle of refraction, we have $\sin\varphi' = k\sin\varphi$, and the equation $y - x\tan\varphi' = \tan\varphi$ of the refracted ray becomes, putting for φ' its value,



$$y - \frac{k\sin\varphi}{\sqrt{1-k^2\sin^2\varphi}}x - \tan\varphi = 0.$$

Differentiating with respect to the variable parameter and combining the two equations, we obtain, after a simple reduction,

$$kx = -\frac{(1-k^2\sin^2\varphi)^{\frac{3}{2}}}{\cos^3\varphi}$$

$$k'y = -\frac{k^3\sin^3\varphi}{\cos^3\varphi},$$

where $k' = \sqrt{1-k^2}$, hence eliminating

$$(kx)^{\frac{2}{3}} - (k'y)^{\frac{2}{3}} = 1,$$

which is the equation of the caustic. When the refraction takes place into a denser medium k is less than 1, and k'^2 is positive, the caustic is therefore the evolute of a hyperbola (see fig. 1); but when the refraction takes place in a rarer medium k is greater than 1, and k'^2 is negative, the caustic is therefore the evolute of an ellipse (see fig. 2). These results appear to have been first obtained by GERGONNE. The conic (hyperbola or ellipse) is the secondary caustic, and as such may be obtained as follows.

VI.

The equation of the variable circle is

$$x^2 + (y - \tan\varphi)^2 - k^2\sec^2\varphi = 0;$$

or reducing, the equation is

$$x^2+y^2-2y\tan\varphi+k'^2\tan^2\varphi-k^2=0.$$

Whence, considering $\tan\varphi$ as the variable parameter, the equation of the envelope is

$$k'^2(x^2+y^2-k^2)-y^2=0,$$

that is,

$$k'^2x^2-k^2y^2-k^2k'^2=0,$$

or

$$\frac{x^2}{k^2}-\frac{y^2}{k'^2}=1$$

is the equation of the secondary caustic or conic having the caustic for its evolute. The radiant point, it is clear, is a focus of the conic.

VII.

Let the equation of the refracted ray be represented by

$$\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}=0,$$

we have

$$\mathrm{X}:\mathrm{Y}:\mathrm{Z}=\frac{-k\sin\varphi}{\sqrt{1-k^2\sin^2\varphi}}:1:-\tan\varphi,$$

from which we obtain

$$\frac{k^2}{\mathrm{X}^2}-\frac{k'^2}{\mathrm{Y}^2}=\frac{1}{\mathrm{Z}^2}$$

for the tangential equation of the caustic; or if we represent the equation of the refracted ray by

$$\mathrm{X}x+\mathrm{Y}y-k=0,$$

then we have

$$\frac{k^2}{\mathrm{X}^2}-\frac{k'^2}{\mathrm{Y}^2}=\frac{1}{k^2}$$

for the tangential equation of the caustic.

Fig. 1.

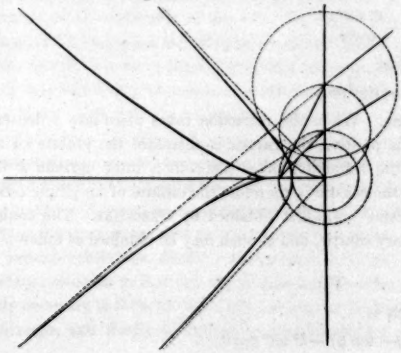
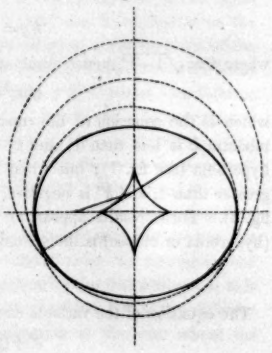


Fig. 2.



VIII.

If a ray be reflected at a circle; we may take a, b as the coordinates of the centre of the circle, and supposing as before that ξ, η are the coordinates of a point Q in the incident ray, α, β the coordinates of the point G of incidence, and x, y the coordinates of a point q in the reflected ray, the equation of the reflected ray, treating x, y as current coordinates, is

$$\{(b-\beta)(x-\alpha)-(a-\alpha)(y-\beta)\}\{(a-\alpha)(\xi-\alpha)+(b-\beta)(\eta-\beta)\}$$
$$+\{(a-\alpha)(x-\alpha)+(b-\beta)(y-\beta)\}\{(b-\beta)(\xi-\alpha)-(a-\alpha)(\eta-\beta)\}=0.$$

Write for shortness,

$$N_{q,\,G}=(b-\beta)(x-\alpha)-(a-\alpha)(y-\beta)$$
$$T_{q,\,G}=(a-\alpha)(x-\alpha)+(b-\beta)(y-\beta),$$

and similarly for $N_{Q,\,G}$, &c; the equation of the reflected ray is

$$N_{q,\,G}T_{Q,\,G}+T_{q,\,G}N_{Q,\,G}=0.$$

Suppose that the reflected ray meets the circle again in G′ and undergoes a second reflexion, and let x', y' be the coordinates of a point q' in the ray thus twice reflected. We see first (G′ being a point in the first reflected ray) that

$$N_{G',\,G}T_{Q,\,G}+T_{G',\,G}N_{Q,\,G}=0.$$

Again, considering G as a point in the ray by the reflexion of which the second reflected ray arises, the equation of the second reflected ray is

$$N_{q',\,G'}T_{G,\,G'}+T_{q',\,G'}N_{G,\,G'}=0;$$

and from the form of the expressions $N_{q,\,G}$, $T_{q,\,G}$ it is clear that

$$N_{G,\,G'}=-N_{G'\,G},\ T_{G,\,G'}=+T_{G',\,G};$$

the equation for the second reflected ray may therefore be written under the form

$$N_{q',\,G'}\,T_{G',\,G}-T_{q',\,G'}N_{G',\,G}=0;$$

or reducing by a previous equation, we obtain finally for the equation of the second reflected ray,

$$N_{q',\,G'}T_{Q,\,G}+T_{q',\,G'}N_{Q,\,G}=0;$$

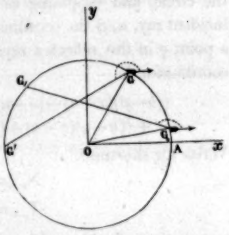
and in like manner the equation for the third reflected ray is

$$N_{q'',\,G''}T_{Q,\,G}+T_{q'',\,G''}N_{Q,\,G}=0,$$

and so on, the equation for the last reflected ray containing, it will be observed, the coordinates of the radiant point and of the first and last points of incidence (the coordinates of the last point of incidence can of course only be calculated from those of the radiant point and the first point of incidence, through the coordinates of the intermediate points of incidence), but not containing explicitly the coordinates of any of the intermediate points of incidence. The form is somewhat remarkable, but the result is really the same with that obtained by simple geometrical considerations, as follows.

IX.

Consider a ray reflected any number of times at a circle; and let G_0G_1 be the ray incident at G_1 and GG' the last reflected ray, the point at which the reflexion takes place or last point of incidence being G. Take the centre O of the circle for the origin, and any two lines Ox, Oy through the centre and at right angles to each other for axes, and let Ox meet the circle in the point A. Write



$$\angle AOG_0 = \theta_0, \quad \angle xG_0G_1 = \psi_0$$
$$\angle AOG = \theta \quad \angle xGG' = \psi$$
$$\angle G_0G_1O = \varphi.$$

Then the radius of the circle being taken as the centre of the circle, the equation of the reflected ray is

$$y - \sin\theta = \tan\psi(x - \cos\theta);$$

and if there have been n reflexions, then

$$\theta = \theta_0 + n(\pi - 2\varphi) = \theta_0 + n\pi - 2n\varphi,$$
$$\psi = \psi_0 - 2n\varphi,$$

and therefore the equation of the reflected ray is

$$y\cos(\psi_0 - 2n\varphi) - x\sin(\psi_0 - 2n\varphi) + (-)^n\sin(\psi_0 - \theta_0) = 0.$$

X.

If a pencil of parallel rays is reflected any number of times at a circle, then taking AO for the direction of the incident rays, we may write $\theta_0 = \varphi$, $\psi_0 = \pi$, and the equation of a reflected ray is

$$x\sin 2n\varphi + y\cos 2n\varphi = (-)^n\sin\varphi.$$

Differentiating with respect to the variable parameter, we find

$$x\cos 2n\varphi - y\sin 2n\varphi = (-)^n\frac{1}{2n}\cos\varphi;$$

and these equations give

$$x = \frac{(-)^n}{4n}\left\{(2n+1)\cos(2n-1)\varphi - (2n-1)\cos(2n+1)\varphi\right\}$$
$$y = \frac{(-)^n}{4n}\left\{-(2n+1)\sin(2n-1)\varphi + (2n-1)\sin(2n+1)\varphi\right\},$$

which may be taken for the equation of the caustic; the caustic is therefore an epicycloid: this is a well-known result.

XI.

If rays proceeding from a point upon the circumference are reflected any number of

times at a circle, then taking the point A for the radiant point, we have $\theta_0=0$, $\psi_0=\pi-\phi$, and the equation of a reflected ray is

$$x\sin(2n+1)\phi+y\cos(2n+1)\phi=(-)^n\sin\phi.$$

Differentiating with respect to the variable parameter, we find

$$x\cos(2n+1)\phi-y\sin(2n+1)\phi=(-)^n\frac{1}{2n+1}\sin\phi;$$

and these equations give

$$x=\frac{(-)^n}{2n+1}\Big\{\quad(n+1)\cos 2n\phi-n\cos(2n+2)\phi\Big\}$$

$$y=\frac{(-)^n}{2n+1}\Big\{-(n+1)\sin 2n\phi+n\cos(2n+2)\phi\Big\},$$

which may be taken as the equation of the caustic; the caustic is therefore in this case also an epicycloid: this is a well-known result.

XII.

Consider a pencil of parallel rays refracted at a circle; take the radius of the circle as unity, and let the incident rays be parallel to the axis of x, then if ϕ, ϕ' be the angles of incidence and refraction, and μ or $\frac{1}{k}$ the index of refraction, so that $\sin\phi'=k\sin\phi$, the coordinates of the point of incidence are $\cos\phi$, $\sin\phi$, and the equation of the refracted ray is

$$y-\sin\phi=\tan(\phi-\phi')(x-\cos\phi),$$

i. e.

$$\cos(\phi-\phi')(y-\sin\phi)=\sin(\phi-\phi')(x-\cos\phi),$$

or

$$y\cos(\phi-\phi')-x\sin(\phi-\phi')=\sin\phi',$$

which may also be written

$$(y\cos\phi-x\sin\phi)\cos\phi'+(y\sin\phi+x\cos\phi-1)\sin\phi'=0;$$

or writing $k\sin\phi$, $\sqrt{1-k^2\sin^2\phi}$ instead of $\sin\phi'$, $\cos\phi'$, and putting for shortness

$$y\cos\phi-x\sin\phi=\mathrm{Y}$$

$$y\sin\phi+x\cos\phi=\mathrm{X}$$

$$\frac{k\sin\phi}{\sqrt{1-k^2\sin^2\phi}}=\Phi,$$

the equation of the refracted ray becomes

$$\mathrm{Y}+\Phi(\mathrm{X}-1)=0.$$

And differentiating with respect to the variable parameter ϕ, observing that

$$\frac{d\mathrm{Y}}{d\phi}=-\mathrm{X},\quad\frac{d\mathrm{X}}{d\phi}=\mathrm{Y}$$

$$\frac{d\Phi}{d\phi}=\frac{k\cos\phi}{(1-k^2\sin^2\phi)^{\frac{3}{2}}}=\frac{\cot\phi}{1-k^2\sin^2\phi}\Phi,$$

we have

$$-\mathrm{X}+\Phi\left(\mathrm{Y}+\frac{\cot\varphi(\mathrm{X}-1)}{1-k^2\sin^2\varphi}\right)=0,$$

and the combination of the two equations gives

$$\mathrm{Y}=-\frac{\Phi(1-k^2\sin^2\varphi)}{\Phi\cot\varphi-1}$$

$$\mathrm{X}=\frac{\Phi\cot\varphi-k^2\sin^2\varphi}{\Phi\cot\varphi-1},$$

and we have therefore

$$y=\mathrm{Y}\cos\varphi+\mathrm{X}\sin\varphi=\frac{k^2\sin^3\varphi(\Phi\cot\varphi-1)}{\Phi\cot\varphi-1}=k^2\sin^3\varphi$$

$$x=\mathrm{X}\cos\varphi-\mathrm{Y}\sin\varphi=\frac{\Phi\left(\frac{1}{\sin\varphi}-k^2\sin^3\varphi\right)-k^2\sin^2\varphi\cos\varphi}{\Phi\cot\varphi-1},$$

i. e.

$$x=\frac{\Phi(1-k^2\sin^4\varphi)-k^2\sin^3\varphi\cos\varphi}{\Phi\cos\varphi-\sin\varphi};$$

or multiplying the numerator and denominator by $(1-k^2\sin^2\varphi)(\Phi\cos\varphi+\sin\varphi)$, the numerator becomes

$$\begin{aligned}(1-k^2\sin^2\varphi)\{&\Phi^2\cos\varphi(1-k^2\sin^4\varphi)-k^2\sin^4\varphi\cos\varphi\\&+\Phi(\sin\varphi(1-k^2\sin^2\varphi)-k^2\sin^3\varphi\cos\varphi)\}\\=k^2\sin^2\varphi\cos\varphi&\{(1-k^2\sin^4\varphi)-\sin^2\varphi(1-k^2\sin^2\varphi)\}\\&+k\sin^2\varphi\sqrt{1-k^2\sin^2\varphi}(1-k^2\sin^2\varphi)\\=k^2\sin^2\varphi\cos^2\varphi&+k\sin^2\varphi(1-k^2\sin^2\varphi)^{\frac{3}{2}},\end{aligned}$$

and the denominator becomes

$$\begin{aligned}&k^2\sin^2\varphi\cos^2\varphi-(1-k^2\sin^2\varphi)\sin^2\varphi\\&=-k'^2\sin^2\varphi,\end{aligned}$$

if $k'^2=1-k^2$.

Hence we have for the coordinates of the point of the caustic,

$$\begin{cases}k'^2x=-k^2\cos^2\varphi-k(1-k^2\sin^2\varphi)^{\frac{3}{2}}\\ y=\quad k^2\sin^3\varphi;\end{cases}$$

and eliminating φ, we obtain for the equation of the caustic,

$$k'^2x=-k^2\{1-k^{-\frac{2}{3}}y^{\frac{2}{3}}\}^{\frac{3}{2}}-k\{1-k^{\frac{2}{3}}y^{\frac{2}{3}}\}^{\frac{3}{2}};$$

or writing $\frac{1}{\mu}$ instead of k, we find

$$(1-\mu^2)x=(1-\mu^{\frac{2}{3}}y^{\frac{2}{3}})^{\frac{3}{2}}+\mu(1-\mu^{-\frac{2}{3}}y^{\frac{2}{3}})^{\frac{3}{2}}$$

for the equation of the caustic by refraction of the circle, for parallel rays. The equation was first obtained by St. Laurent.

XIII.

The discussion of the preceding equation presents considerable interest. In the first place to obtain the rational form write

$$\alpha=(1-\mu^2)x,\ \beta=(1-\mu^{\frac{4}{3}}y^{\frac{4}{3}})^{\frac{1}{2}},\ \gamma=\mu(1-\mu^{-\frac{4}{3}}y^{\frac{4}{3}})^{\frac{1}{2}};$$

this gives

$$\alpha^4-2\alpha^2(\beta^3+\gamma^3)+(\beta^3-\gamma^3)^2=0,$$

and we have

$$\beta^3=1-3\mu^{\frac{4}{3}}y^{\frac{4}{3}}+3\mu^{\frac{8}{3}}y^{\frac{8}{3}}-\mu^4y^4$$

$$\gamma^3=\mu^3-3\mu^{\frac{4}{3}}y^{\frac{4}{3}}+3\mu^{\frac{8}{3}}y^{\frac{8}{3}}-y^4,$$

and consequently

$$\beta^3-\gamma^3=(1-\mu^3)\{1-3\mu^{\frac{4}{3}}y^{\frac{4}{3}}+(1+\mu^3)y^3\}.$$

Hence dividing out by the factor $(1-\mu^3)^2$, the equation becomes

$$(1-\mu^3)^2x^4-2\big(1+\mu^3-6\mu^{\frac{4}{3}}y^{\frac{4}{3}}+3\mu^{\frac{8}{3}}(1+\mu^3)y^{\frac{8}{3}}-(1+\mu^4)y^3\big)2x^2+\big(1-3\mu^{\frac{4}{3}}y^{\frac{4}{3}}+(1+\mu^3)y^3\big)^2=0;$$

or reducing and arranging,

$$(1-\mu^3)^2x^4-2(1+\mu^3)x^3+2(1+\mu^4)x^2y^2+1+2(1+\mu^3)y^2+(1+\mu^3)^2y^4$$
$$+(12\mu^{\frac{4}{3}}x^2+9\mu^{\frac{4}{3}}y^2)y^{\frac{4}{3}}-\big(6\mu^{\frac{4}{3}}(1+\mu^3)x^2+6\mu^{\frac{8}{3}}+6\mu^{\frac{8}{3}}(1+\mu^3)y^2\big)y^{\frac{4}{3}}=0,$$

which is of the form

$$A+3\mu^{\frac{4}{3}}By^{\frac{4}{3}}-6\mu^{\frac{4}{3}}Cy^{\frac{4}{3}}=0;$$

and the rationalized equation is

$$A^3+27\mu^4B^3y^4-216\mu^3C^3y^4+54\mu^3ABCy^2=0,$$

where the values of A, B, C may be written

$$A=(x^3+y^3)\{(1-\mu^3)^2x^3+(1+\mu^3)^2y^3\}-2(1+\mu^3)(x^3-y^3)+1$$
$$B=4x^3+3y^3$$
$$C=(1+\mu^3)(x^3+y^3)+1;$$

the caustic is therefore a curve of the 12th order.

To find where the axis of x meets the curve, we have

$$y=0,\ A_0^3=0,$$

where

$$A_0=(1-\mu^3)^2x^4-2(1+\mu^3)x^3+1$$
$$=\{(1-\mu)^2x^3-1\}\{(1+\mu)^2x^3-1)\},$$

i. e.

$$\begin{cases} y=0 \\ x=\pm\frac{1}{1-\mu},\ x=\pm\frac{1}{1+\mu}, \end{cases}$$

or there are in all four points each of them a point of triple intersection.

To find where the line ∞ meets the curve, we have

$$\infty,\ A^{12}=0,$$

where

$$A'=(x^2+y^2)\{(1-\mu^2)^2x^2+(1+\mu^2)^2y^2\},$$

i. e.

$$\begin{cases}\infty \\ x=\pm iy, \quad x=\pm\frac{1+\mu^2}{1-\mu^2}iy,\end{cases}$$

or the curve meets the line ∞ in four points, each of them a point of triple intersection: two of these points are the circular points at ∞.

To find where the circle $x^2+y^2=1$ meets the curve, this gives $x^2=1-y^2$, and thence

$$A=\mu^2(\mu^2-4)+4(1+2\mu^2)y^2$$
$$B=4-y^2$$
$$C=\mu^2+2,$$

and the equation becomes

$$\{\mu^2(\mu^2-4)+4(1+2\mu^2)y^2\}^3+27\mu^4(4-y^2)^2y^2-216(\mu^2+2)^3y^4$$
$$+54\mu^2(\mu^2+2)y^2(4-y^2)\{\mu^2(\mu^2-4)+4(1+2\mu^2)y^2\}=0,$$

which is only of the eighth order; it follows that each of the circular points at ∞ (which have been already shown to be points upon the curve) are quadruple points of intersection of the curve and circle. The equation of the eighth order reduces itself to

$$(y^2-\mu^2)^2\{27\mu^4y^2+(\mu^2-4)^2\}=0;$$

the values of x corresponding to the roots $y=\pm\mu$ are obtained without difficulty, and those corresponding to the other roots are at once found by means of the identical equation

$$(\mu^2-4)^3+27\mu^4+(1-\mu^2)(\mu^2+8)^2=0;$$

we thus obtain for the coordinates of the points of intersection of the curve with the circle $x^2+y^2=1$, the values

$$\begin{cases}\infty \\ x=\pm iy,\end{cases} \quad \begin{cases}x=\pm\sqrt{1-\mu^2} \\ y=\pm\mu,\end{cases} \quad \begin{cases}x=\pm\frac{(\mu^2+8)\sqrt{1-\mu^2}}{3\sqrt{3}\mu^2}i \\ y=\pm\frac{(\mu^2-4)^2}{3\sqrt{3}\mu^2}i,\end{cases}$$

each of the points of the first system being a quadruple point of intersection, each of the points of the second system a triple point of intersection, and each of the points of the third system a single point of intersection.

Next, to find where the circle $x^2+y^2=\frac{1}{\mu^2}$ meets the curve; writing $x^2=\frac{1}{\mu^2}-y^2$, we obtain for y an equation of the eighth order, which after all reductions is

$$\left(y^2-\frac{1}{\mu^4}\right)^2\{27\mu^4y^2+(1-4\mu^2)^2\}=0,$$

and we have for the coordinates of the points of intersection,

$$\begin{cases} \infty \\ x=\pm iy, \end{cases} \quad \begin{cases} x=\pm\frac{1}{\mu}\sqrt{1-\frac{1}{\mu^2}} \\ y=\pm\frac{1}{\mu^2}, \end{cases} \quad \begin{cases} x=\pm\frac{(1+8\mu^2)\sqrt{1-\frac{1}{\mu^2}}}{3\sqrt{3}\mu}i \\ y=\pm\frac{(1-4\mu^2)^{\frac{1}{2}}}{3\sqrt{3}\mu}i, \end{cases}$$

each of the points of the first system being a quadruple point of intersection, each of the points of the second system a triple point of intersection, and each of the points of the third system a single point of intersection.

The points of intersection with the axes of x, and the points of triple intersection with the circles $x^2+y^2=1$ and $x^2+y^2=\frac{1}{\mu^2}$, are all of them cuspidal points; the two circular points at ∞ are, I think, triple points, and the other two points of intersection with the line ∞, cuspidal points, but I have not verified this: assuming that it is so, there will be a reduction 54 accounted for in the class of the curve, but the curve is, in fact, as will be shown in the sequel, of the class 6; there is consequently a reduction 72 to be accounted for by other singularities of the curve.

XIV.

It is obvious from the preceding formulæ that the caustic stands to the circle radius $\frac{1}{\mu}$ in a relation similar to that in which it stands to the circle radius 1, *i. e.* to the refracting circle. In fact, the very same caustic would have been obtained if the circle radius $\frac{1}{\mu}$ had been taken for the refracting circle, the index of refraction being $\frac{1}{\mu}$ instead of μ. This may be shown very simply by means of the irrational form of the equation as follows.

The equation of the caustic by refraction of the circle radius 1, index of refraction μ, is, we have seen,

$$(1-\mu^2)x=(1-\mu^{\frac{2}{3}}y^{\frac{2}{3}})^{\frac{3}{2}}+\mu(1-\mu^{-\frac{2}{3}}y^{\frac{2}{3}})^{\frac{3}{2}}.$$

Hence the equation of the caustic by refraction of the circle radius c', index of refraction μ', is

$$(1-\mu'^2)\frac{x}{c}=\left\{1-\mu'^{\frac{2}{3}}\left(\frac{y}{c}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}}+\mu'\left\{1-\mu'^{-\frac{2}{3}}\left(\frac{y}{c}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}},$$

or, what is the same thing,

$$(1-\mu'^2)\frac{x}{c'\mu'}=\left\{1-\mu'^{-\frac{2}{3}}c'^{-\frac{2}{3}}y^{\frac{2}{3}}\right\}^{\frac{3}{2}}+\frac{1}{\mu'}\left\{1-\mu'^{\frac{2}{3}}c'^{-\frac{2}{3}}y^{\frac{2}{3}}\right\}^{\frac{3}{2}},$$

which becomes identical with the equation of the first-mentioned caustic if $\mu'=c'=\frac{1}{\mu}$. Hence taking c instead of 1 as the radius of the first circle, we find,—

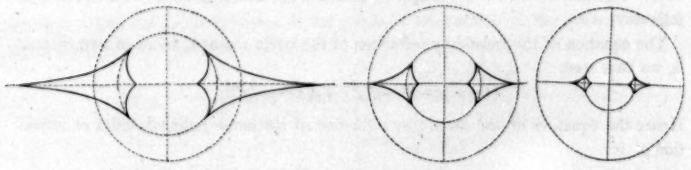
Theorem. The caustic by refraction for parallel rays of a circle radius c, index of

refraction μ, is the same curve as the caustic by refraction for parallel rays of a concentric circle radius $\frac{c}{\mu}$, index of refraction $\frac{1}{\mu}$.

XV.

We may consequently in tracing the caustic confine our attention to the case in which the index of refraction is greater than unity. The circle radius $\frac{c}{\mu}$ will in this case be within the refracting circle, and it is easy to see that if from the extremity of the diameter of the refracting circle perpendicular to the direction of the incident rays, tangents are drawn to the circle radius $\frac{c}{\mu}$, the points of contact are the points of triple intersection of the caustic with the last-mentioned circle, and these points of intersection being, as already observed, cusps, the tangents in question are the tangents to the caustic at these cusps. The points of intersection with the axis of x are also cusps of the caustic, the tangents at these cusps coinciding with the axis of x: two of the last-mentioned cusps, viz. those whose distances from the centre are $\pm\frac{1}{\mu+1}$, lie within the circle radius $\frac{c}{\mu}$, the other two of the same four cusps, viz. those whose distances from the centre are $\pm\frac{1}{\mu-1}$, lie without the circle radius $\frac{c}{\mu}$; the last-mentioned two cusps lie without the refracting circle, when $\mu<2$, upon this circle when $\mu=2$, and within it, and therefore between the two circles when $\mu>2$. The caustic is therefore of the forms in the annexed figures 3, 4, 5, in each of which the outer circle is the refracting circle, and μ is >1, but the

Fig. 3. Fig. 4. Fig. 5.



three figures correspond respectively to the cases $\mu<2$, $\mu=2$ and $\mu>2$. The same three figures will represent the different forms of the caustic when the inner circle is the refracting circle and μ is <1, the three figures then respectively corresponding to the cases $\mu>\frac{1}{2}$, $\mu=\frac{1}{2}$, and $\mu<\frac{1}{2}$.

XVI.

To find the tangential equation, I retain k instead of its value $\frac{1}{\mu}$, the equation of the refracted ray then is

$$x(k\cos\varphi-\sqrt{1-k^2\sin^2\varphi})+y(k\sin\varphi+\cot\varphi\sqrt{1-k^2\sin^2\varphi})-k=0,$$

and representing this by

$$Xx+Yy-k=0,$$

we have

$$X=k\cos\varphi-\sqrt{1-k^2\sin^2\varphi}$$

$$Y=k\sin\varphi+\cot\varphi\sqrt{1-k^2\sin^2\varphi},$$

equations which give

$$X\cos\varphi+Y\sin\varphi=k$$

$$X^2+Y^2=\frac{1}{\sin^2\varphi},$$

and consequently

$$\sin\varphi=\frac{1}{\sqrt{X^2+Y^2}}$$

$$\cos\varphi=\frac{\sqrt{X^2+Y^2-1}}{\sqrt{X^2+Y^2}},$$

and we have

$$X\sqrt{X^2+Y^2-1}+Y-k\sqrt{X^2+Y^2}=0,$$

which gives

$$(X^2+Y^2)(X^2-1-k^2)=-2kY\sqrt{X^2+Y^2};$$

or dividing out by the factor $\sqrt{X^2+Y^2}$, the equation becomes

$$\sqrt{X^2+Y^2}(X^2-1-k^2)=-2kY,$$

from which

$$(X^2+Y^2)(X^2-1-k^2)^2-4k^2Y^2=0;$$

or reducing and arranging, we obtain

$$X^2(X^2-1-k^2)^2+Y^2(X+1+k)(X+1-k)(X-1+k)(X-1-k)=0$$

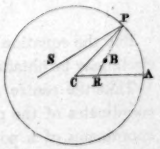
for the tangential equation of the caustic by refraction of a circle for parallel rays. The caustic is therefore of the class 6.

XVII.

Suppose next that rays proceeding from a point are reflected at a circle.

A very elegant solution of the problem is given by LAGRANGE in the Mém. de Turin; the investigation, as given by Mr. P. SMITH in a note in the Cambridge and Dublin Mathematical Journal, t. ii. p. 237, is as follows:—

Let B be the radiant point, RBP an incident ray, and PS a reflected ray; CA a fixed radius; ACP=α, ACB=ε, reciprocal of CB=c, reciprocal of CP=a. The equations of the incident and reflected ray, where $u=\frac{1}{r}$, may be written



$$u=A\sin\theta+B\cos\theta \text{ incident ray},$$

$$u=A\sin(2\alpha-\theta)+B\cos(2\alpha-\theta) \text{ reflected},$$

the conditions for determining A and B being

$$a=A\sin\alpha+B\cos\alpha$$

$$c=A\sin\varepsilon+B\cos\varepsilon,$$

whence,

$$A=\frac{a\cos\varepsilon-c\cos\alpha}{\sin(\alpha-\varepsilon)},\ B=\frac{c\sin\alpha-a\sin\varepsilon}{\sin(\alpha-\varepsilon)}.$$

Substituting these values, the equation of the reflected ray becomes

$$a\sin(2\alpha-\theta-\varepsilon)=u\sin(\alpha-\varepsilon)+c\sin(\alpha-\theta),$$

from which, and its differential with respect to the arbitrary parameter α, the equation of the caustic or envelope of the reflected rays will be found by eliminating α.

In this, α being the only quantity treated as variable in the differentiation, let

$$2\alpha-\theta-\varepsilon=2\varphi.$$

Therefore

$$\alpha=\varphi+\tfrac{1}{2}(\theta+\varepsilon),$$

and the equation becomes

$$a\sin 2\varphi=u\sin\{\varphi+\tfrac{1}{2}(\theta-\varepsilon)\}+c\sin\{\varphi-\tfrac{1}{2}(\theta-\varepsilon)\}$$

Make

$$P=\frac{(u+c)\cos\frac{1}{2}(\theta-\varepsilon)}{2a}$$

$$Q=\frac{(u-c)\sin\frac{1}{2}(\theta-\varepsilon)}{2a},$$

also

$$x=\frac{1}{\cos\varphi},\ y=\frac{1}{\sin\varphi},$$

and the equation becomes

$$Px+Qy=1,$$

with the condition

$$x^{-2}+y^{-2}=1.$$

Hence

$$P=\lambda x^{-3}$$

$$Q=\lambda y^{-3}.$$

Multiplying by x and y, and adding, we find $\lambda=1$; therefore

$$x^{-2}=P^{\frac{2}{3}},\ y^{-2}=Q^{\frac{2}{3}}.$$

Hence

$$P^{\frac{2}{3}}+Q^{\frac{2}{3}}=1;$$

or restoring the values of P and Q,

$$\{(u+c)\cos\tfrac{1}{2}(\theta-\varepsilon)\}^{\frac{2}{3}}+\{(u-c)\sin\tfrac{1}{2}(\theta-\varepsilon)\}^{\frac{2}{3}}=1,$$

the equation of the caustic.

XVIII.

But the equation of the caustic for rays proceeding from a point and reflected at a circle may be obtained by a different method, as follows:—

Take the centre of the circle for origin; let c be the radius of the circle, a, b the coordinates of the radiant point, α, β the coordinates of the point of incidence, x, y the coordinates of a point in the reflected ray. Then we have from the equation of the

circle $\alpha^2+\beta^2=c^2$, and the equation of the reflected ray is by the general formula,

$$(b\alpha-a\beta)(\alpha x+\beta y-c^2)+(y\alpha-x\beta)(a\alpha+b\beta-c^2)=0;$$

or arranging the terms in a different order,

$$(bx+ay)(\alpha^2-\beta^2)+2(by-ax)\alpha\beta-c^2(b+y)\alpha+c^2(a+x)\beta=0.$$

Writing now $\alpha=c\cos\theta$, $\beta=c\sin\theta$, the equation becomes

$$(bx+ay)\cos 2\theta+(by-ax)\sin 2\theta-(b+y)c\cos\theta+(a+x)c\sin\theta=0,$$

where θ is a variable parameter.

Now in general to find the envelope of

$$A\cos 2\theta+B\sin 2\theta+C\cos\theta+D\sin\theta+E=0,$$

we may put $e^{i\theta}=z$, which gives the equation

$$(A-Bi)z^4+(C-Di)z^3+2Ez^2+(C+Di)z+(A+Bi)=0,$$

and equate the discriminant to zero: this gives

$$(4I)^3-27(-8J)^2=0,$$

where

$$4I=4(A^2+B^2)-(C^2+D^2)+\tfrac{4}{3}E^2$$

$$-8J=A(C^2-D^2)+2BCD-\{8(A^2+B^2)+(C^2+D^2)\}\tfrac{1}{3}E+\tfrac{8}{27}E^3,$$

and consequently

$$\{4(A^2+B^2)-(C^2+D^2)+\tfrac{4}{3}E^2\}^3-27\{A(C^2-D^2)+2BCD$$

$$-\left(8(A^2+B^2)+(C^2+D^2)\right)\tfrac{1}{3}E+\tfrac{8}{27}E^3\}^2=0;$$

and substituting for A, B, C, D, E their values, we find

$$\{4(a^2+b^2)(x^2+y^2)-c^2\left((x+a)^2+(y+b)^2\right)\}^3-27(bx-ay)^2(x^2+y^2-a^2-b^2)=0$$

for the equation of the caustic in the case of rays proceeding from a point and reflected at a circle: the equation was first obtained by St. Laurent.

It will be convenient to consider the axis of x as passing through the radiant point: this gives $b=0$; and if we assume also $c=1$, the equation of the caustic becomes

$$\{(4a^2-1)(x^2+y^2)-2ax-a^2\}^3-27a^2y^2(x^2+y^2-a^2)^2=0.$$

XIX.

Reverting to the equation of the reflected ray, and putting, as before, $c=1$, $b=0$, this becomes

$$(-2a\cos\theta+1)x+\frac{a\cos 2\theta-\cos\theta}{\sin\theta}y+a=0.$$

Differentiating with respect to θ, we have

$$(-2a\sin\theta)x+\frac{-a\cos\theta(1+2\sin^2\theta)+1}{\sin^2\theta}y=0;$$

and from these equations

$$x=\frac{a^2\cos\theta(1+2\sin^2\theta)-a}{1-3a\cos 2\theta+2a^2}$$

$$y=\frac{2a^2\sin^3\theta}{1-3a\cos 2\theta+2a^2},$$

which give the coordinates of a point of the caustic in terms of the angle θ, which determines the position of the point of incidence. The values in question satisfy, as they should do, the equation

$$\{(4a^2-1)(x^2+y^2)-2ax-a^2\}^3-27a^2y^2(x^2+y^2-a^2)^2=0.$$

We have, in fact,

$$x^2+y^2-a^2=\frac{4a^2(\cos\theta-a)^2}{(1-3a\cos 2\theta+2a^2)^2}$$

$$(4a^2-1)(x^2+y^2)-2ax-a^2=\frac{12a^4(\cos\theta-a)^2}{(1-3a\cos 2\theta+2a^2)^3}$$

from which it is easy to derive the equation in question.

XX.

If we represent the equation of the reflected ray by

$$Xx+Yy+a=0,$$

then we have

$$X=-2a\cos\theta+1$$

$$Y=\frac{a\cos 2\theta-\cos\theta}{\sin\theta},$$

and thence

$$(X-1)^2-4a^2=-4a^2\sin^2\theta$$

$$X^2+Y^2=\frac{1}{\sin^2\theta}(1-2a\cos\theta+a^2)$$

$$X+a^2=1-2a\cos\theta+a^2,$$

and consequently

$$(X^2+Y^2)\{(X-1)^2-4a^2\}+4a^2X+4a^4=0,$$

or, what is the same thing,

$$\{X(X-1)-2a^2\}^2+Y^2\{(X-1)^2-4a^2\}=0,$$

which may be considered as the tangential equation of the caustic by reflexion of a circle; or if we consider X, Y as the coordinates of a point, then the equation may be considered as that of the polar of the caustic. The polar is therefore a curve of the fourth order, having two double points defined by the equations $X(X-1)-2a^2=0$, $Y=0$, and a third double point at infinity on the axis of Y, *i. e.* three double points in all; the number of cusps is therefore 0, and there are consequently 4 double tangents and 6 inflections, and the curve is of the class 6. And as Y is given as an explicit function of X, there is of course no difficulty in tracing the curve. We thus see that the caustic by reflexion of a circle is a curve of the order 6, and has 4 double points and 6 cusps (the circular points at infinity are each of them a cusp, so that the number of cusps at a finite distance is 4): this coincides with the conclusions which will be presently obtained by considering the equation of the caustic.

XXI.

The equation of the caustic by reflexion of a circle is

$$\{(4a^2-1)(x^2+y^2)-2ax-a^2\}^2-27a^2y^2(x^2+y^2-a^2)^2=0.$$

Suppose first that $y=0$, we have

$$\{(4a^2-1)x^2-2ax-a^2\}^2=0,$$

i. e.

$$x=\frac{-a}{2a+1},\ x=\frac{a}{2a-1},$$

or the curve meets the axis of x in two points, each of which is a triple point of intersection.

Write next $x^2+y^2=a^2$, this gives

$$\{(4a^2-1)a^2-2ax-a^2\}^2=0,$$

and consequently

$$x=-a(1-2a^2)$$
$$y=\pm 2a^2\sqrt{1-a^2},$$

or the curve meets the circle $x^2+y^2-a^2=0$ in two points, each of which is a triple point of intersection.

To find the nature of the infinite branches, we may write, retaining only the terms of the degrees six and five,

$$(4a^2-1)^2(x^2+y^2)^2-6(4a^2-1)^2a(x^2+y^2)^2x-27a^2y^2(x^2+y^2)^2=0;$$

and rejecting the factor $(x^2+y^2)^2$, this gives

$$(4a^2-1)^2x^2+\{(4a^2-1)^2-27a^2\}y^2-6(4a^2-1)^2ax=0;$$

or reducing,

$$(4a^2-1)^2x^2-(1-a^2)(8a^2+1)^2y^2-6(4a^2-1)^2ax=0;$$

and it follows that there are two asymptotes, the equations of which are

$$y=\frac{(4a^2-1)^2}{\sqrt{1-a^2}(8a^2+1)}\left\{x-\frac{3a}{4a^2-1}\right\}.$$

Represent for a moment the equation of one of the asymptotes by $y=\mathrm{A}(x-a)$, then the perpendicular from the origin or centre of the reflecting circle is $\mathrm{A}a\div\sqrt{1+\mathrm{A}^2}$, and

$$\mathrm{A}a=\frac{3a\sqrt{4a^2-1}}{\sqrt{1-a^2}(1+8a^2)}$$

$$1+\mathrm{A}^2=\frac{(1-a^2)(1+8a^2)^2+(4a^2-1)^2}{(1-a^2)(1+8a^2)^2}=\frac{27a^4}{(1-a^2)(1+8a^2)^2}$$

$$\sqrt{1+\mathrm{A}^2}=\frac{3\sqrt{3}a}{\sqrt{1-a^2}(1+8a^2)},$$

and the perpendicular is $\frac{1}{\sqrt{3}}\sqrt{4a^2-1}$, which is less than a if only $a^2<1$, *i. e.* in every case in which the asymptote is real.

The tangents parallel and perpendicular to the axis of x are most readily obtained from the equation of the reflected ray, viz.

$$(-2a\cos\theta+1)x+\frac{a\cos 2\theta-\cos\theta}{\sin\theta}y+a=0;$$

the coefficient of x (if the equation is first multiplied by $\sin\theta$) vanishes if $\sin\theta=0$, which gives the axis of x, or if $\cos\theta=\frac{1}{2a}$, which gives $y=\pm\frac{\sqrt{4a^2-1}}{2a}$, for the tangents parallel to the axis of x.

The coefficient of y vanishes if $a\cos 2\theta-\cos\theta=0$; this gives

$$\cos\theta=\frac{1\pm\sqrt{8a^2+1}}{4a},\ \sin\theta=\frac{1}{8a^2}(4a^2-1\mp\sqrt{8a^2+1}),$$

and the tangents perpendicular to the axis of x are given by

$$x=\frac{-2a}{1\mp\sqrt{8a^2+1}};$$

these tangents are in fact double tangents of the caustic. In order that the point of contact may be real, it is necessary that $\sin\theta$, $\cos\theta$ should be real; this will be the case for both values of the ambiguous sign if $a>$ or $=1$, but only for the upper value if $a<1$.

It has just been shown that for the tangents parallel to the axis of x, we have

$$y=\pm\frac{\sqrt{4a^2-1}}{2a},$$

the values of y being real for $a>\frac{1}{2}$: it may be noticed that the value $y=\frac{\sqrt{4a^2-1}}{2a}$ is greater, equal, or less than, or to $y=2a^2\sqrt{1-a^2}$, according as $a>=$ or $<\frac{1}{\sqrt{2}}$; this depends on the identity $(4a^2-1)-16a^4(1-a^2)=(2a^2-1)^2(2a^2+1)$.

To find the points of intersection with the reflecting circle, $x^2+y^2-1=0$, we have

$$(3a^2-1-2ax)^2-27a^2(1-x^2)(1-a^2)^2=0;$$

or reducing

$$8a^2x^2+(-27a^4+18a^2-15)a^2x^2+(54a^4-36a^2+6)ax+(-27a^4+18a^2+1)=0,$$

i. e.

$$(ax-1)^2(8ax-27a^4+18a^2+1)=0.$$

The factor $(ax-1)^2$ equated to zero shows that the caustic touches the circle in the points $x=\frac{1}{a}$, $y=\pm\sqrt{1-\frac{1}{a^2}}$, *i. e.* in the points in which the circle is met by the polar of the radiant point, and which are real or imaginary according as $a>$ or <1. The other factor gives

$$x=\frac{27a^4-18a^2-1}{8a}.$$

Putting this value equal to ±1, the resulting equation is $(a\mp1)(27a^2+9a+1)=0$, and it follows that x will be in absolute magnitude greater or less than 1, *i. e.* the points in question will be imaginary or real, according as $a>1$ or $a<1$.

It is easy to see that the curve passes through the circular points at infinity, and that these points are cusps on the curve; the two points of intersection with the axis of x are cusps (the axis of x being the tangent), and the two points of intersection with the circle $x^2+y^2-a^2=0$ are also cusps, the tangent at each of the cusps coinciding with the tangent of the circle; there are consequently in all six cusps.

XXII.

To investigate the position of the double points we may proceed as follows: write for shortness $P=(4a^2-1)(x^2+y^2)-2ax-a^2$, $Q=ayS$, $S=x^2+y^2-a^2$; the equation of the caustic is

$$P^3-27Q^2=0.$$

Hence, at a double point,

$$P^2\frac{dP}{dx}-18Q\frac{dQ}{dx}=0$$

$$P^2\frac{dP}{dy}-18Q\frac{dQ}{dy}=0;$$

one of which equations may be replaced by

$$\frac{dP}{dx}\frac{dQ}{dy}-\frac{dP}{dy}\frac{dQ}{dx}=0.$$

Now

$$\frac{dP}{dx}=2\{(4a^2-1)x-a\},\quad \frac{dP}{dy}=2(4a^2-1)y$$

$$\frac{dQ}{dx}=2axy,\qquad \frac{dQ}{dy}=a(x^2+3y^2-a^2)=a(S+2y^2).$$

Substituting these values in the last preceding equation, we find

$$\frac{(4a^2-1)x-a}{(4a^2-1)y}=\frac{2xy}{S+2y^2},$$

or reducing

$$(4a^2-1)x-a=\frac{2ay^2}{S};$$

and using this to simplify the equation

$$P^2\frac{dP}{dx}-18Q\frac{dQ}{dx}=0,$$

we have

$$P^2\frac{4ay^2}{S}-18ayS.2axy=0,$$

i. e.

$$\frac{P^2}{S}-9axS=0,$$

and therefore

$$9x=\frac{P^2}{aS^2}.$$

Multiplying by P and writing for P^3 its value $27a^2y^2S^2$, we have

$$Px=3ay^2,$$

and thence

$$P=\frac{3ay^4}{x},\ P^3=\frac{27a^3y^6}{x^3}=27a^3y^3S^3,$$

whence

$$\frac{ay^4}{x^3}=S^3,\ \frac{a^{\frac{1}{3}}y^{\frac{4}{3}}}{x^{\frac{1}{3}}}=S,\text{ or }\frac{y^3}{S}=\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}};$$

and substituting in the equation

$$x=\frac{a}{4a^3-1}\left(1+\frac{2y^3}{S}\right),$$

we find

$$x=\frac{a^2}{4a^3-1}\left(1+\frac{2x^{\frac{1}{3}}}{a^{\frac{1}{3}}}\right)$$

and rationalising

$$4ax^3-\{(4a^3-1)x-a\}^3=0,$$

or, what is the same thing,

$$(4ax-1)\left(x-a(a+i\sqrt{1-a^3})^2\right)\left(x-a(a-i\sqrt{1-a^3})^2\right)=0.$$

The factor $4ax-1$ equated to zero gives $x=\frac{1}{4a}$ from which y may be found, but the resulting point is not a double point, the other factors give each of them double points; and if we write

$$x=a(a+i\sqrt{1-a^3})^2,$$

we find

$$y=\frac{2a^2i(a+i\sqrt{1-a^3})^{\frac{1}{3}}}{(3a-i\sqrt{1-a^3})^{\frac{1}{3}}},$$

values which, in fact, belong to one of the four double points. It is easy to see that the points in question are always imaginary.

It may be noticed, by way of verification, that the preceding values of x, y give

$$(4a^3-1)(x^2+y^2)-2ax-a^2=\frac{12a^4}{1+8a^3}(1-4a^3-4ai\sqrt{1-a^3})$$

$$x^2+y^2-a^2=\frac{-4a^3}{1+8a^3}(3a+i\sqrt{1-a^3})$$

$$y^2=\frac{4a^4}{1+8a^3}\left(-1+14a^3-16a^4+2a(3-8a^3)i\sqrt{1-a^3}\right);$$

and if the quantities within ( ) on the right-hand side are represented by A, B, C, then

$$\frac{A}{B}=-(a+i\sqrt{1-a^3})$$

$$\frac{C}{B}=-(a+i\sqrt{1-a^3})^2,$$

whence we have identically,

$$\left(\frac{A}{B}\right)^2=\frac{C}{B},\text{ or }A^2=B^3C,$$

by means of which it appears that the values of x, y satisfy, as they should do, the equation of the caustic; and by forming the expressions for $(4a^3-1)x-a$ and $x^2+3y^2-a^2$, it might be shown, *à posteriori*, that the point in question was a double point.

XXIII.

The equation

$$\{(4a^3-1)(x^2+y^2)-2ax-a^2\}^3-27a^2y^2(x^2+y^2-a^2)^2=0$$

becomes when $a=1$ (*i. e.* when the radiant point is in the circumference),

$$\{3y^2+(x-1)(3x+1)\}^3-27y^2(y^2+x^2-1)^2=0;$$

it is easy to see that this divides by $(x-1)^2$; and throwing out this factor, we have for the caustic the equation of the fourth order,

$$27y^4+18y^2(3x^2-1)+(x-1)(3x+1)^2=0.$$

XXIV.

The equation

$$\{(4a^3-1)(x^2+y^2)-2ax-a^2\}^3-27a^2y^2(x^2+y^2-a^2)^2=0$$

becomes when $a=\infty$ (*i. e.* in the case of parallel rays),

$$(4x^2+4y^2-1)^3-27y^2=0,$$

which may also be written

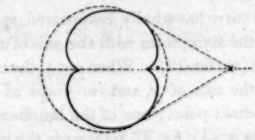
$$64x^6+48x^4(4y^2-1)+12x^2(4y^2-1)^2+(8y^2+1)^2(y^2-1)=0.$$

XXV.

It is now easy to trace the curve. Beginning with the case $a=\infty$, the curve lies wholly within the reflecting circle, which it touches at two points; the line joining the points of contact, being in fact the axis of y, divides the curve into two equal portions; the curve has in the present, as in every other case (except one limiting case), two cusps on the axis of x (see fig. 6). Next, if a be positive and >1, the general form of the curve is the same as before, only the line joining the points of contact with the reflecting circle divides the curve into unequal portions, that in the neighbourhood of the radiant point being the smaller of the two portions (see fig. 7). When $a=1$, the two

Fig. 6. $a=\infty$.

Fig. 7. $a>1$.



points of contact with the reflecting circle unite together at the radiant point; the curve throws off, as it were, the two coincident lines $x=1$, and the order is reduced from 6 to 4. The curve has the form fig. 8, with only a single cusp on the axis of x. If a be further diminished, $a<1>\frac{1}{\sqrt{2}}$, the curve takes the form shown by fig. 9, with two infinite branches, one of them having simply a cusp on the axis of x, the other having a cusp on the axis of x, and a pair of cusps at its intersection with the circle through the radiant point, there are two asymptotes equally inclined to the axis of x. In the case

$a=\frac{1}{\sqrt{2}}$, the form of the curve is nearly the same as before, only the cusps upon the circle through the radiant point lie on the axis of y (see fig. 10). The case $a<\frac{1}{\sqrt{2}}>\frac{1}{2}$ is shown, fig. 11. For $a=\frac{1}{2}$, the two asymptotes coincide with the axis of x; one of the

Fig. 8. $a=1$.

Fig. 9. $a<1>\frac{1}{\sqrt{3}}$

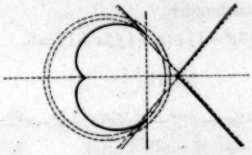


Fig. 13. $a=\frac{1}{3}$.

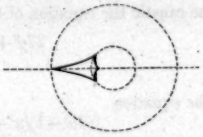


Fig. 11. $a<\frac{1}{\sqrt{3}}>\frac{1}{2}$.

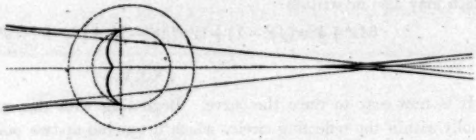


Fig. 10. $a=\frac{1}{\sqrt{3}}$.

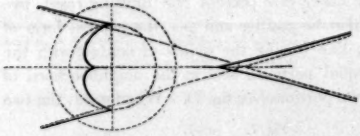
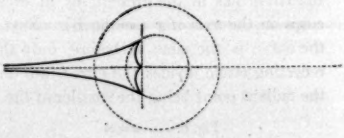


Fig. 12. $a=\frac{1}{2}$.



branches of the curve has wholly disappeared, and the form of the other is modified by the coincidence of the asymptotes with the axis of x; it has in fact acquired a cusp at infinity on the axis of x (see fig. 12). When $a<\frac{1}{3}$, the curve consists of a single finite branch, with two cusps on the axis of x, and two cusps at the points of intersection with the circle through the radiant point; one of the last-mentioned cusps will be outside the reflecting circle as long as $a>\frac{1}{3}$; fig. 13 represents the case $a=\frac{1}{3}$, for which this cusp is upon the reflecting circle. For $a<\frac{1}{3}$, the curve lies wholly within the reflecting circle, one of the cusps upon the axis of x being always within, and the other always without the circle through the radiant point, and as a approaches O the curve becomes smaller and smaller, and ultimately disappears in a point. The case a negative is obviously included in the preceding one.

Several of the preceding results relating to the caustic by reflexion of a circle were obtained, and the curve is traced in a memoir by the Rev. HAMNET HOLDITCH, Quarterly Mathematical Journal, t. i.

XXVI.

Suppose next that rays proceeding from a point are *refracted* at a circle. Take the centre of the circle as origin, let the radius be c, and take ξ, η as the coordinates of the radiant point, α, β the coordinates of the point of incidence, x, y the coordinates of a point in the refracted ray: then the general equation

$$-\overline{qG}^2\ \overline{\nabla QGN}^2+\mu^2\ \overline{QG}^2\ \overline{\nabla qGN}^2=0$$

becomes, taking the centre of the circle as the point N on the normal, or writing $a=0$, $b=0$,

$$-\{(x-\alpha)^2+(y-\beta)^2\}(\beta x-\alpha\eta)^2+\mu^2\{(\xi-\alpha)^2+(\eta-\beta)^2\}(\beta x-\alpha y)^2=0;$$

or putting $\alpha^2+\beta^2=c^2$, and expanding,

$$\begin{aligned}
&\alpha^2 &&\{2(\eta^2x-\mu^2y^2\xi)\}\\
+&\alpha^2\beta &&\{-4(\xi\eta x-\mu^2xy\xi)+2(\eta^2y-\mu^2y^2\eta)\}\\
+&\alpha\beta^2 &&\{-4(\xi\eta y-\mu^2xy\eta)+2(\xi^2x-\mu^2x^2\xi)\}\\
+&\beta^3 &&\{2(\xi^2y-\mu^2x^2\eta)\}\\
-&\alpha^3 &&\{(x^2+y^2+c^2)\eta^2-\mu^2(\xi^2+\eta^2+c^2)y^2\}\\
+2&\alpha\beta &&\{(x^2+y^2+c^2)\xi\eta-\mu^2(\xi^2+\eta^2+c^2)xy\}\\
-&\beta^3 &&\{(x^2+y^2+c^2)\xi^2-\mu^2(\xi^2+\eta^2+c^2)x^2\}\\
&=0,
\end{aligned}$$

which may be represented by

$$A\alpha^3+B\alpha^2\beta+C\alpha\beta^2+D\beta^3+F\alpha^2+G\alpha\beta+H\beta^2=0.$$

Now $\alpha^2+\beta^2=c^2$, and we may write

$$\alpha=\tfrac{1}{2}c\left(z+\frac{1}{z}\right),\ \beta=-\tfrac{1}{2}ci\left(z-\frac{1}{z}\right).$$

The equation thus becomes

$$A\left(z+\frac{1}{z}\right)^3-Bi\left(z+\frac{1}{z}\right)^2\left(z-\frac{1}{z}\right)-C\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)^2-Di\left(z-\frac{1}{z}\right)^3$$
$$+\frac{2}{c}F\left(z+\frac{1}{z}\right)^2-\frac{2}{c}Gi\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)-\frac{2}{c}H\left(z-\frac{1}{z}\right)^2=0;$$

or expanding,

$$\left.\begin{aligned}
&(A-Bi-C-Di)z^3\\
+&\tfrac{2}{c}(F-Gi-H)z^2\\
+&(3A-Bi+C+3Di)z\\
+&\tfrac{4}{c}(F+H)\\
+&\left(3A+Bi+C-3Di\right)\tfrac{1}{z}\\
+&\tfrac{2}{c}(F+Gi-H)\tfrac{1}{z^2}\\
+&(A+Bi-C+Di)\tfrac{1}{z^3}
\end{aligned}\right\}=0,$$

in which z may be considered as the variable parameter; hence the equation of the caustic may be obtained by equating to zero the discriminant of the above function of z; but the discriminant of a sextic function has not yet been calculated. The equation would be of the order 20, and it appears from the result previously obtained for parallel rays, that the equation must be of the order 12 at the least; it is, I think, probable that there is not any reduction of degree in the general case. It is however practicable, as will presently be seen, to obtain the tangential equation of the caustic by refraction, and the curve is thus shown to be only of the class 6.

XXVII.

Suppose that rays proceeding from a point are refracted at a circle, and let it be required to find the equation of the secondary caustic: take the centre of the circle as origin, let c be the radius, ξ, η the coordinates of the radiant point, α, β the coordinates of a point upon the circle, μ the index of refraction; the secondary caustic will be the envelope of the circle,

$$\mu^2\{(x-\alpha)^2+(y-\beta)^2\}-\{(\xi-\alpha)^2+(\eta-\beta)^2\}=0,$$

where α, β are variable parameters connected by the equation $\alpha^2+\beta^2-c^2=0$; the equation of the circle may be written in the form

$$\mu^2(x^2+y^2+c^2)-(\xi^2+\eta^2+c^2)-2(\mu^2x-\xi)\alpha-2(\mu^2y-\eta)\beta=0.$$

But in general the envelope of $A\alpha+B\beta+C=0$, where α, β are connected by the equation $\alpha^2+\beta^2-c^2=0$, is $c^2(A^2+B^2)-C^2=0$, and hence in the present case the equation of the envelope is

$$\{\mu^2(x^2+y^2+c^2)-(\xi^2+\eta^2+c^2)\}^2=4c^2\{(\mu^2x-\xi)^2+(\mu^2y-\eta)^2\},$$

which may also be written

$$\{\mu^2(x^2+y^2-c^2)-(\xi^2+\eta^2-c^2)\}^2=4c^2\mu^2\{(x-\tfrac{\xi}{\mu^2})^2+(y-\tfrac{\eta}{\mu^2})^2\}.$$

If the axis of x be taken through the radiant point, then $\eta=0$, and writing also $\xi=\alpha$, the equation becomes

$$\{\mu^2(x^2+y^2-c^2)-a^2+c^2\}^2=4c^2\mu^2\{(x-a)^2+y^2\};$$

or taking the square root of each side,

$$\{\mu^2(x^2+y^2-c^2)-a^2+c^2\}=2c\mu\sqrt{(x-a)^2+y^2};$$

whence multiplying by $1-\frac{1}{\mu^2}$ and adding on each side $c^2\left(\mu-\frac{1}{\mu}\right)^2+(x-a)^2+y^2$, we have

$$\mu^2\left\{\left(x-\frac{a}{\mu^2}\right)^2+y^2\right\}=\left\{\sqrt{(x-a)^2+y^2}+c\left(\mu-\frac{1}{\mu}\right)\right\}^2,$$

or

$$\mu\sqrt{\left(x-\frac{a}{\mu^2}\right)^2+y^2}=\sqrt{(x-a)^2+y^2}+c\left(\mu-\frac{1}{\mu}\right),$$

which shows that the secondary caustic is the Oval of Descartes, or as it will be convenient to call it, the Cartesian.

It is proper to remark, that the Cartesian consists in general of two ovals, one of which is the orthogonal trajectory of the refracted rays, the other the orthogonal trajectory of the false refracted rays. In the case of reflexion, the secondary caustic is a Cartesian having a double point; this may be either a conjugate point, or a real double point arising from the union and intersection of the two ovals; the same secondary caustic may arise also from refraction, as will be presently shown.

XXVIII.

Reverting to the original form of the equation of the secondary caustic, multiplying by $\frac{1}{\mu^2}\left(1-\frac{c^2}{a^2}\right)$ and adding on each side $\frac{a^2}{\mu^2}\left(1-\frac{c^2}{a^2}\right)^2+\frac{c^2}{a^2}\{(x-a)^2+y^2\}$, the equation becomes

$$\left\{\left(x-\frac{c^2}{a}\right)^2+y^2\right\}=\left\{\frac{c}{a}\sqrt{(x-a)^2+y^2}+\frac{a}{\mu}\left(1-\frac{c^2}{a^2}\right)\right\}^2,$$

or extracting the square root,

$$\sqrt{\left(x-\frac{c^2}{a}\right)^2+y^2}=\frac{c}{a}\sqrt{(x-a)^2+y^2}+\frac{a}{\mu}\left(1-\frac{c^2}{a^2}\right).$$

Combining this with the former result, we see that the equation may be expressed indifferently in any one of the four forms,

$$\sqrt{\left(x-\frac{a}{\mu^2}\right)^2+y^2}=\frac{1}{\mu}\sqrt{(x-a)^2+y^2}+\frac{c}{\mu}\left(\mu-\frac{1}{\mu}\right)$$

$$\sqrt{\left(x-\frac{c^2}{a}\right)^2+y^2}=\frac{c}{a}\sqrt{(x-a)^2+y^2}+\frac{1}{\mu}\left(a-\frac{c^2}{a}\right)$$

$$\sqrt{\left(x-\frac{c^2}{a}\right)^2+y^2}=\frac{c\mu}{a}\sqrt{\left(x-\frac{a}{\mu^2}\right)^2+y^2}+\frac{a}{\mu}-\frac{c^2\mu}{a}$$

$$c\left(\mu-\frac{1}{\mu}\right)\sqrt{\left(x-\frac{c^2}{a}\right)^2+y^2}+\left(-a+\frac{c^2}{a}\right)\sqrt{\left(x-\frac{a}{\mu^2}\right)^2+y^2}+\left(\frac{a}{\mu}-\frac{c^2\mu}{a}\right)\sqrt{(x-a)^2+y^2}=0.$$

It follows, that if we write successively

$$a'=a,\quad c'=c,\quad \mu'=\mu \qquad (1)$$

$$a'=\frac{c^2}{a},\quad c'=\frac{c}{\mu},\quad \mu'=\frac{c}{a} \qquad (\alpha)$$

$$a'=\frac{a}{\mu^2},\quad c'=\frac{c}{\mu},\quad \mu'=\frac{1}{\mu} \qquad (\beta)$$

$$a'=a\quad c'=\frac{a}{\mu},\quad \mu'=\frac{a}{c} \qquad (\gamma)$$

$$a'=\frac{c^2}{a},\quad c'=c,\quad \mu'=\frac{c\mu}{a} \qquad (\delta)$$

$$a'=\frac{a}{\mu^2},\quad c'=\frac{a}{\mu},\quad \mu'=\frac{a}{c\mu} \qquad (\epsilon),$$

or what is the same thing,

$$a=a', \quad c=c', \quad \mu=\mu' \qquad (1)$$

$$a=\frac{a'}{\mu'^2}, \quad c=\frac{a'}{\mu'}, \quad \mu=\frac{a'}{c'\mu'}, \qquad (\alpha)$$

$$a=\frac{a'}{\mu'^2}, \quad c=\frac{c'}{\mu'}, \quad \mu=\frac{1}{\mu'}, \qquad (\beta)$$

$$a=a', \quad c=\frac{a'}{\mu'}, \quad \mu=\frac{a'}{c'} \qquad (\gamma)$$

$$a=\frac{c'^2}{a'}, \quad c=c', \quad \mu=\frac{c'\mu'}{a'} \qquad (\delta)$$

$$a=\frac{c'^2}{a'}, \quad c=\frac{c'}{\mu'}, \quad \mu=\frac{c'}{a'} \qquad (\varepsilon),$$

or what is again the same thing,

$$a'=a, \quad \frac{c'^2}{a'}=\frac{c^2}{a}, \quad \frac{a'}{\mu'^2}=\frac{a}{\mu^2} \qquad (1)$$

$$a'=\frac{c^2}{a}, \quad \frac{c'^2}{a'}=\frac{a}{\mu^2}, \quad \frac{a'}{\mu'^2}=a \qquad (\alpha)$$

$$a'=\frac{a}{\mu^2}, \quad \frac{c'^2}{a'}=\frac{c^2}{a}, \quad \frac{a'}{\mu'^2}=a \qquad (\beta)$$

$$a'=a, \quad \frac{c'^2}{a'}=\frac{a}{\mu^2}, \quad \frac{a'}{\mu'^2}=\frac{c^2}{a} \qquad (\gamma)$$

$$a'=\frac{c^2}{a}, \quad \frac{c'^2}{a'}=a, \quad \frac{a'}{\mu'^2}=\frac{a}{\mu^2} \qquad (\delta)$$

$$a'=\frac{a}{\mu^2}, \quad \frac{c'^2}{a'}=a, \quad \frac{a'}{\mu'^2}=\frac{c^2}{a} \qquad (\varepsilon),$$

we have in each case identically the same secondary caustic, and therefore also identically the same caustic; in other words, the same caustic is produced by six different systems of a radiant point and refracting circle. It is proper to remark that if we represent the six systems of equations by $(a', c', \mu')=(a, c, \mu)$, $(a', c', \mu')=\alpha(a, c, \mu)$, &c., then $\alpha, \beta, \gamma, \delta, \varepsilon$ will be functional symbols satisfying the conditions

$$1=\alpha\beta=\beta\alpha=\gamma^2=\delta^2=\varepsilon^2$$
$$\alpha=\beta^2=\delta\gamma=\varepsilon\delta=\gamma\varepsilon$$
$$\beta=\alpha^2=\gamma\delta=\delta\varepsilon=\varepsilon\gamma$$
$$\gamma=\delta\alpha=\alpha\varepsilon=\varepsilon\beta=\beta\delta$$
$$\delta=\varepsilon\alpha=\alpha\gamma=\gamma\beta=\beta\varepsilon$$
$$\varepsilon=\gamma\alpha=\alpha\delta=\delta\beta=\beta\gamma.$$

XXIX.

The preceding formulæ, which were first given by me in the Philosophical Magazine, December 1853, include as particular cases a preceding theorem with respect to the

caustic by refraction of parallel rays, and also two theorems of St. Laurent, *Gergonne*, t. xviii., viz. if we suppose first that $a=c$, *i. e.* that the radiant point is in the circumference of the refracting circle, then the system (α) shows that the same caustic would be obtained by writing $c, \frac{c}{\mu}, 1$ (or what is the same thing -1) in the place of c, c, μ, and we have

Theorem. The caustic by refraction for a circle when the radiant point is in the circumference is also the caustic by reflexion for the same radiant point, and for a reflecting circle concentric with the refracting circle, but having its radius equal to the quotient of the radius of the refracting circle by the index of refraction.

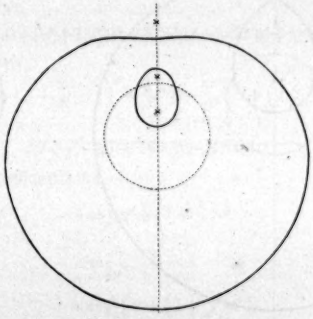
Next, if we write $a=c\mu$, then the refracted rays all of them pass through a point which is a double point of the secondary caustic, the entire curve being in this case the orthogonal trajectory, not of the refracted rays, but of the false refracted rays; the formula (δ) shows that the same caustic is obtained by writing $\frac{c^2}{a}, c, 1$ (or what is the same thing -1) in the place of $a, c, \mu\left(=\frac{a}{c}\right)$, and we have

Theorem. The caustic by refraction for a circle when the distance of the radiant point from the centre is to the radius of the circle in the ratio of the index of refraction to unity, is also the caustic by reflexion for the same circle considered as a reflecting circle, and for a radiant point the image of the former radiant point.

XXX.

The curve is most easily traced by means of the preceding construction; thus if we take the radiant point outside the refracting circle, and consider μ as varying from a small to a large value (positive or negative values of μ give the same curve), we see that when μ is small the curve consists of two ovals, one of them within and the other

Fig. 14.



without the refracting circle (see fig. 14). As u increases the exterior oval continually

increases, but undergoes modifications in its form; the interior oval in the first instance diminishes until we arrive at a curve, in which the interior oval is reduced to a conjugate point (see fig. 15); then as μ continues to increase the interior oval reappears (see fig. 16),

Fig. 15.

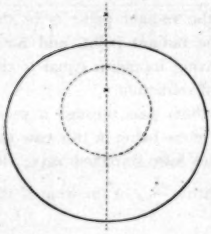
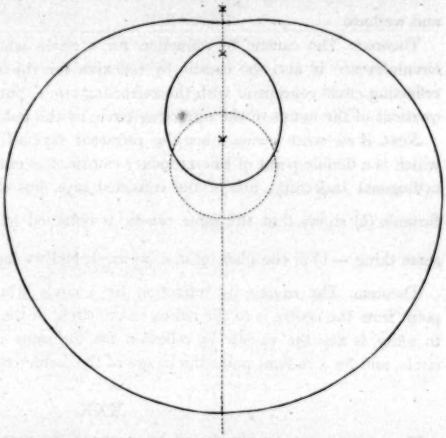


Fig. 16.



and at last connects itself with the exterior oval, so as to form a curve with a double point (see fig. 17); and as μ increases still further the curve again breaks up into an

Fig. 17.

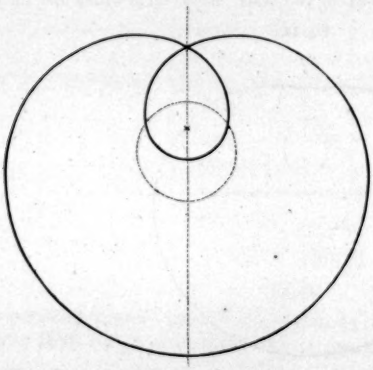
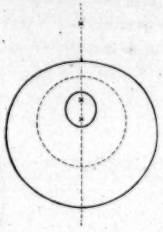


Fig. 18.



exterior and an interior oval (see fig. 18); and thenceforward as μ goes on increasing

consists always of two ovals; the shape of the exterior oval is best perceived from the figures. An examination of the figures will also show how the same curves may originate from a different refracting circle and radiant point.

XXXI.

The theorem, "If a variable circle have its centre upon a circle S, and its radius proportional to the tangential distance of the centre from a circle C, the envelope is a Cartesian,"
is at once deducible from the theorem—

"If a variable circle have its centre upon a circle S and its radius proportional to the distance of the centre from a point C', the locus is a Cartesian,"
which last theorem was in effect given in discussing the theory of the secondary caustic. In fact, the locus of a point P such that its tangential distances from the circles C, C' are in a constant ratio, is a circle S. Conversely, if there be a circle C, and the locus of P be a circle S, then the circle C' may be found such that the tangential distances of P from the two circles are in a constant ratio, and the circle C' may be taken to be a point, *i. e.* if there be a circle C and the locus of P be a circle S, then a point C' may be found such that the tangential distance of P from the circle C is in a constant ratio to the distance from the point C'.

Hence treating P as the centre of the variable circle, it is clear that the variable circle is determined in the two cases by equivalent constructions, and the envelope is therefore the same in both cases.

XXXII.

The equation of the secondary caustic developed and reduced is

$$\mu^4(x^2+y^2)^2-2\mu^2\left(a^2+(\mu^2+1)c^2\right)(x^2+y^2)+8c^2\mu^2ax$$
$$+a^4-2a^2c^2(\mu^2+1)+(\mu^2-1)^2c^4=0,$$

or, what is the same thing,

$$\left\{\mu^2(x^2+y^2)-\left(a^2+(\mu^2+1)c^2\right)\right\}^2+8c^2\mu^2ax-4c^2\left(c^2\mu^2+(\mu^2+1)a^2\right)=0,$$

which may also be written

$$\left(x^2+y^2-\left(\frac{a^2}{\mu^2}+\left(1+\frac{1}{\mu^2}\right)c^2\right)\right)^2+\frac{8}{\mu^2}c^2ax-\frac{4c^2}{\mu^2}\left(c^2+\left(1+\frac{1}{\mu^2}\right)a^2\right)=0,$$

which is of the form

$$(x^2+y^2-\alpha)^2+16\mathrm{A}(x-m)=0;$$

and the values of the coefficients are

$$\alpha=\frac{1}{\mu^2}a^2+\left(1+\frac{1}{\mu^2}\right)c^2$$

$$\mathrm{A}=\frac{c^2a}{2\mu^2}$$

$$m=\frac{1}{2a}\left(c^2+\left(1+\frac{1}{\mu^2}\right)a^2\right).$$

The equation just obtained should, I think, be taken as the standard form of the equation of the Cartesian, and the form of the equation shows that the Cartesian may be defined as the locus of a point, such that the fourth power of its tangential distance from a given circle is in a constant ratio to its distance from a given line.

XXXIII.

The Cartesian is a curve of the fourth order, symmetrical about a certain line which it intersects in four arbitrary points, and these points determine the curve. Taking the line in question (which may be called the axis) as the axis of x, and a line at right angles to it as the axis of y, let a, b, c, d be the values of x corresponding to the points of intersection with the axis, then the equation of the curve is

$$y^4+y^2[2x^2-(a+b+c+d)x-\tfrac{1}{4}(a^2+b^2+c^2+d^2-2ab-2ac-2ad-2bc-2bd-2cd)]$$
$$+(x-a)(x-b)(x-c)(x-d)=0.$$

It is easy to see that the form of the equation is not altered by writing $x+\theta$ for x, and $a+\theta$, $b+\theta$, $c+\theta$, $d+\theta$ for a, b, c, d, we may therefore without loss of generality put $a+b+c+d=0$, and the equation of the curve then becomes

$$y^4+y^2(2x^2+ab+ac+ad+bc+bd+cd)+(x-a)(x-b)(x-c)(x-d)=0,$$

where

$$a+b+c+d=0.$$

The curve is in this case said to be referred to the centre as origin.

The last-mentioned equation may be written

$$(x^2+y^2)^2+(ab+ac+ad+bc+bd+cd)(x^2+y^2)$$
$$-(abc+abd+acd+bcd)x+abcd=0,$$

or

$$\{x^2+y^2+\tfrac{1}{2}(ab+ac+ad+bc+bd+cd)\}^2$$
$$-(abc+abd+acd+bcd)x$$
$$-\tfrac{1}{4}\left\{\begin{array}{l}a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2\\+2a^2bc+2a^2bd+2a^2cd+2b^2ac+2b^2ad+2b^2cd\\+2c^2ab+2c^2ad+2c^2bd+2d^2ab+2d^2ac+2d^2bc\\+2abcd\end{array}\right\}=0,$$

or observing that

$$a^2bc+a^2bd+a^2cd+b^2ac+b^2ad+b^2cd$$
$$+c^2ab+c^2ad+c^2bd+d^2ab+d^2ac+d^2bc$$
$$=abc(a+b+c)+abd(a+b+d)+acd(a+c+d)+bcd(b+c+d)$$
$$=-4abcd,$$

the equation becomes

$$\{x^2+y^2+\tfrac{1}{2}(ab+ac+ad+bc+bd+cd)\}^2$$
$$-(abc+abd+acd+bcd)x$$
$$-\tfrac{1}{4}(a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2-6abcd)=0,$$

which is of the form

$$(x^2+y^2-\alpha)^2+16\mathrm{A}(x-m)=0,$$

and, as already remarked, signifies that the fourth power of the tangential distance of a point in the curve from a given circle, is proportional to the distance of the same point from a given line. The circle in question (which may be called the dirigent circle) has for its equation

$$x^2+y^2+\tfrac{1}{3}(ab+ac+ad+bc+bd+cd)=0.$$

The line in question, which may be called the directrix, has for its equation

$$x+\frac{a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2-6abcd}{4(abc+abd+acd+bcd)}=0;$$

the multiplier of the distance from the directrix is

$$abc+abd+acd+bcd.$$

It may be remarked that a, b, c, d being real, the dirigent circle is real; the equation may, in fact, be written

$$x^2+y^2=\tfrac{1}{3}[(a+b)^2+(a+c)^2+(a+d)^2+(b+c)^2+(b+d)^2+(c+d)^2].$$

XXXIV.

Considering the equation of the Cartesian under the form

$$(x^2+y^2-\alpha)^2+16\mathrm{A}(x-m)=0,$$

the centre of the dirigent circle $x^2+y^2-\alpha=0$ must be considered as a real point, but α may be positive or negative, *i. e.* the radius may be either a real or a pure imaginary distance: the coefficients A, m must be real, the directrix is therefore a real line. The equation shows that for all points of the curve $x-m$ is always negative or always positive, according as A is positive or negative, *i. e.* that the curve lies wholly on one side of the directrix, viz. on the same side with the centre of the dirigent circle if A is positive, but on the contrary side if A is negative. In the former case the curve may be said to be an 'inside' curve, in the latter an 'outside' curve. If $m=0$, or the directrix passes through the centre of the dirigent circle, then the distinction between an inside curve and an outside curve no longer exists. It is clear that the curve touches the directrix in the points of intersection of this line and the dirigent circle, and that the points in question are the only points of intersection of the curve with the directrix or the dirigent circle; hence if the directrix and dirigent circle do not intersect, the curve does not meet either the directrix or the dirigent circle.

XXXV.

To discuss the equation

$$(x^2+y^2-\alpha)^2+16\mathrm{A}(x-m)=0,$$

I write first $y=0$, which gives

$$x^4-2\alpha x^2+16\mathrm{A}x+\alpha^2-16\mathrm{A}m=0$$

for the points of intersection with the axis of x. If this equation has equal roots, there will be a double point on the axis of x, and it is important to find the condition that this may be the case. The equation may be written in the form

$$(3,\ 0,\ -\alpha,\ 12\mathrm{A},\ 3\alpha^2-48\mathrm{A}m\,\Big|\,x,\ 1)^4=0,$$

the condition for a part of equal roots is then at once seen to be

$$-(\alpha^2-12\mathrm{A}m)^3+(\alpha^3-18\mathrm{A}m\alpha+54\mathrm{A}^2)^2=0;$$

or reducing and throwing out the factor A^2, this is

$$27\mathrm{A}^2+2m(8m^3-9\alpha)\mathrm{A}-\alpha^2(m^3-\alpha)=0.$$

This equation will give two equal values for A if

$$m^2(8m^3-9\alpha)^2+27\alpha^2(m^3-\alpha)=0,$$

an equation which reduces itself to

$$(4m^3-3\alpha)^2=0.$$

Whence, if $4m^3-3\alpha$ be negative, *i. e.* if $\alpha>\frac{4m^3}{3}$, the values of A will be imaginary, but if $4m^3-3\alpha$ be positive, or $\alpha<\frac{4m^3}{3}$, the values of A will be real. If $\alpha=\frac{4m^3}{3}$, then there will be two equal values of A, which in fact corresponds to a cusp upon the axis of x. Whenever the curve is real there will be at least two real points on the axis of x; and when $\alpha<\frac{4m^3}{3}$, but not otherwise, then for properly selected values of A there will be four real points on the axis of x.

Differentiating the equation of the curve, we have

$$\big((x^2+y^2-\alpha)x+4\mathrm{A}\big)dx+(x^2+y^2-\alpha)ydy=0;$$

and if in this equation we put $dx=0$, we find $y=0$, or $x^2+y^2-\alpha=0$, *i. e.* that the points on the axis of x, and the points of intersection with the circle $x^2+y^2-\alpha=0$, are the only points at which the curve is perpendicular to the axis of x. To find the points at which the curve is parallel to the axis of x, we must write $dx=0$, this gives

$$(x^2+y^2-\alpha)x+4\mathrm{A}=0,$$

and thence

$$x^2+y^2-\alpha=-\frac{4\mathrm{A}}{x},$$

and

$$\mathrm{A}+x^2(x-m)=0:$$

this equation will have three real roots if $\mathrm{A}<\frac{4m^3}{27}$, and only a single real root if $\mathrm{A}>\frac{4m^3}{27}$, for $\mathrm{A}=\frac{4m^3}{27}$, the equation in question will have a pair of equal roots. It is easy to see that there is always a single real root of the equation which gives rise to a real value of y, i. e. to a real point upon the curve; but when the equation has three real roots, two of the roots may or may not give rise to real points upon the curve.

XXXVI.

It is now easy to trace the curve. First, when $m=0$, or the directrix passes through the centre of the dirigent circle, the curve is here an oval bent in so as to have double contact with the directrix, and lying on the one or the other side of the directrix according to the sign of A. See fig. *a*.

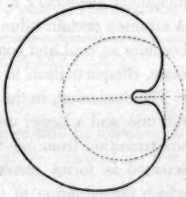
Fig. *a*.

Next, when the directrix does not pass through the centre of the dirigent circle, it will be convenient to suppose always that m is positive, and to consider A as passing first from 0 to ∞ and then from 0 to $-\infty$, *i. e.* to consider first the different inside curves, and then the different outside curves. Suppose $\alpha > \frac{4m^2}{3}$, the inside curve is at first an oval, as in fig. *b*, where (attending to one side only of the axis) it will be noticed that there are three tangents parallel to the axis, viz. one for the convexity of the oval, and two for the concavity. For $A=\frac{4m^3}{27}$ the two tangents for the concavity come together, and give rise to a stationary tangent (*i. e.* a tangent at an inflection) parallel to the axis, and for $A>\frac{4m^3}{27}$ the two tangents for the concavity disappear. The outside curve is an oval (of course on the opposite side of, and) bent in so as to have double contact with the directrix.

Fig. *b*.

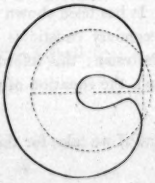
Next, if $\alpha=\frac{4m^2}{3}$, the inside curve is at first an oval, as in fig. *c*, and there are, as before, three tangents parallel to the axis: for $A=\frac{4m^3}{27}$, the tangents for the concavity of the oval come to coincide with the axis, and are tangents at a cusp, and for $A>\frac{4m^3}{27}$ the cusp disappears, and there are not for the concavity of the oval any tangents parallel to the axis. The outside curve is an oval as before, but smaller and more compressed.

Fig. *c*.



Next, $\alpha < \frac{4m^2}{3} > m^2$, then the inside curve is at first an oval, as in fig. *d*, and there are, as before, three tangents parallel to the axis; when A attains a certain value which is less than $\frac{4m^3}{27}$, the curve acquires a double point; and as A further increases, the curve breaks up into two separate ovals, and there are then only two tangents parallel to the axis, viz. one for the exterior oval and one for the interior oval. As A continues to increase, the interior oval decreases; and when A attains a certain value which is less than $\frac{4m^3}{27}$, the interior oval

Fig. *d*.



reduces itself to a conjugate point, and it afterwards disappears altogether. The outside curve is an oval as before, but smaller and more compressed.

Next, if the directrix touch the dirigent circle, *i. e.* if $\alpha=m^2$. Then the inside curve is at first composed of an exterior oval which touches the dirigent circle, and of an interior oval which lies wholly within the dirigent circle. As A increases the interior oval decreases, reduces itself to a conjugate point, and then disappears. The outside curve is an oval which always touches the dirigent circle, at first very small (it may be considered as commencing from a conjugate point corresponding to $A=0$), but increasing as A increases negatively.

Next, when the directrix does not meet the dirigent circle, *i. e.* if $\alpha<m^2$. The inside curve consists at first of two ovals, an exterior oval lying without the dirigent circle, and an interior oval lying within the dirigent circle. As A increases the interior oval decreases, reduces itself to a conjugate point and disappears. The outside curve is at first imaginary, but when A attains a sufficiently large negative value, it makes its appearance as a conjugate point and afterwards becomes an oval, which gradually increases.

Next, when the dirigent circle reduces itself to a point, *i. e.* if $\alpha=0$. The inside curve makes its appearance as a conjugate point (corresponding to $A=0$), and as A increases it becomes an oval and continually increases. The outside curve comports itself as in the last preceding case.

Finally, when the dirigent circle becomes imaginary, or has for its radius a pure imaginary distance, *i. e.* if α is negative. The inside curve is at first imaginary, but when A attains a certain value it makes its appearance as a conjugate point, and as A increases becomes an oval and continually increases. The outside curve, as in the preceding two cases, comports itself in a similar manner.

The discussion, in the present section, of the different forms of the curve is not a very full one, and a larger number of figures would be necessary in order to show completely the transition from one form to another. The forms delineated in the four figures were selected as forms corresponding to imaginary values of the parameters by means of which the equation of the curve is usually represented, *e. g.* the equations in Section XXVIII.

XXXVII.

It has been shown that for rays proceeding from a point and refracted at a circle, the secondary caustic is the Cartesian; the caustic itself is therefore the evolute of the Cartesian; this affords a means of finding the tangential equation of the caustic. In fact, the equation of the Cartesian is

$$(x^2+y^2-\alpha)^2+16A(x-m)=0;$$

and if we take for the equation of the normal

$$X\xi+Y\eta+Z=0,$$

(where ξ, η are current coordinates), then

$$\begin{aligned} X : Y : Z &= -y(x^2+y^2-\alpha) \\ &: x(x^2+y^2-\alpha)+4A \\ &: 4Ay, \end{aligned}$$

equations which give

$$\begin{aligned} Z^2Yx &= Y(mZ^2-AX^2) \\ -Z^2Yy &= Z^2+X(mZ^2-AX^2) \\ Z^4Y^2(x^2+y^2-\alpha) &= 4AZ^2XY^2, \end{aligned}$$

whence eliminating, we have

$$\{Z^2+X(mZ^2-AX^2)\}^2+Y^2(mZ^2-AX^2)^2-Z^2Y^2(\alpha Z+4AX)=0,$$

where if, as before, c denotes the radius of the refracting circle, a the distance of the radiant point from the centre, and μ the index of refraction, we have

$$\alpha = \frac{1}{\mu^2}a^2+\left(1+\frac{1}{\mu^2}\right)c^2$$

$$A = \frac{c^2a}{2\mu^2}$$

$$m = \frac{1}{2a}\left(c^2+\left(1+\frac{1}{\mu^2}\right)a^2\right).$$

The above equation is the condition in order that the line $Xx+Yy+Z=0$ may be a normal to the secondary caustic $(x^2+y^2-\alpha)^2+16A(x-m)=0$, or it is the tangential equation of the caustic, which is therefore a curve of the class 6 only. The equation may be written in the more convenient form

$$Z^6+2Z^2X(mZ^2-AX^2)+(X^2+Y^2)(mZ^2-AX^2)^2-Z^2Y^2(\alpha Z+4AX)=0.$$

XXXVIII.

To compare the last result with that previously obtained for the caustic by reflexion, I write $\mu=-1$, and putting also $c=1$ and $Z=a$ (for the equation of the reflected ray was assumed to be $Xx+Yy+a=0$), we have

$$\alpha=a^2+2,\ A=\tfrac{1}{2}a,\ m=\frac{1}{2a}(1+2a^2),$$

and the equation becomes, after a slight reduction,

$$4a^4+4a^2X(2a^2+1-X^2)+(X^2+Y^2)(2a^2+1-X^2)^2-4a^2Y^2(a^2+2+2X)=0$$

which may be written

$$\left(2a^2+X(2a^2+1-X^2)\right)^2+Y^2\left(-4a^2+1-8a^2X-2(2a^2+1)X^2+X^4\right)=0;$$

this divides out by the factor $(X+1)^2$, and the equation then becomes,

$$(X^2-X-2a^2)^2+Y^2\left((X-1)^2-4a^2\right)=0,$$

which agrees with the result before obtained.

XXXIX.

Again, to compare the general equation with that previously obtained for parallel rays refracted at a circle, we must write $\mu=\frac{1}{k}$, $c=1$, $a=\infty$, $Z=k$ (for the equation of the refracted ray was taken to be $Xx+Yy+k=0$); we have then

$$\alpha=1+k^2+k^2a^2,\ A=\tfrac{1}{2}k^2a^2,\ m=\frac{1}{2a}\left(1+(1+k^2)a^2\right),$$

and after the substitution $a=\infty$. The equation becomes in the first instance

$$k^2+2k^2X\left\{\frac{1}{2a}\left(1+(1+k^2)a^2\right)k^2-\tfrac{1}{2}k^2aX^2\right\}+(X^2+Y^2)\left\{\frac{1}{2a}\left(1+(1+k^2)a^2\right)k^2-\tfrac{1}{2}k^2aX^2\right\}^2$$
$$-k^2Y^2(1+k^2+k^2a^2+2k^2aX)=0;$$

and then putting $a=\infty$, or, what is the same thing, attending only to the terms which involve a^2, and throwing out the constant factor k^4, we obtain

$$(X^2+Y^2)(X^2-1-k^2)^2-4k^2Y^2=0,$$

or

$$X^2(X^2-1-k^2)^2+Y^2(X+1+k)(X-1-k)(X+1-k)(X-1-k)=0,$$

which agrees with the former result

XL.

It was remarked that the ordinary construction for the secondary caustic could not be applied to the case of parallel rays (the entire curve would in fact pass off to an infinite distance), and that the simplest course was to measure the distance GQ from a line through the centre of the refracting circle perpendicular to the direction of the rays. To find the equation of the resulting curve, take the centre of the circle as the origin and the direction of the incident rays for the axis of x; let the radius of the circle be taken equal to unity, and let μ denote, as before, the index of refraction. Then if α, β are the coordinates of the point of incidence of a ray, we have $\alpha^2+\beta^2=1$, and considering α, β as variable parameters connected by this equation, the required curve is the envelope of the circle,

$$\mu^2\{(x-\alpha)^2+(y-\beta)^2\}-\alpha^2=0.$$

Write now $\alpha=\cos\theta$, $\beta=\sin\theta$, then multiplying the equation by -2, and writing $1+\cos 2\theta$ instead of $2\cos^2\theta$, the equation becomes

$$1+\cos 2\theta-2\mu^2(x^2+y^2-2x\cos\theta-2y\sin\theta+1)=0,$$

which is of the form

$$A\cos 2\theta+B\sin 2\theta+C\cos\theta+D\sin\theta+E=0,$$

and the values of the coefficients are

$$A=1$$
$$B=0$$
$$C=4\mu^2 x$$
$$D=4\mu^2 y$$
$$E=-2\mu^2(x^2+y^2)-2\mu^2+1.$$

Substituting these values in the equation

$$\{12(A^2+B^2)-3(C^2+D^2)+4E^2\}^3-\{27A(C^2-D^2)+54BCD$$
$$-\big(72(A^2+B^2)+9(C^2+D^2)\big)E+8E^3\}^2=0,$$

the equation of the envelope is found to be

$$16\{(1-\mu^2+\mu^4)-(\mu^2+\mu^4)(x^2+y^2)+\mu^4(x^2+y^2)^2\}^3$$
$$-\begin{Vmatrix} 4-6\mu^2-6\mu^4+4\mu^6 \\ -(6\mu^2+3\mu^4+6\mu^6)(x^2+y^2)-27\mu^4(x^2-y^2) \\ -(6\mu^4+6\mu^6)(x^2+y^2)^2 \\ +4\mu^6(x^2+y^2)^3 \end{Vmatrix}^2=0,$$

which is readily seen to be only of the 8th order. But to simplify the result, write first $(x^2+y^2-1)+1$, and $2x^2-1-(x^2+y^2-1)$ in the place of x^2+y^2 and x^2-y^2 respectively, the equation becomes

$$4\{(1-\mu^2)^2-\mu^2(1-\mu^2)(x^2+y^2-1)+\mu^4(x^2+y^2-1)^2\}^3$$
$$-\begin{Vmatrix} 2(1-\mu^2)^3 \\ -3\mu^2(1-\mu^2)^2(x^2+y^2-1)-27\mu^4x^2 \\ -3\mu^4(1-\mu^2)(x^2+y^2-1)^2 \\ +2\mu^6(x^2+y^2-1)^3 \end{Vmatrix}^2=0.$$

Write for a moment $1-\mu^2=q$, $\mu^2(x^2+y^2-1)=\varrho$, the equation becomes

$$4(q^2-q\varrho+\varrho^2)^3-(2q^3-3q^2\varrho-3\varrho^2+2\varrho^3-27\mu^4x^2)^2=0;$$

or developing,

$$4(q^2-q\varrho+\varrho^2)^3-(2q^3-3q^2\varrho-3q\varrho^2+2\varrho^3)^2$$
$$+54(2q^3-3q^2\varrho-3q\varrho^2+2\varrho^3)\mu^4x^2-729\mu^8x^4=0,$$

and reducing and dividing out by 27, this gives

$$q^2\varrho^2(\varrho-q)^2+2(\varrho+q)(2\varrho-q)(\varrho-2q)\mu^4x^2-27\mu^8x^4=0,$$

whence replacing q, ϱ by their values, the required equation is

$$(1-\mu^2)^2(x^2+y^2-1)^2\big(\mu^2(x^2+y^2)-1\big)^2$$
$$+2\big(\mu^2(x^2+y^2)-2\mu^2+1\big)\big(2\mu^2(x^2+y^2)-\mu^2-1\big)\big(\mu^2(x^2+y^2)-2+\mu^2\big)x^2-27\mu^4x^4=0,$$

which is the equation of an orthogonal trajectory of the refracted rays.

In the case of reflexion, $\mu=-1$, and the equation becomes

$$4(x^2+y^2-1)^3-27x^2=0.$$

Comparing this with the equation of the caustic, it is easy to see,

Theorem. In the case of parallel rays and a reflecting circle, there is a secondary caustic which is a curve similar to and double the magnitude of the caustic, the position of the two curves differing by a right angle.

XLI.

The entire system of the orthogonal trajectories of the refracted rays might in like manner be determined by finding the envelope of the circle (where, as before, α, β are variable parameters connected by the equation $\alpha^2+\beta^2=1$),

$$\mu^2\{(x-\alpha)^2+(y-\beta)^2\}-(\alpha+m)^2=0.$$

[The result, as far as I have worked it out, is as follows, viz.—

$$\begin{aligned}&\big(3-12[m^2+2m\mu^2x+\mu^4(x^2+y^2)]+[1-2\mu^2+2m^2-2\mu^2(x^2+y^2)]^2\big)^3\\&-\big([1-2\mu^2+2m^2-2\mu^2(x^2+y^2)][9+18m^2+36m\mu^2x+18\mu^4(x^2+y^2)]\\&-54[m^2+2m\mu^2x+\mu^4(x^2-y^2)]-[1-2\mu^2+2m^2-2\mu^2(x^2+y^2)]^2\big)^2=0,\end{aligned}$$

which, it is easy to see, is an equation of the order 8 only. Added Sept. 12.—A. C.]

XIV. *On the Diœcious Character of the Rotifera.* By Philip Henry Gosse.

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1. When Dr. Ehrenberg, in 1838, published his 'Infusionsthierchen,' containing his matured conclusions respecting the Rotifera, he defined the Class as consisting of animals all the individuals of which are hermaphrodite, oviparous, and ovipositing. The hermaphroditism, or "dualism of the sexual system," he described as consisting of "an ovary, simultaneously developing a few great eggs; two thread-like male sexual glands, thicker and club-shaped in front, and a contractile bladder, placed near the cloacal orifice, uniting the hermaphrodite organs for self-impregnation."

2. These conclusions remained unchallenged till 1848, when Mr. Brightwell announced his important discovery of separate sexes in a Rotiferous animal, since named *Asplanchna*. The presence of the tortuous threads in both sexes, and of the contractile bladder in the female, at once negatived the conclusions of Ehrenberg, that these were male impregnating organs, though their functions were still undetermined.

3. This single example, however, was by no means sufficient to overthrow the presumed hermaphroditism of the Class as a whole; though it naturally threw doubt on statements as yet unhesitatingly received. The case might have been one of those solitary exceptions, which occasionally mock our generalizations; like the exception to the law of metamorphosis in *Gecarcinus* and *Astacus*; or to that of immediate relation of impregnation to birth, in *Aphis* and *Daphnia*.

4. In the "Annals and Magazine of Natural History" for July 1850, I published some observations on a second species of *Asplanchna* (*A. priodonta*), in which I described and figured the male. Soon afterwards I had the pleasure of repeating and verifying those of Mr. Brightwell and Mr. Dalrymple on *A. Brightwellii*.

5. The diœcious character was thus extended from a species to a genus. Since then, I believe, no advance has been made in our knowledge of the sexual relations of the Rotifera, from actual observation, save the discovery of a third species of the same genus. Dr. Franz Leydig, in a recent able memoir* on this Class of animals, announces that *Notommata Sieboldii* has distinct sexes; and that the male, while exhibiting some curious peculiarities of external form, agrees with that figured by Mr. Dalrymple, in all essential points. He appears not to have been aware of my observations on *A. priodonta*.

* "Ueber den Bau, und die systematische Stellung, der *Räderthiere*" (Siebold and Kölliker's Zeitschrift, July 1854).

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It is manifest, however, that the animal which the German zoologist describes as a *Notommata* must be referred to my genus *Asplanchna*; and that its alliance with *A. Brightwellii* is still more intimate than is that of *A. priodonta*. Thus we have still no absolute proof that the diœcious character is not confined to a single genus.

6. Dr. Leydig, however, conjectures, and supports his suggestion with many plausible reasons, that several animals, described and figured by Ehrenberg as distinct species, may be really the males of other recognized forms. Thus *Enteroplea hydatina* he ventures to assign as the male of *Hydatina senta*, grounding his conclusions on the descriptions and figures of Ehrenberg and Dujardin. The close resemblance of *Enteroplea* to *Hydatina*, its smaller dimensions, and the absence of a manducatory apparatus, had been noticed by Ehrenberg, who found the eggs which produced the animal, among those from which were evolved *Hydatinæ*. What Ehrenberg described as an undeveloped ovary, Dr. Leydig considers to have been analogous to the sperm-bag of Mr. Dalrymple; in other words, a true testicle. This last interpretation is further confirmed by the figure given by Dujardin; in which, moreover, additional evidence is found, in the absence of a true alimentary canal, in the presence of certain "*touffes des granules pédicellées*" in the posterior part of the body (which Dr. Leydig interprets as "masses of spermatozoids"), and of "un organ cilié entre les muscles de la queue," which Dr. Leydig supposes to have been the excretory duct of the testicle, *i. e.* the penis.

7. To these conclusions I entirely assent (perhaps with the exception of his interpretation of Dujardin's "*touffes pédicellées*"); and they are still further confirmed by the observations which I have presently to offer.

8. For reasons of a similar character, Dr. Leydig presumes *Notommata granularis* of Ehrenberg to be the male of *N. Brachionus*, and *Diglena granularis* of Weisse to be the male of *D. catellina*.

9. The first occasion on which I saw a male of this Class was on the 9th of August, 1849, when, however, I was too inexperienced in the study of the Rotifera to be aware of the true character of the animal which I was observing. Among many specimens of *Brachionus pala*, which I had just obtained from a pond near London, was one with seven or eight eggs excluded and attached to the posterior extremity of the lorica. One of these eggs was discharged from the ovary in my sight, so that I was sure of their origin. When extruded, the ovum displayed a substance slightly granular, and almost colourless, except that a slight tinge of smoky black pervaded the whole. In an hour or two this pellucid interior became turbid, and soon after was marked with indistinct corrugations, which gradually acquired more definiteness. The waves of the frontal cilia were next seen at one extremity of the egg, sometimes moving transversely, sometimes longitudinally, but conveying, in either case, no accurate idea of the organs which produced them. The vacant space occupied by these waves became larger, and presently the eye became visible, as an undefined pale red spot, near the middle of the egg. About the same time two or three amorphous masses of opake substance were seen near the posterior extremity. See Plate XV. fig. 1.

10. After about five hours from extrusion the motions of the enclosed embryo become very vigorous: the folds of the body are continually changing their places, and the working of the cilia is strong and rapid. Presently the regularly elliptical form of the egg undergoes a slight alteration, becoming more elongated, then slightly constricted in the middle, apparently by the pushing outwards of the two extremities of the body, and by the inflation of the fore-parts. At this moment the anterior half of the egg-shell is pushed off; the cilia are instantly seen expanded and rotating for a second or two; the little new-born *Brachionus* (see fig. 2) sits in the hinder half of the shell, as in a nest, and then glides away.

11. The infant *Brachionus* swims rather feebly at first; but, acquiring vigour as it goes, presently darts along with amazing swiftness, so that the eye can scarcely follow it. When several of the young were hatched, they whirled and shot round and round the live-box in which they were enclosed, in the most headlong manner, and were every instant coming into contact with the parent, when, almost invariably, the course of the little creature would be arrested, and it would play around the body of the mother, or grasp with its pincer-like toes her cilia. As I have already said, I had at that time no notion of these being males, having seen no offspring but these produced from this species. I had, however, seen the birth of several young of *Brachionus urceolaris*, which were exactly like the parent, whereas these were very dissimilar; a circumstance which excited my curiosity, as did also the fact that both eggs and young were notably smaller than those of *B. urceolaris*, in proportion to the adult. The actual dimensions of the eggs in question were $\frac{1}{350}$th of an inch in the longer diameter; of the new-born male, $\frac{1}{150}$th; of the parent, $\frac{1}{100}$th, exclusive of the foot.

12. I shall not here describe the structure of the male, because I shall presently have occasion to recur to the species again, in recording observations made with the advantage of more experience.

13. In April 1850 I first saw eggs of two different dimensions, attached to the same species. This was *Brachionus rubens*, several specimens of which carried each seven or eight small eggs (fig. 3), measuring $\frac{1}{350}$th of an inch in length, while others carried few eggs (two or three at most), nearly twice as large, viz. $\frac{1}{190}$th of an inch in length. One of the latter, under slight pressure, freed prematurely a young one, enveloped in a spined lorica, agreeing in all respects with the parent. I then crushed several of the smaller eggs. From such as were nearly mature, numbers of spermatozoa were expressed: these had a round body, and a short tail which was vibrated in a serpentine manner. They measured about $\frac{1}{3500}$th of an inch in the diameter of the body, and the length of the tail was about the same (fig. 4). These were sufficiently numerous to make it appear as if the eggs had been entirely filled with them. The less mature eggs yielded spermatozoa also, but without motion, with the tail undeveloped, adhering together, and somewhat resembling blood-disks.

14. Others of the smaller eggs produced the young by natural maturation. These were quite diverse from the parent, having no perceptible lorica, nor any spines; no

mastax or jaws; no alimentary canal; nor any distinct organization within, except a large red eye near the middle of the body. The whole interior, with this exception, appeared to be filled with a granular mass, in the midst of which, near the posterior region, were two or three amorphous lumps of opake substance. The foot was short; the frontal cilia very large and strong, and the little animal shot to and fro with great swiftness (fig. 5).

15. My next observations were made on *Brachionus amphiceros*. Here again I met with large and small eggs in individuals of the same species; the former $\frac{1}{170}$th of an inch in length, the latter $\frac{1}{317}$th. As before, the larger eggs (fig. 7) produced young exactly like their parent; the smaller (fig. 6) yielded animals quite dissimilar, but agreeing in all essentials with the small young of *B. pala* and *B. rubens*.

16. One of these latter was hatched under my eye. The little animal (figs. 8, 9, 10) was several seconds in escaping, after the broad ciliated head was outside the egg-shell. Then it swam swiftly away, and remained traversing the live-box with indefatigable perseverance all the time I looked at it. It was totally different from the other young of the same species, and from the parent: it had no lorica, no jaws, no mastax, no stomach, nor any viscera, except a large mass of opake matter, apparently surrounded by traces of a vesicle near the hinder part. It had a red eye placed near the middle of the body, and an ample rotatory organ, of which the central cilia were sometimes made to converge, as in the parent (fig. 9). The body was wide, nearly cylindrical, gibbous at the posterior part, where it was abruptly attenuated to a stout cylindrical foot (fig. 8). But this organ exhibited a remarkable peculiarity; for, from behind the foot, proceeded a stout cylindrical organ, truncated and ciliated at its extremity, whence a blunt point was sometimes protruded: this organ, from its direction and size, might easily be mistaken for the foot itself, for the latter was much shorter and smaller, and projected almost like the thumb of a mitten, but it was readily known by its two little soft toes (fig. 10). The body seemed to be hollow, with thick walls, but a great number of bubbles hindered the transparency. Pressure did not reveal the slightest trace of jaws. The colour of the fore-parts was buff, like that of the parent; but the rest was colourless, except the opake mass and the red eye. I put this specimen and a newly-born female together, and watched them for an hour; but though, in their devious wanderings, they once or twice touched each other, no apparently voluntary communication took place. I left them together all night, and in the morning the female was dead. The male, however, was still active, and lived till the middle of the day.

17. About the same time *B. pala* occurred again to my notice, and this time with both male and female eggs. The observations now made may perhaps be most distinctly given in the words of my journal.

12^{h}45^m P.M. I isolated a specimen with five small eggs attached, in two of which the embryos were vibrating the cilia (fig. 1). There was in the ovary, nearly ready for exclusion, another egg, similar in size and appearance to the least advanced of the five attached.

1^h 5^m. The egg last named was extruded under my eye, but took its place with the

other five, not differing from them. The small egg, when deposited (and for some time before), is pellucid and colourless, except that many yellow oil-globules are contained in it, which are aggregated into two (or three) masses. The pellucid interior has a delicately granular appearance, which after a while begins more distinctly to form cells, and becomes more turbid.

3h 30m. A male was hatched from the most advanced of the small eggs. It is almost indistinguishable from that of *amphiceros*; but the toes are rather more developed (fig. 2).

The opake masses in the posterior region are aggregated together, and very conspicuous. This aggregate mass seems to be contained in a central irregular bladder, which is connected with the protrusile organ. I distinctly saw, many times in succession, the penis behind the foot everted and protruded beyond the toes, when the tip was seen to be ciliated: a sort of thick (two-lobed?) piston (probably the thickened retractor muscles) was thrust rapidly down when it was protruded. It is frequently retracted with the foot into the abdomen.

18. I put it, almost as soon as born, into a live-box, with a half-grown female, and managed to enclose them together, by means of cotton fibres, in a cell sufficiently small to be contained in the field. I watched them long: the male was very wild, trying to escape: several times, in darting round, he stopped near the hinder part of the female, and momentarily protruded the penis, and turned it towards her, and once, as I thought, actually introduced it into the cloaca, but only for an instant. The female appeared conscious of excitement while the male was near, whisking about the foot and feeling him with it.

19. The next day I collected about a dozen females, half-grown and adult (which I could easily take up with a tube from the numbers in the water), and placed with them two lively males that had been hatched during the night. I directed my attention principally to one of these, as I could not watch them both. It soon came near some of the females, when it seemed to become animated by a sort of frenzy; describing, with excessive rapidity, a circle, of which its head formed the circumference, and its foot the centre: the extremities were incurved in the direction of its circular movement. After a while it left off and began to play about the body of a female, moving over and round the lorica, while she whisked about the foot, as if to lay hold of him; at length she drew in her foot, and that of the male appeared to adhere to it, and I distinctly saw the thick penis presented to the cloaca, and for a moment inserted about half its own length; then it was instantly drawn out, and the male began his frenzied gyrations again. Two or three times I saw similar play between him and females, but could not see any actual intromission again. Sometimes he moved his body in quick waves, the foot being the fixed point.

20. Next I examined the little new species that I have named *B. angularis*, which was swarming in a bottle of water dipped from a pond near Tottenham. Many carried two eggs each, measuring $\frac{1}{100}$th of an inch in length, others several, which measured only $\frac{1}{150}$th. One of the latter produced a little male (figs. 13, 14), which appeared

covered with a tortoise-like lorica. Its length, exclusive of the foot, was $\frac{1}{300}$th of an inch. A constriction indicates the distinction between the head and the body. The foot has a thick penis united to it behind (fig. 14), the tip of which is protrusile and ciliated. No internal organs are visible, except a red eye near the middle of the back, and a mass of opake matter at the hinder part. The interior is occupied with a vast number of minute granules or globules irregularly clustered. Its movements were very rapid; it darted incessantly about the live-box, so that it was impossible to keep it for more than a moment in the field; several females were enclosed with it, some of which were without eggs, while others carried them. By means of a small microscope I watched them awhile, and had the satisfaction of seeing what I had no doubt was a sexual conjunction, though it was but momentary, with one of the females without eggs.

21. All the observations I have mentioned (except the first) were made in April and May 1850. Leaving what I have to say of other genera for the present, I shall go on with *Brachionus*, detailing all I have learned of the males of this genus together.

In August of the same year I examined many specimens of *B. Bakeri*. One of these had two eggs attached to its body, which, from their size and appearance, I judged to be male eggs (fig. 11): they measured $\frac{1}{350}$th × $\frac{1}{440}$th of an inch. After a while, one of these began to show the ciliary action; then the eye became visible; and at length the well-defined fœtus moved vigorously in its transparent prison (fig. 11 *c*). Meanwhile, the egg that had been developing was at length extruded, and took its place beneath its two fellows (fig. 11 *b*). The evolution of the fœtus from these eggs being delayed considerably beyond the usual period, I endeavoured to promote it by artificial pressure, and in one case succeeded in cracking the egg, so that the animal was enabled to escape (fig. 12). It was very inert and feeble, did not swim rapidly away as usual, but remained in the same place. It resembled other male *Brachioni*; there was manifestly a simple enveloping lorica, cup-like, without points. On killing it by pressure, the viscera were forced out in the form of globose vesicles of various sizes; the eye was resolved into three or four atoms of pigment, and the opake white spots that are so characteristic of males *seemed* composed of an immense number of moving atoms, excessively minute, which, when they were freed from the body, continued for a moment or two to move spontaneously. These were probably spermatozoa; but, from subsequent observations, I have reason to think I was mistaken in supposing them to have proceeded from the opake masses.

22. About the same time I procured from Walthamstow a very fine and elegant species, described by myself under the name of *B. Dorcas*. One specimen carried a *female* egg nearly matured, in the ovisac, and the empty shell of another (fig. 15) attached to its lorica. Another had three *male* eggs attached, and one maturing: the dimensions of male eggs were $\frac{1}{300}$th × $\frac{1}{310}$th of an inch: fig. 16 represents one of these, with the fœtus nearly ready for exclusion; the thick penis-foot is seen turned up to one side.

23. The female is represented at fig. 17, an hour after birth. I had the pleasure of seeing this hatched. The egg is of a bluish-grey hue, and shows the large mastax with

the jaws, long before hatching. I was surprised at the great size of the newly-born young, not only in proportion to the parent, but even to the egg from which it was just escaped. For, when I delineated it an hour afterwards (during which I did not perceive any manifest increase), its length, from the tip of the frontal antlers to the posterior tubercles, was $\frac{1}{28}$th of an inch. The dimensions of the empty shell (fig. 15), which still retained its form, were $\frac{1}{113}$th × $\frac{1}{330}$th of an inch: this continued attached to the parent for hours afterwards. This increase I suppose to be effected by the expansion of the lorica (at first flexible and membranous), which in the egg had lain in many folds, and by the imbibition of water into the cavity of the body.

24. Soon afterwards I had a young male of the same species hatched. I selected a specimen with two small eggs attached, and isolated it in the live-box, examining it at intervals. At length I perceived the young male whirling rapidly about the box, and on looking at the adult, saw that one of the eggs was but an empty shell. By the aid of cotton filaments and the compressorium, I was enabled to keep it steady, and to examine and draw it with care (see figs. 18, 19). It is of the usual shape, with a well-marked neck, up to which a flexible lorica appears to reach. The head is broad, truncate, or subconical, with vibrating bristles set on the cone, and rotating bristly cilia around the margin. A head-mass, of irregular, rounded, or sacculated lobes, bears a lozenge-shaped red eye on its dorso-posterior angle (fig. 19). From each side of the head a chain of irregular masses runs down to the posterior part of the body, probably answering to the tortuous glands of the female. A large mass, similar in appearance to the head, but distinct from it, occupies the middle of the body, extending down into the penis-foot. This great mass is continually moved by quick contractions up and down, but it does not appear to be tubular and hollow. The common white, opake, granular masses are enveloped in it, near the hind part of the body.

25. From this time I made no further investigation into the subject until the autumn of the present year (1855), when I had an opportunity of examining the beautiful and interesting marine species *B. Mülleri*, and of seeing, with satisfaction, that in the point of its œconomy, which is the subject of this memoir, it differs not from its congeners of fresh water. The males, however, displayed an internal organization more developed than in any species that I had yet seen, with the exception of the *Asplanchnæ*.

26. In July last, my kind friend Mr. BRIGHTWELL of Norwich sent me a small phial of sea-water much filled with this species. I immediately divided the contents among my Aquaria, where they increased rapidly. Many of the females have two or three male eggs attached, which are $\frac{1}{110}$th of an inch in length, and others carry one or two female eggs $\frac{1}{113}$th of an inch long. The male young (fig. 20) is $\frac{1}{330}$th of an inch long, exclusive of the foot, and about $\frac{1}{500}$th of an inch wide, with a distinct, but very flexible lorica, of which the anterior edges seem to become membranous. There is a great conical head (fig. 20 *a*), set with large setiform cilia all over the front, and behind descending into a rounded mass of several lobes, with a red eye (*b*) on the truncate apex of the posterior lobe. Several muscle-threads (*c*) pass longitudinally from this head-

mass to the hinder mass (*d*), which is connected with the foot (*g*) and the great protrusile penis. In the midst of this latter mass is situated the group of opake white granules (*f*), which are always so conspicuous in male ROTIFERA. The central part of the body is occupied by the spermatic sac (*e*),—a great pyriform vesicle, connected by a bottle-like neck with the head-mass, and apparently terminating posteriorly in a round form, where, however, a distinct corrugation probably marks an orifice closed by a sphincter, which leads to the discharging duct. On pressure, this sac is seen to be filled with bodies having a vermicular motion, and, on the pressure being continued, it bursts, giving issue to a number (about thirty) of spermatozoa (fig. 22). These spermatozoa are comparatively of great size, being $\frac{1}{300}$th of an inch long; they consist of a slender, nearly cylindrical body, merging into a long whip-like tail, which keeps up a quivering undulatory motion for several minutes after exclusion. With these are also extruded many spicula-like bodies (fig. 23), much more minute, somewhat fusiform, slightly curved, and motionless.

The penis (fig. 21) is, as usual, thick, and united to the foot, terminating in a short truncate tube, which I could not see to be ciliated. A few crimson pigment-cells, exactly agreeing with those of the eye, are usually seen in some part or other of the body, apparently in contact with the inner surface of the integument.

27. In the middle of the summer of 1850 I discovered the males of two or three other genera. The first of these was the curious form which I have named *Sacculus viridis** (fig. 24). I have had no opportunity of seeing the birth of females of this species, all the specimens that came under my observation (that bore eggs at all) carrying either one or two eggs of like size and appearance. From one of these eggs (fig. 25), which, before maturity, are much clouded and spotted with granules and globules, a young was produced in my live-box, which was, I doubt not, a male (fig. 26). I could not detect any eye (though this organ is conspicuous in the parent), nor any internal organization; nothing but a confused assemblage of granules and globules; even the ordinary opake masses were not present. The form somewhat resembled that of an amphora with a short wide neck; the frontal cilia were very large, but the motion was not rapid, nor was the animal wild, as male ROTIFERA usually are.

28. *Polyarthra platyptera* is another form, which, from its habit of carrying its deposited eggs, affords facilities for investigations regarding its sexuality. Finding it numerous in July, with eggs attached, I discovered that these presented the same diversity that I have mentioned above (§ 13, &c.). I isolated one with six small eggs attached, measuring $\frac{1}{700}$th × $\frac{1}{815}$th of an inch. Presently afterwards I found another, with a single egg attached, measuring $\frac{1}{313}$nd × $\frac{1}{437}$th of an inch. This egg was truly oval, being sharper at one end than at the other. Several of the smaller eggs were hatched together, about twenty-four hours after isolation. They

* From the description and figure of Dr. LEYDIG, I have no doubt this is identical with the genus *Asco-morpha* of PERTY. I had, however, characterized and named it in the 'Annals and Magazine of Natural History' for September 1851, a year before M. PERTY's work was published.

each produced a little animalcule (figs. 27, 28, 29), which, in its form, brightness, and motions, greatly resembled a free *Vorticella*. Its length was $\frac{1}{150}$th of an inch. The head (fig. 27) is very large, and the body tapers quickly to the posterior part, but both extremities are truncate. The front bears two warts, between which the rotatory cilia are placed, but the cilia are longer (perhaps *setæ*) on the warts. The hinder part is bifid (figs. 28, 29), the smaller division being the caudal extremity or toeless foot, and the larger a protrusile truncate penis, ciliated at the tip. No internal organization is discoverable. In one there was a globule in the midst of the great head, but under sunlight it reflected no colour, and it was wanting in another. Towards the posterior dorsal parts, a few irregular dark specks were visible, but generally the whole animal was clear, colourless, highly refracting, and showing an indistinct granulation. Its motions were swift and impatient, gliding about the field with headlong speed, occasionally remaining in one place for a few minutes, but not in stillness, for it was rapidly oscillating to and fro, and quivering.

29. The last observation that I have to offer is on *Synchæta tremula*. This genus does not carry about its eggs after their escape from the ovary, and as I did not see the hatching of the male I am about to describe, I cannot identify it with positive certainty.

30. In a phial of water obtained near Leamington, this *Synchæta* was very numerous, but I am not aware that any other Rotiferous animal was present. Among the hundreds which congregated, in their playful gyrations, just above the sediment, I found a specimen (figs. 30, 31) which was certainly a male, and, from its form, was probably of this genus. It was about $\frac{1}{150}$th of an inch long, obconical, with a rounded front, set with many long cilia, and surrounded by a prominent ring, also set with long cilia. The body was capable of taking a swollen or round form, when the separate foot was more distinct (fig. 31). The organization differed not from that of males of other species. A red eye sharply defined, a central granular viscus, contained in a longitudinal cavity, in the bottom of which lay a large, irregular, opake white mass, was all that could be traced internally. The foot seemed to carry a great protrusile penis, and to terminate in two minute toes. No spermatozoa were visible within, nor were expressed by the compressorium. The animal was active and swift. A constriction behind the wreath of cilia formed a well-marked neck.

31. From the following observations which I published in the 'Journal of Microscopical Science' for January 1853, I think I am justified in concluding that *Melicerta ringens* also produces separate male and female young :—

"Opening one or two cases [of *Melicerta*], I freed one and another very curious egg-like bodies, not symmetrical in shape, being much more gibbous on one side than on the opposite, and measuring $\frac{1}{150}$th × $\frac{1}{100}$th of an inch. Each was encircled by five or six raised ribs, running parallel to each other longitudinally, somewhat like the varices of a wentle-trap. Viewed perpendicularly to the ribs, the form is symmetrical—a long narrow oval. The whole surface between the ribs appeared punctured or granulate, and the colour was a dull brownish-yellow. Under pressure it was ruptured, and discharged an

infinity of atoms of an excessive minuteness, but every one of which, for a few seconds, displayed spontaneous motion. Their whole appearance, and the manner in which they presently changed to motionless disks, were exactly the same as of the spermatozoa which the male eggs of other ROTIFERA contain, except that these were so minute.

"From another I extracted an egg of the ordinary form and appearance. It was very long, measuring $\frac{1}{145}$th × $\frac{1}{350}$th of an inch. The contained embryo was well-advanced; two red eyes were plainly seen by reflected and by transmitted light: the gizzard was transverse, very large in proportion, and the jaws worked vigorously: a little opake body, white under sunlight, was in the posterior part. This embryo died without hatching."

32. Thus there is evidence of the diœcious character in the following genera and species of ROTIFERA:—

Asplanchna Brightwellii.	*Brachionus amphiceros.*
Asplanchna priodonta.	*Brachionus angularis.*
Asplanchna Sieboldii.	*Brachionus Bakeri.*
? *Synchæta tremula.*	*Brachionus Dorcas.*
Polyarthra platyptera.	*Brachionus Mülleri.*
Brachionus Pala.	*Sacculus viridis.*
Brachionus rubens.	*Melicerta ringens.*

And it is presumed, with a high degree of probability, to mark also the following:—

Hydatina senta. *Diglena catellina**.

33. It must be borne in mind, too, that there is no direct evidence on the other side; for no species has, I believe, been actually proved to be hermaphrodite. Hence, I think, we may reasonably conclude a distinction of sex to be the normal condition of the Class, or at least of that great group which is most typical, viz. such as have articulated *mallei* working upon a separate articulated *incus*. Whether the same rule prevails so generally in those which have the *mallei* and *incus* fused together into quadrantic masses, and in those in which these organs exist in a rudimentary condition, is a question yet to be determined. As these are certainly the lowest forms of their Class, it is possible that hermaphroditism *may* be found in them—in the *Philodinadæ* for instance.

34. In summing up the results of the herein-recorded, as well as of former observations, the most prominent thing that strikes us is the absolute and universal atrophy (so far) of the digestive system in male ROTIFERA. It was first observed with admiration in *Asplanchna Brightwellii*, and has been found to belong, without exception, to every male since observed. Mr. DALRYMPLE, indeed, remarked a little isolated congeries of vesicles in the cavity of the male of the species just named, and I have seen it also. Possibly this may be the lingering rudiment of a stomach, but it is scarcely likely; and

* Since this memoir was read, I have been enabled to add to the above list *Notommata Brachionus*. The male of this species much resembles those of the *Brachioni*, equalling in its organization the male of *B. Mülleri*. Its form, however, is as conical as that of *Synchæta*.

whatever it be, no semblance of such viscera is to be traced in *A. priodonta*, or in any other male that has come under my observation.

35. Another curious peculiarity is the dissimilarity that always subsists between the sexes. In *Asplanchna* and *Hydatina* the resemblance is at its highest point: in every other instance observed, the sexes are so unlike, that they would be taken for widely remote genera. The male is always inferior in size, and also in organization, to the female.

36. Whether certain individuals produce only male, and others only female young, or whether separate impregnations are required for the production of the separate sexes, I do not know; but from all my observations, I gather that the development of the one sex never takes place coetaneously with that of the other; for male and female eggs are never seen attached to the same parent, and the immature eggs in the ovary invariably develope themselves into the same sex as those which are already extruded.

37. The duration of life in the male is always very brief. I have never been able to preserve one alive for twenty-four hours. Their one business is to impregnate the females, which is the work of a few minutes, probably, in a state of freedom; and for this momentary occupation no supply of loss by assimilation of food is wanted; and hence we can understand the lack of the nutritive organism.

38. *Some* organs are found, with greater or less distinctness, in all. The (presumed) male of *Hydatina senta* received its names of *Enteroplea* and *Organ-fisch* from Ehrenberg, on account of the copiousness of its internal organization. A muscular system is well-developed there, and in the males of *Asplanchna* and of *Brachionus Mülleri*; and, from the varied movements of all, its existence may be inferred where it is not detected. The frontal cilia are, in almost all cases, much more developed than in the females; the result of which endowment is seen in the excessive rapidity with which the male shoots in all directions through the water. The great head-mass of granular substance is generally distinct; and in several cases (as in the *Asplanchnæ* and in *Br. Dorcas* and *Mülleri*) the great occipital ganglion is well-defined, with the red eye seated on it as in the other sex. Even where the ganglion is not apparent, the eye is conspicuous, with the exception of *Sacculus* and *Polyarthra*; and in this last instance the small size of the animal must be borne in mind, and the density of the anterior parts.

39. In the (presumed) male of *Hydatina*, in those of all the *Asplanchnæ*, and of *Brach. Dorcas*, there are organs answering to the lateral convoluted threads of the female; and, in *Aspl. Brightwellii*, at least, these are accompanied by tremulous tags, and by a contractile bladder.

40. A large mass of substance, which, being perfectly opake, appears black by transmitted light, but is white when the rays are reflected, is so generally found in male Rotifera, as to be characteristic, though it is not universally present. I do not find it in the *Asplanchnæ*, nor in *Sacculus*. On the other hand, I have observed it in the young of *Stephanoceros*, *Floscularia complanata*, and *F. cornuta*; and Ehrenberg mentions it in *F. ornata* and *Lacinularia*. In *Stephanoceros*, it was certainly associated with well-developed jaws; and hence I presume it is not exclusively an indication of the male sex.

The mass is sometimes broken up into fragments of irregular size and shape, and sometimes apparently pulverulent. In general, it appears to lie loosely in the midst of the granular amorphous matter that occupies the posterior region of the body-cavity; but in *Brach. Pala*, and especially in *B. amphiceros*, I have fancied that I discerned traces of a vesicle, within which the white substance seems to be contained.

41. On the nature of this substance I have no light from personal research. Dr. LEYDIG, however, considers it to be a urinary concretion (*Harnconcremente*), analogous to the chalky fluid which is discharged by many insects, immediately after their evolution from pupa.

42. In the male of *Asplanchna Brightwellii*, there is, as its discoverer observes, "a conspicuous round sperm-vessel, or testis, in which ..... spermatozoa in active vibratile motion may be seen*." Mr. DALRYMPLE, and subsequently myself, also saw these, both within the sac and discharged by pressure. Each spermatozoon, according to my own observation, consists of an oblong body, $\frac{1}{1750}$th of an inch long, and an abrupt, slender, vibratile tail, of equal length. In the sperm-sac of *A. Sieboldii*, Dr. LEYDIG finds various seminal elements, viz. round cells; pyriform cells, drawn out to a fine point, and adhering to each other by their rounded ends, in a stellate manner; oblong bodies, with one side dilated into a free, undulating, membranous border; and slender, stiff, rod-like bodies, with a central swelling; all containing nucleated nuclei†. On the male of *A. priodonta*, my observations were too limited to determine more than the existence of the globular sperm-sac.

43. In *Brachionus rubens* and *B. Mülleri* I found spermatozoa, which I have above described (§§ 13 and 26). In the latter, the sperm-bag is of great size, and contains, besides the spermatozoa of unusual development, slender spiculiform bodies (fig. 23), which may be the equivalents of the little rods described by Dr. LEYDIG in *Aspl. Sieboldii*. The sperm-bag (in *Br. Mülleri*) is closed posteriorly, as it is also in *Aspl. Brightwellii*, by what appears to be a true sphincter (fig. 20); and such I conjecture to be the explanation of those diverging lines which M. DUJARDIN saw in *Enteroplea* (so-called), which he considered to be pedicels of his "*touffes de granules*," while the "*touffes*" themselves I take to have been the masses of urinary concrement. Dr. LEYDIG, however, considers the whole to have been masses of spermatozoids.

44. The outlet of the sperm-bag is, in all cases, by a thick protrusile and retractile penis. Wherever a foot exists, this intromittent organ is continuously united to its dorsal side, and is often so greatly developed that the foot itself appears as an appendage. The protrusion of the organ, at least in most of the examples that I have noticed, is by the eversion of the integuments. When these are evolved to the utmost, the organ is seen to be a thick column, conical or nearly cylindrical, with the extremity truncate, and surrounded by a wreath of vibratile cilia. It was, doubtless, the extremity of the penis that M. DUJARDIN saw, as "*un organ cilié entre les muscles de la queue*," in the (so-called) *Enteroplea*. The male of *Sacculus viridis* (fig. 26), a species which is footless in

* BRIGHTWELL in 'Ann. Nat. Hist.,' Sept. 1848.

† *Op. cit.* p. 32.

both the sexes, is the only example in which I have not seen the penis; but the organ is probably wholly retractile within the body, and my observations, on the only individual of this sex that I saw, were insufficient to determine anything concerning it.

45. For a parallel to the curious facts thus established we must look only to the CRUSTACEA. The economy of the *Hectocotylus* of certain Cephalopod MOLLUSCA, though perhaps even still more abnormal, is only remotely analogous. Nor is the parallelism very close of those ENTOZOA in which the males are organically united to the females, as the genera *Heteroura* and *Syngamus*, described by Professor OWEN*.

46. In the Class CRUSTACEA, however, many examples occur of a sexual difference, which may instructively be compared with the one before us. Thus, among the *Isopoda*, we find the parasitic genera *Bopyrus*, *Phryxus*, and *Ione*, in which the males are notably smaller than the females, very diverse in form, and in some respects inferior in structure. In the *Siphonostoma* "the males are extremely small, and do not in the least resemble the females†," though those of different genera bear a strong resemblance *inter se*, even when the females are very dissimilar. So low is their grade of organization, that BURMEISTER has attempted to prove these minute creatures to be embryonic or larval forms. And, finally, in the CIRRIPEDIA, Mr. DARWIN has proved the existence of males in the genera *Ibla* and *Scalpellum*, which are very minute as compared with their females, excessively abnormal in form, and in some respects in an embryonic condition, though unquestionably mature, as shown by the spermatozoa‡. And, what is still more interesting, the same accurate zoologist observes,—"After the most careful dissection of very many specimens ..... I can venture positively to assert that there is no vestige of a mouth, or masticatory organs, or stomach§." Again, he describes the internal structure as "a pulpy mass with numerous oil-globules," and the sperm-vesicle as "a pear-shaped bag at the very bottom of the sack-formed animal, containing either pulpy matter, or a great mass of spermatozoa‖,"—terms which might have been employed in describing some of the male *Brachioni* (§ 26, &c.).

47. In all these analogies I conceive we may find additional reasons, to those that have been before adduced, for assigning to the ROTIFERA a zoological position among the ARTICULATA.

EXPLANATION OF THE PLATE.

PLATE XV.

Fig. 1. *Brachionus Pala*; male egg, nearly mature.
Fig. 2. *Brachionus Pala*; male, newly born.
Fig. 3. *B. rubens*; male egg, nearly mature.
Fig. 4. *B. rubens*; spermatozoa from the same egg.
Fig. 5. *B. rubens*; male, newly born.

* Cycl. of Anat. and Physiol. p. 142.
† BAIRD's 'Entomostraca,' 325.
‡ Monograph of Lepadidæ, 198.
§ Ibid. 235.
‖ Ibid. 236.

Fig. 6. *B. amphiceros*; male egg, just laid.
Fig. 7. *B. amphiceros*; female egg, nearly mature.
Figs. 8, 9, 10. *B. amphiceros*; males.
Fig. 11. *B. Bakeri*; hinder part of an adult female. *a*, male egg in the ovisac; *b*, egg just laid; *c*, egg nearly mature; *d*, empty egg-shell.
Fig. 12. *B. Bakeri*; male.
Figs. 13, 14. *B. angularis*; males.
Fig. 15. *B. Dorcas*; empty shell of female egg.
Fig. 16. *B. Dorcas*; male egg, mature.
Fig. 17. *B. Dorcas*; female, newly born.
Figs. 18, 19. *B. Dorcas*; males, newly born.
Fig. 20. *B. Mülleri*; male. *a*, head-mass; *b*, eye; *c*, muscles; *d*, posterior mass; *e*, sperm-sac; *f*, urinary concretion; *g*, foot.
Fig. 21. *B. Mülleri*; penis and foot.
Fig. 22. *B. Mülleri*; spermatozoa.
Fig. 23. *B. Mülleri*; spiculiform bodies.
Fig. 24. *Sacculus viridis*; adult female, carrying male eggs.
Fig. 25. *Sacculus viridis*; male egg, just laid.
Fig. 26. *Sacculus viridis*; male, newly born.
Figs. 27, 28, 29. *Polyarthra platyptera*; males.
Figs. 30, 31. *Synchæta tremula* (presumed); male.

XV. *On the Structure and Motion of Glaciers.*

By JOHN TYNDALL, *F.R.S., Professor of Natural Philosophy, Royal Institution; and* THOMAS H. HUXLEY, *F.R.S., Fullerian Professor of Physiology, Royal Institution.*

Received and Read, January 15, 1857.

§ 1.

IN a lecture given at the Royal Institution on the 6th of June, by Mr. TYNDALL, 1856, certain views regarding the origin of slaty cleavage were brought forward, and afterwards reported in the 'Proceedings' of the Institution. A short time subsequently, the attention of the lecturer was drawn by Mr. HUXLEY to the observations of Professor J. D. FORBES on the veined or laminar structure of glacier ice, and the surmise was expressed, that the same explanation might apply to it as to slaty cleavage. On consulting the observations referred to, the probability of the surmise seemed apparent, and the result was a mutual arrangement to visit some of the Swiss glaciers, for the purpose of observing the structure of the ice. This arrangement was carried out, the field of observation comprising the glaciers of Grindelwald, the Aar, and the Rhone. After returning to England, the one in whose department it more immediately lay, followed up the inquiry, which gradually expanded, until at length it touched the main divisions of the problem of glacier structure and motion. An account of the experiments and observations, and our joint reflections on them, are embodied in the memoir now submitted to the Royal Society.

§ 2. *On the Viscous Theory of Glaciers.*

A glacier is a mass of ice which, connected at its upper extremity with the snow which fills vast mountain basins, thrusts its lower extremity into the warm air which lies below the snow-line. The glacier moves. It yields in conformity with the sinuosities of its walls, and otherwise accommodates itself to the inequalities of the valley which it fills. It is not therefore surprising that the glacier should have been regarded as an ice-river by those who dwelt in its vicinity, or that this notion should have found a place in the speculations of writers upon the subject. The statements of M. RENDU in connexion with this point are particularly distinct:—"There are," he writes, "a multitude of facts which seem to necessitate the belief that the substance of glaciers enjoys a kind of ductility which permits it to model itself on the locality which it occupies, to become thin and narrow, and to elongate itself like a soft paste*." But this observer put forward his speculations with great caution, and often in the form of questions which he confessed his inability to answer. "M. RENDU," says Pro-

* Théorie des Glaciers de la Savoie, p. 84.

fessor FORBES, "has the candour not to treat his ingenious speculations as leading to any certain result, not being founded on experiments worthy of confidence..... My theory of glacial motion, then, is this:—A GLACIER IS AN IMPERFECT FLUID OR VISCOUS BODY, WHICH IS URGED DOWN SLOPES OF A CERTAIN INCLINATION BY THE MUTUAL PRESSURE OF ITS PARTS."

"The sort of consistency to which we refer," proceeds Professor FORBES, "may be illustrated by that of moderately thick mortar, or the contents of a tar-barrel poured into a sloping channel." Treacle and honey are also referred to as illustrative of the consistency of a glacier. The author of the theory endeavours, with much ability, to show that the notion of semifluidity, as applied to ice, is not an absurdity, but on the contrary, that the motion of a glacier exactly resembles that of a viscous body. Like the latter, he urges, it accommodates itself to the twistings of valleys, and moves through narrow gorges. Like a viscous mass, it moves quickest at its centre, the body there being most free from the retarding influence of the lateral walls. He refers to the "Dirt-Bands" upon the surface of the glacier, and shows that they resemble what would be formed on the surface of a sluggish river. In short, the analogies are put forth so clearly, so ably, and so persistently, that it is not surprising that this theory stands at present without a competitor. The phenomena, indeed, are really such as to render it difficult to abstain from forming some such opinion as to their cause. The resemblance of many glaciers to "a pail of thickish mortar poured out;" the gradual changing of a straight line transverse to the glacier into a curve, in consequence of the swifter motion of the centre; the bent grooves upon the surface; the disposition of the dirt; the contortions of the ice, a specimen of which, as sketched near the Heisseplatte upon the Lower Grindelwald glacier, is given in fig. 1, and of which other striking examples have been adduced by M. ESCHER, in proof of the plasticity of the substance,—are all calculated to establish the conviction, that the mass must be either viscous, *or endowed with some other property mechanically equivalent to viscosity*. The question then occurs, is the viscosity real or apparent? Does any property equivalent to viscosity exist, in virtue of which ice can move and mould itself in the manner indicated, and which is still in harmony with our experience of the non-viscous character of the substance? If such a property can be shown to exist, the choice will rest between a quality which ice is *proved* to possess, and one which, in opposition to general experience, it is assumed to possess, in accounting for a series of phenomena which either the real or the hypothetical property might be sufficient to produce. In the next section, the existence of a true cause will be pointed out, which reconciles the properties of ice, exhibited even by hand specimens, with the apparent evidences

Fig. 1.

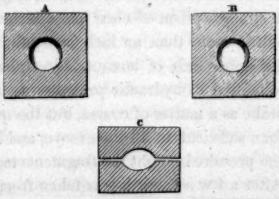
of viscosity already referred to, and which, though it has been overlooked hitherto, must play a part of the highest importance in the phenomena of the glacier world.

§ 3. *On the Regelation of Ice, and its application to Glacial Phenomena.*

In a lecture given by Mr. FARADAY at the Royal Institution on the 7th of June, 1850, and briefly reported in the 'Athenæum' and 'Literary Gazette' for the same month, it was shown that when two pieces of ice, at 32° FAHR., with moistened surfaces, were placed in contact, they became cemented together by the freezing of the film of water between them. When the ice was below 32°, and therefore dry, no adhesion took place between the pieces. Mr. FARADAY referred, in illustration of this point, to the well-known experiment of making a snowball. In frosty weather the dry particles of ice will scarcely cohere, but when the snow is in a thawing condition, it may be squeezed into a hard compact mass. On one of the warmest days of last July, when the thermometer stood at upwards of 80° FAHR. in the shade and above 100° in the sun, a pile of ice-blocks was observed by one of us in a shop window, and he thought it interesting to examine whether the pieces were united at their places of contact. Laying hold of the topmost block, the whole heap, consisting of several large lumps, was lifted bodily out of its vessel. Even at this high temperature the pieces were frozen together at the places of contact, though the ice all round these places had been melted away, leaving the lumps in some cases united by slender cylinders of the substance. A similar experiment may be made in water as hot as the hands can bear; two pieces of ice will freeze together, and sometimes continue so frozen in the hot water, until, as in the case above mentioned, the melting of the ice around the points of contact leaves the pieces united by slender columns of the substance.

Acquainted with these facts, the thought arose of examining how far, in virtue of the property referred to, the *form* of ice could be changed without final prejudice to its continuity. It was supposed that though crushed by great pressure, new attachments would be formed by the cementing, through regelation, of the severed surfaces; and that a resemblance to an effect due to viscosity might be produced. To test this conjecture the following experiments were made:—Two pieces of seasoned boxwood, A and B, fig. 2, 4 inches square and 2 deep, had two cavities hollowed out, so that when one was placed upon the other, a lenticular space, shown in section at C, was enclosed between them. A *sphere* of compact, transparent ice, of a volume rather more than sufficient to fill the cavity, was placed between the pieces of wood, and subjected to the pressure of a small hydraulic press. The ice broke, as was expected, but it soon re-attached itself; the pressure was continued, and in a few seconds *the sphere was*

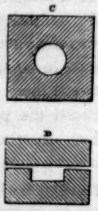
Fig. 2.



reduced to a transparent lens of the shape and size of the mould in which it had been formed.

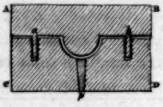
This lens was placed in a cylindrical cavity, two inches wide and half an inch deep, hollowed out in a piece of boxwood, C, fig. 3, as before; a *flat* plate, D, of the wood being placed over the lens, it was submitted to pressure. The lens broke as the sphere did, but the fragments attached themselves in accordance with their new conditions, *and in less than half a minute the mass was taken from the mould a transparent cake of ice.*

Fig. 3.



The substance was subjected to a still severer test. A hemispherical cavity was hollowed out in a block of boxwood, and a protuberant hemisphere was turned upon a second slab of the wood, so that, when the protuberance and the cavity were concentric, a distance of a quarter of an inch separated the convex surface of the former from the concave surface of the latter. Fig. 4 shows the arrangement in section. The pins of brass, *ab*, fixed in the slab AB, and entering suitable apertures in the mould CD, served to keep the two surfaces concentric. A lump of clear ice was placed in the cavity, the protuberance was brought down upon it, and the mould submitted to hydraulic pressure. After a short interval, it was taken from the press, and when the upper slab was removed, a smooth concave surface of ice was exposed. By tapping the conical plug *p*, this ice was lifted from the cavity, *the lump having been converted by pressure into a hard transparent cup of ice.*

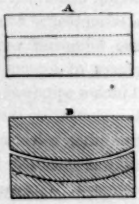
Fig. 4.



The application of the results here obtained to the "viscous flow" of glaciers, will perhaps be facilitated by the following additional experiments.

A block of boxwood (A, fig. 5), 4 inches long, 3 wide and 3 deep, had its upper surface slightly curved, and a longitudinal groove (shown in dots in the figure), an inch wide and an inch deep, worked into it. A slab of the wood was prepared, the under surface of which was that of a convex cylinder, curved to the same degree as the concave surface of the former piece. The arrangement is shown in section at B. A straight prism of clear ice, 4 inches in length, an inch wide, and a little more than an inch in depth, was placed in the groove, and the upper slab of boxwood was placed upon it. The mould was submitted to hydraulic pressure, as in the former cases; the prism broke as a matter of course, but the quantity of ice being rather more than sufficient to fill the groove, and hence projecting above its edge, the pressure brought the fragments together and re-established the continuity of the ice. After a few seconds it was taken from the mould, bent as if it had been a plastic mass. Three other moulds similar to the last, but of augmenting curvature, were afterwards made use of, the same prism being passed through all of them in succession. *At the*

Fig. 5.



conclusion of the experiments the prism came out, bent to a transparent semi-ring of solid ice.

In this way, by the proper application of force, all the bendings and contortions observed in glacier ice, and adduced in proof of its viscosity, can be accurately imitated. Any observer, seeing a straight bar of ice converted into a continuous semi-ring without being aware of the quality referred to, and having his attention fixed on the changes of external form alone, would be naturally led to the conclusion that the substance is viscous. But it is plainly not viscosity, properly so called, which enables it to change its shape in this way, but a property which has hitherto been entirely overlooked by writers upon glaciers.

It has been established by observation, that a vertical layer of ice originally plane, and perpendicular to the axis of a glacier, becomes bent, because the motion of its ends is retarded in comparison with that of its centre. This is the fact upon which the viscous theory principally rests.

In the experiments with the straight prism of ice, four successive moulds, gradually augmenting in curvature, were made use of. In passing suddenly from the shape of one to that of the other, the ice was fractured, but the pressure brought the separated surfaces again into contact and caused them to freeze together, thus restoring the continuity of the mass. The fracture was in every case both audible and tangible; it could be heard and it could be felt. A series of cracks occurred in succession as the different parts of the ice-prism gave way, and towards the conclusion of the experiment, the crackling in some instances melted into an almost musical tone. But if instead of causing the change to take place by such wide steps as those indicated; if instead of four moulds, forty, or four hundred were made use of; or better still, suppose a single mould to have the power of gradually changing its curvature from a straight line to a semicircle under the hydraulic press; the change in the curvature of the ice would closely approximate to that of a truly plastic or viscous body. This represents the state of things in a glacier. A transverse plate of ice, situated between the mass in front of it and the mass behind, is virtually squeezed in a press of the description which has been just imagined. The curvature of the ice-mould *does* change in the manner indicated, and so slowly, that the bending closely resembles what would take place if the substance were viscous. The gradual nature of the change of curvature may be inferred from an experiment made by Professor FORBES on an apparently compact portion of the Mer de Glace. He divided a distance of 90 feet transverse to the axis of the glacier into spaces of two feet each, and observed with a theodolite the gradual passage of this straight line into a bent one. The row of pins bent gradually so as to form a curve convex towards the lower extremity of the glacier; their deviations from a perfect curve were slight and irregular, nor was any great dislocation to be observed throughout their whole extent. After six days the summit of the curve formed by the forty-five pins was one inch in advance of the straight chord which united its two ends. It is not surprising if, with this extremely gradual change, the motion should have appeared to be the result of

viscosity. It may, however, be remarked, that the slight and irregular variations to which Professor FORBES alludes, and which are such as would occur if the motion were such as we suppose it to be, are likely to throw much light upon the problem. It is also extremely probable that the motion, if effected in the manner referred to, will be sometimes accompanied by an audible crackling of the mass. To this we paid but little attention when on the ground; for the significance of this as well as of many other points was first suggested by the experiments made after our return. It is, however, we believe, a phenomenon of common occurrence. Professor FORBES calls the glacier a "crackling mass;" he speaks of the ice "cracking and straining forwards;" and in that concluding passage of his 'Travels' which has excited such general admiration, he says of the glacier, "it yields groaning to its fate." Other observers make use of similar expressions. M. DESOR also speaks of the sudden change of the colour of the blue veins of the ice where a portion of the central moraine near the Abschwung is cleared away; the observation is very remarkable. "Au moment," says M. DESOR, "où on la met à découvert, la glace des bandes bleues est parfaitement transparente, l'œil y plonge jusqu'à une profondeur de plusieurs pieds, mais cette pureté ne dure qu'un instant, et l'on voit bientôt se former des petites fêlures d'abord superficielles, qui se combinent en réseau de manière à enlever peu à peu à la glace bleue toute sa transparence. Ces fêlures propagent également dans les bandes blanches, et lorsqu'on approche l'oreille de la surface de la glace, en entend distinctement un *leger bruit de crépitation* qui les accompagnent au moment de leur formation." These facts appear to be totally at variance with the idea of viscosity.

In a chapter on the "Appearance of the larger Glaciers," in an interesting little work by M. MOUSSON of Zürich, for which one of us has to thank the kindness of Professor CLAUSIUS, the phenomena which they exhibit are thus described*:—"The appearance of a large glacier of the first order has been compared, not without reason, with that of a high swelled, and suddenly solidified stream. It winds itself in a similar manner through the curving of the valley, is deflected by obstacles, contracts its width, or spreads itself out..... In short, the form is modified in the most complete manner to suit the character and irregularities of its bed. To this capacity to change its form, the ice of glaciers unites another property, which reminds us of the fluid condition; namely, the capability of joining and blending with other ice. Thus we see separate glacier branches perfectly uniting themselves to a single trunk; regenerated glaciers formed from crushed fragments; fissures and chasms closed up, and other similar appearances. These phenomena evidently point to a slow movement of the particles of which the glacier consists; strange as the application of such an idea to a solid brittle mass such as glacier ice may appear to be. The solution of this enigma constitutes one of the most difficult points in the explanation of glaciers."

When the appearances here enumerated are considered with reference to the experiments on the regelation of ice above described, the enigma referred to by the writer appears to have received a satisfactory solution. The glacial valley is a mould through

* Die Gletscher des Jetztzeit, by ALBERT MOUSSON. Zürich, 1854.

which the ice is pressed by its own gravity, and to which it will accommodate itself, while preserving its general continuity, as the hand specimens do to the moulds made use of in the experiments. Two glacial branches unite to form a single trunk, by the regelation of their pressed surfaces of junction. Crevasses are cemented for the same reason; and the broken ice of a cascade is reconstituted, as a heap of fragments under pressure become consolidated to a single mass. To those who occupy themselves with the external conditions merely of a glacier, it may appear of little consequence whether the flexures exhibited by the ice be the result of viscosity or of the principle demonstrated by the experiments above described. But the natural philosopher, whose vocation it is to inquire into the inner mechanism concerned in the production of the phenomena, will discern in the yielding of a glacier a case of simulated fluidity hitherto unexplained, and perhaps without a parallel in nature.

§ 4. *On the Veined Structure of Glacier Ice.*

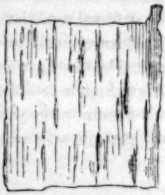
This structure has been indifferently called the "veined structure," the "banded structure," the "ribboned structure," and the "laminar structure" of glacier ice. In a communication to the Geological Society of France assembled at Porrentruy in September 1838, M. Guyot gave the following interesting description of the phenomenon:—"Since the word layer has escaped me, I cannot help recording as a subject of investigation for future observers a fact, regarding which I dare not hazard an explanation; especially as I have not encountered it more than once. It was at the summit of the Gries, at a height of about 7500 feet, a little below the line of the first or high nevé, where the ice passes into a state of granular snow..... In ascending to the origin of this latter (the glacier of Bettelmatten), for the purpose of examining the formation and direction of the great transverse fissures, I saw under my feet the surface of the glacier entirely covered with regular furrows, from 1 to 2 inches in width, hollowed in a half snowy mass, and separated by protruding plates of an ice more hard and transparent. It was evident that the mass of the glacier was here composed of two sorts of ice, one that of the furrows, still snowy and more easily melted, the other that of the plates, more perfect, crystalline, glassy and resistent; and that it was to the unequal resistance which they presented to the action of the atmosphere that was due the hollowing of the furrows and the protrusion of the harder plates. After having followed them for several hundred yards, I reached the edge of a great fissure, 20 or 30 feet wide; which cutting the plates and furrows perpendicularly to their direction, and exposing the interior of the glacier to a depth of 30 or 40 feet, permitted the structure to be observed on a beautiful transverse section. As far down as my vision could reach I saw the mass of the glacier composed of a multitude of layers of snowy ice, each two separated by one of the plates of ice of which I have spoken, and forming a whole regularly laminated in the manner of certain calcareous slates."

A description of this structure, as observed upon the glacier of the Aar, was communicated by Professor Forbes to the Royal Society of Edinburgh on the 6th of De-

cember 1841, and published in the Edinburgh New Philosophical Journal for 1842*. He was undoubtedly the first to give the phenomenon a theoretic significance.

While engaged in the Lower Grindelwald glacier, we separated plates of ice perpendicular to the lamination of the glacier. The appearance presented on looking through them, was that sketched in fig. 6. The layers of transparent ice seemed imbedded in a general milky mass; through the former the light reached the eyes, while it was intercepted by the latter. Some of the transparent portions were sharply defined, and exhibited elongated oval sections, resembling that of a double convex lens, and we therefore called this disposition of the veins "*the lenticular structure.*" In other cases, however, the sharpness of outline did not exist, but still the tendency to the lenticular form could be discerned, the veins in some cases terminating in washy streaks of blue. This structure is probably the same as that observed by Professor FORBES on the Glacier des Bossons, and described in the following words:—"The veins and bands ..... are not formed in this glacier by a simple alternation of parallel layers, but the icy bands have all the appearance of posterior infiltration, occasioned by fissures, *thinning off both ways*†."

Fig. 6.



In 1842 Professor FORBES undertook the survey and examination of the Mer de Glace, and finally arrived at a theory of glacier lamination, which both in his 'Travels' and in a series of letters, extending over a period of several years, he has expounded and illustrated with great skill. The theory is summed up in the following words:—"The whole phenomena in the case of any of the semifluids I have mentioned (treacle, tar, &c.), are such as, combined with the evidence which I have given, that the motion of a glacier is actually such as I have described that of a viscid fluid to be, can leave, I think, no reasonable doubt, *that the crevices formed by the forced separation of a half rigid mass, whose parts are compelled to move with different velocities, becoming infiltrated with water, and frozen during winter, produce the bands which we have described*‡."

This theory has been opposed by Mr. HOPKINS, whose excellent papers, published in the 26th volume of the Philosophical Magazine, are replete with instruction as to the mechanical conditions of glaciers. On the other hand, the theory of Professor FORBES is defended in the same journal by Dr. WHEWELL§. We will leave the points discussed

* This communication gave rise to a discussion as to priority between Professor FORBES and M. AGASSIZ, for the details of which we must refer to the original papers on the subject.

† Travels, p. 181.

‡ Ibid. p. 377. M. AGASSIZ also seems disposed to regard the blue bands as the result of the freezing up of fissures, which, however, are supposed to be formed in a manner different from that assumed by Professor FORBES. But M. AGASSIZ calls the attention of future observers to some of the related phenomena; and gives it as his opinion, "qu'il n'est aucune phénomène dont l'explication offre plus des difficultés." See his important work, 'Système Glacière,' which, until quite recently, we had not the opportunity of examining.

§ Philosophical Magazine, S. 3. vol. xxvi. pp. 171, 217.

in their communications for the present untouched, and confine ourselves to stating a few of the circumstances which appear to us to render the theory doubtful.

1. It is not certain that the colds of winter penetrate to depths sufficient to produce the blue veins, which, it is affirmed, are "an integral part of the inmost structure" of the ice. SAUSSURE was of opinion that the frosts of winter did not penetrate to a greater depth than 10 feet, even at the summit of Mont Blanc, and Professor FORBES considers this opinion to be a just one. But if so, there would be some difficulty in referring to the frosts of winter the blue veins which M. AGASSIZ observed at a depth of 120 feet below the surface of the glacier of the Aar.

2. It will be remembered that M. GUYOT's statement regarding the blue veins is, that he saw the mass of the glacier composed of a multitude of layers of white ice, separated, each from the other, by a plate of transparent ice. The description of Professor FORBES is briefly this:—"Laminæ or thin plates of transparent blue ice, alternate in most parts of every glacier with laminæ of ice, not less hard and perfect, but filled with countless air-bubbles which give it a frothy semitransparent look." But there is another form of the blue veins, already referred to, which consists in transparent lenticular masses imbedded in the general substance of the white ice. Horizontal sections of these transparent lenses were exposed upon the surface of the Grindelwald glacier, and vertical sections of them upon the perpendicular sides of the water-courses, and upon the walls of the crevasses. The following measurements, taken on the spot, will give an idea of their varying dimensions:—

Fig. 7.

Dimensions.

	in.	in.		in.	in.
No. 1.	*ab* 24	*cd* 2	No. 4.	*ab* 3½	*cd* ½
2.	*ab* 10	*cd* 1	5.	*ab* 1½	*cd* ¼
3.	*ab* 6	*cd* 1	6.	*ab* 1	*cd* $\frac{1}{12}$

Such masses as these here figured were distributed in considerable numbers through the glacier; they had all the appearance of flattened cakes, and the smaller ones resembled the elongated green spots exhibited by sections of ordinary roofing-slate cut perpendicular to the planes of cleavage. Now it appears mechanically impossible that a solution of continuity, such as that supposed, could take the form of the detached lenticular spaces above figured.

3. The fissures to which the blue veins owe their existence are stated to be due to the motion of the glacier; and as this motion takes place both in summer and winter, it is to be inferred that the fissures are produced at both seasons of the year. Now as the fissures formed in winter cannot be filled with ice during that season for want of *water*, and as those formed in the ensuing summer cannot, while summer continues, be

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frozen for want of *cold*, we ought at the end of each summer to have *a whole year's fissures* in the ice. These fissures, which the ensuing winter is, according to the theory, to fill with blue ice, must, in summer, be filled with *blue water*. *Why then are they not seen in summer?* The fissures are such as can produce plates of ice varying "from a small fraction of an inch to several inches in thickness," which, according to our own observations, produce lenticular masses of ice 2 feet long and 2 inches thick, or even (for we have seen pieces of this description) 10 feet long and 10 inches thick; and M. Desor informs us in the memoir from which we have already quoted, that under the medial moraine of the Aar glacier, there are bands 10 inches and even a foot in thickness. Such fissures could not escape observation if they existed, but they never have been observed, and hence the theory which makes their pre-existence necessary to the production of the blue veins appears to us improbable.

§ 5. *On the Relation of Slaty Cleavage to the Veined Structure.*

Within the last few years a mechanical theory of the cleavage of slate rocks has been gradually gaining ground among those who have reflected upon the subject. The observations of the late Daniel Sharpe appear to have originated this theory. He found that fossils contained in slate rocks were distorted in a manner which proved that they had suffered compression in a direction at right angles to the planes of cleavage. His specimens of shells, which are preserved in the Museum of Practical Geology, and other compressed fossils in the same collection, illustrate in a remarkable manner his important observations. The subsequent microscopic observations of Mr. Sorby, carried out with so much skill and patience, show convincingly that the effects of compression may be traced to the minutest constituents of the rocks in which cleavage is developed. More recently, Professor Haughton has endeavoured to give numerical accuracy to this theory, by computing, from the amount of the distortion of fossils, the magnitude of the change which cleaved rocks have undergone. By the united testimony of these and other observers, whose researches have been carried out in different places, the association of cleavage and compression has been established in the most unequivocal manner; and hence the question naturally arises, "Is the pressure sufficient to produce the cleavage?" Sharpe appears to have despaired of an experimental answer to this question. "If," says he, "to this conclusion it should be objected, that no similar results can be produced by experiment, I reply, that we have never tried the experiment with a power at all to be compared with that emplbyed; and that this may be one of the many cases where our attempts to imitate the operations of nature fail, owing to the feebleness of our means, and the shortness of the period during which we can employ them." The same opinion appears to have been entertained by Professor Forbes:—"The experiment," he says, "is one which the boldest philosopher would be puzzled to repeat in his laboratory; it probably requires acres for its scope, and years for its accomplishment."

While one of us was engaged in 1855 in examining the influence of pressure upon

magnetism, he was fortunate enough to discover that in white wax, and other bodies, a cleavage of surpassing fineness may be developed by pressure, and he afterwards endeavoured, in a short paper*, to show the application of this result, both to slaty cleavage and to a number of other apparently unrelated phenomena. The theory propounded in this paper may be thus briefly stated. If a piece of clay, wax, marble or iron be broken, the surface of fracture will not be a plane surface, nor will it be a surface dependent only on the form of the body and the strain to which it has been subjected; the fracture will be composed of innumerable indentations, or small facets, each of which marks a surface of weak cohesion. The body has yielded, where it could yield, most easily, and in exposing these facets, in some cases crystalline, in others purely mechanical, wherever the mass is broken, it is shown to be composed of an aggregate of irregularly-shaped parts, which are separated from each other by surfaces of weak cohesion. Such a quality must, in an eminent degree, have been possessed by the mud of which slate-rocks are composed, after the water with which the mud had at first been saturated had drained away; and the result of the application of pressure to such a mass would be, to develope in it a lamination similar to that so perfectly produced on a small scale in white wax. Thus one cause of cleavage may be stated, in general terms, to be the conversion by pressure of irregularly-formed surfaces of weak cohesion into parallel planes. To produce lamination in a compact body such as wax, it is manifest that while it yields to the compression in one direction, it must have an opportunity of expanding in a direction at right angles to that in which the pressure is exerted; a second cause is the lateral sliding of the particles which thus takes place, and which may be very influential in producing the cleavage†.

Before attempting to show the connexion between this theory and the case at present under consideration, a mode of experiment may be described which was found to assist in forming a conception of the mechanical conditions of a glacier, and which has already been resorted to by Professor FORBES in demonstration of the viscous theory. Owing to

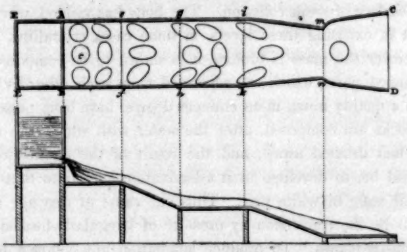
* Proceedings of the Royal Institution, June 1856; Philosophical Magazine for July 1856.

† Three principal causes may operate in producing cleavage:—1st, the reducing of surfaces of weak cohesion to parallel planes; 2nd, the flattening of minute cavities; and 3rd, the weakening of cohesion by tangential action. The third action is exemplified by the state of the rails near a station where the break is applied. In this case, while the weight of the train presses vertically, its motion tends to cause longitudinal sliding of the particles of the rail. Tangential action does not however necessarily imply a force of the latter kind. When a solid cylinder, an inch in height, is squeezed by vertical pressure to a cake a quarter of an inch in height, it is impossible, physically speaking, that the particles situated in the same vertical line shall move laterally with the same velocity; but if they do not, the cohesion between them will be weakened or ruptured. The pressure will produce new contact, and if the new contact have a cohesive value equal to that of the old, no cleavage from this cause can arise. The relative capacities of different substances for cleavage, appears to depend in a great measure upon their different properties in this respect. In butter, for example, the new attachments are equal, or nearly so, to the old, and the cleavage is consequently indistinct; in wax this does not appear to be the case, and hence may arise in a great degree the perfection of its cleavage. The further examination of this subject promises interesting results.

the property of ice described in § 3, the resemblance between the motion of a substance like mud and that of a glacier is so great, that considerable insight regarding the deportment of the latter may be derived from a study of the former. From the manner in which mud yields when subjected to mechanical strain, we may infer the manner in which ice would be *solicited to yield* under the same circumstances.

To represent then the principal accidents of a glacial valley, a wooden trough, ABCD, fig. 8, of varying width and inclination, was made use of. From A to C the

Fig. 8.



trough measures 6 feet, and from A to B, 15 inches. It is divided into five segments; that between AB and *ef* is level, or nearly so, that between *ef* and *gh* is inclined; from *gh* to *ik* is again nearly level; from *ik* to *mn* inclined, while from *mn* to CD the inclination is less than between *ik* and *mn*. The section of the bottom of the trough is figured underneath the plan. ABEF is a box supported at the end of the trough, and filled with a mixture of water and fine pipe-clay. The front, AB, can be raised, like a sluice, and the mud permitted to flow regularly into the trough. While the mud is in slow motion, a coloured circle, *c*, is stamped upon the white clay between AB and *ef*; the changes of shape which this circle undergoes in its passage downwards will indicate the forces acting upon it. The circle first moves on, being rather compressed, in the direction of the length of the trough until it reaches *ef*, on crossing which, and passing down the subsequent slope, it elongates as in the figure. Between *gh* and *ik* the figure passes through the circular form, and assumes that of an ellipse, whose shorter axis is parallel to the length of the trough. It is manifest from this that the mud between *ef* and *gh* is in a state of longitudinal tension, while between *gh* and *ik* its state is that of longitudinal compression. On crossing *ik* and descending the second incline, the figure is again drawn out longitudinally, while between *mn* and CD the ellipse widens on account of the permission given to lateral expansion by the augmented width of the trough.

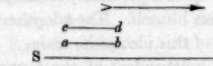
The side circles in the same figure will enable us to study the influence of lateral friction upon the descending stream. These circles are distorted into ellipses, whose major axes are oblique to the direction of the trough's length. Above the line *ef* central fissures perpendicular to the axis of the trough cannot be formed; for here, instead of

tending to open into fissures, the flattening of the central circle shows that the mud is longitudinally compressed. On the slope below *ef*, the distortion of the circles into ellipses is very pronounced; and as the longer axis of each ellipse marks the line of maximum tension, and as the tendency of the mass is to form a fissure at right angles to such a line, we should have here, if the substance were not so plastic as to prevent the formation of fissures, the state of things observed upon the corresponding portion of the glacier; namely, central fissures perpendicular to the longitudinal axis of the trough, and side fissures inclined to the same axis because pointing in the direction of the shorter axis of each ellipse. Between *gh* and *ik* the longitudinal tension is changed to compression; the central figure is flattened, while the side ones remain stretched. In the corresponding portion of the glacier we should expect the central fissures formed between *ef* and *gh* to be squeezed together and closed up, while the lateral ones would remain open. This is also the case*. Between *ik* and *mn* we have again longitudinal tension, and at the corresponding portions of the glacier the transverse central crevasses ought to reappear, which they actually do. Below the line corresponding to *mn*, the widening of the valley, in the case now in our recollection, causes the ridges produced at the previous slope to break across and form prismatic blocks; while lower down the valley these prisms are converted by the action of sun and rain into shining minarets of ice. These results appear to be in perfect accordance with those arrived at by Mr. Hopkins on strict mechanical reasoning†.

We will now seek to show the analogy of slaty cleavage to the laminar structure of glacier ice. Referring to fig. 8, it will be seen that in the distortion of the side circles one diameter is elongated to form the transverse axis of the ellipse, while another is compressed to form the conjugate axis. In a substance like mud, as the elongation of the major axis continues, its inclination to the axis of the glacier continually changes; but were the substance one of limited extensibility like ice, fissures would be formed when the tension had reached a sufficient amount, or in other words, when the major axis of the ellipse had assumed a definite inclination to the axis of the glacier.

Thus, in a glacier of the form represented by our trough, owing to the swifter motion

* The possibility of the coexistence of lateral crevasses and compression at the centre may, perhaps, be thus rendered manifest:—let *ab*, *cd* be two linear elements of a glacier, situated near its side S I.



Suppose, on passing downward, the line *ab* becomes shortened by longitudinal pressure to *a'b'*, and *cd* to *c'd'*, which latter has passed *a'b'* on account of its greater distance from the side of the glacier. Taking the figure to represent the true change both of dimension and position, it is plain, that though each element has been *compressed*, the differential motion has been such as to *distend* the line of particles joining *a* and *d*, in the ratio $\frac{ad}{a'd'}$. If this ratio be more than that which the extensibility of ice can permit of, a side fissure will be formed.

† Philosophical Magazine, 1845, vol. xxvi.

of the centre, we have a line of maximum pressure oblique to the wall of the glacier, and a line of maximum tension perpendicular to the former; crevasses are formed at right angles to the direction of tension, and *it is approximately at right angles to the direction of pressure, as in the case of slate rocks, that the lamination of glacier ice is developed.*

Under ordinary circumstances, therefore, the lamination near the sides of the glacier would, in accordance with the theory of compression, be oblique to the sides, which it actually is. It would be transverse to the crevasses wherever they occur, which it actually is. If the bed of a glacier at any place be so inclined as to cause its central portions to be longitudinally compressed, the lamination, if due to compression, ought to be carried across the glacier at such a place, being transverse to the axis of the glacier at its centre, which is actually the case. This relation of the planes of lamination to the direction of pressure is constant under a great variety of conditions. A local obstacle which produces a thrust and compression is also instrumental in developing the veined structure. In short, so far as our observations reach, wherever the necessary pressure comes into play, the veined structure is developed; being always approximately at right angles to the direction in which the pressure is exerted.

But we will not rely in the present instance upon our own observations alone. Before he formed any theory of the structure, and in his first letter upon the subject, Professor Forbes remarks, that "the whole phenomenon has a good deal the air of a structure induced *perpendicularly to the lines of greatest pressure.*" His later testimony is in substance the same. In his thirteenth letter, read before the Royal Society of Edinburgh on the 2nd of December, 1846, he says that the blue veins are formed *where the pressure is most intense.* In his reference to the development of the laminar structure on the glacier of the Brenva, the pressure is described as being "*violent*," the effect being such as to produce "*a true cleavage when the ice is broken with a hammer or cut with an axe.*" So also with regard to the glacier of Allalein*, he says "the veined structure is especially developed in front, *i. e.* against the opposing side of the valley, where the pressure is greater than laterally." In fact, the parallelism of the phenomenon to that of slaty cleavage struck Professor Forbes himself, as is evident from the use of the term "now" in the following passage:—"It will be understood that I do not *now* suppose that there is any parallelism between the phenomenon of rocky cleavage and the ribboned structure of the ice." This reads like the giving up of a previously held opinion; the term *now* being printed in italics by Professor Forbes himself. The adoption of the viscous theory appears to have carried the renunciation of this idea in its train.

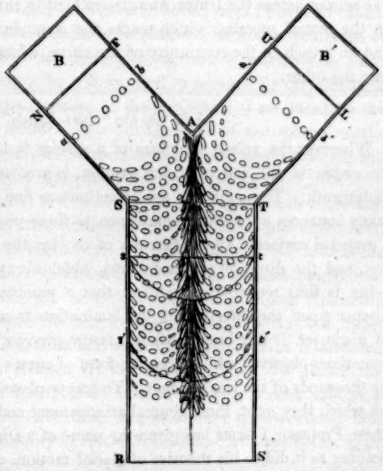
Later still, and from a source wholly independent of the former, we have received additional testimony on the point in question. The following quotation is from a letter, dated 16th November, 1856, received by one of us from Professor Clausius of Zurich, so well known in this country through his important memoirs on the Mechanical Theory of Heat:—"I must now," writes M. Clausius, "describe to you another singular coincidence. I had read your paper upon the cleavage of rocks..... and it occurred to

* Travels, p. 352.

me at the time that the blue veins of glaciers, which indeed I had not seen, but which had been the subject of repeated conversations between Professor STUDER of Berne, Professor ESCHER VON DER LINTH, and myself, might be explained in the same manner. When, therefore, I reached the Rhone glacier for the first time, I walked along it for a considerable extent, and directed my attention particularly to the structure. I repeated this on the other glaciers which I visited during my excursion. I did not indeed pursue the subject so far into detail as to be able in all cases to deduce the blue veins from the existing conditions of pressure, but the correctness of the general explanation impressed itself upon me more and more. This was particularly the case in the glacier of the Rhone, where I saw the blue bands most distinctly, and where also their position harmonized with the pressure endured by the glacier when it was forced to change the direction of its motion. You can therefore imagine how astonished I was to learn that at the same time, and on this very glacier among others, you had been making the same investigations." It ought also to be remarked, that a similar thought occurred to Mr. SORBY, from whom after his return from Switzerland one of us received a note, in which pressure was referred to as the possible cause of the veined structure of glacier ice.

A fine example of ice lamination is that produced by the mutual thrust of two confluent glaciers. The junction of the Lauter Aar and Finster Aar glaciers to form the glaciers of the Unter Aar is a case in point, and the results obtained with a model of this glacier were highly interesting. Fig. 9 is a sketch of the trough in which the experiments were made. The branch terminating at UL is meant to represent the Lauter Aar glacier; that ending at FN the Finster Aar branch. The point at A represents the "Abschwung," so often referred to in the works of M. AGASSIZ. B and B′ are two boxes with sluice fronts, from which the mud flows into the trough, The object was to observe the mechanical state of the mass along the line of junction of the two streams, and along their respective centres, and compare the result with the observations upon the glacier itself. The mud was first permitted to flow simultaneously from both

Fig. 9.



boxes, and after it had covered the bottom of the trough to some distance below the line ST, the end of a glass tube was dipped into a fine mixture of the red oxide of iron and water, and the two arms of the glacier were covered all over with small circles similar to those between the points *ab* and *a'b'*. The mud was then permitted to flow, and the mechanical strains exerted on it were inferred from the distortion of the small circles. The figure represents the result of the experiment. The straight rows of circles bent in the first place into curves; at the point A both streams met, and by their mutual push actually squeezed the circles into lines. Along this central portion in the glacier itself the great medial moraine stands, and under it and beside it, as already stated, the lamination is most strikingly developed; the blue veins being parallel to the axis of the glacier, or, in other words, coinciding with the direction of the central moraine. Midway between the moraine and the sides of the glacier the structure is very imperfectly developed, and the deportment of our model, which shows that the circles here scarcely change their form, tells us that this is the result which ought to be expected. It may be urged, that the structure is here developed, because of the sliding motion produced by the swifter flow of one of the glaciers; but some of the experiments with the model were so arranged, that both of the branch streams flowed with the same velocity; the distortions, however, were such as are shown in the figure. The case is precisely the same in nature. On reference to the map of M. Agassiz, we find a straight line set out across the Unter Aar glacier bent in three successive years into a curve; but on the central moraine, which marks the common limit of the constituent streams, we find no breach in the continuity of the curve, which must be the case if one glacier slid past the other.

§ 6. *On the "Dirt-Bands" of Glaciers.*

Wherever the veined structure of a glacier is highly developed, the surface of the ice, owing to the action of the weather, is grooved in accordance with the lamination underneath. These grooves are sometimes as fine as if drawn by a pencil, and bear in many instances a striking resemblance to those produced by the passage of a rake over a graveled surface. In the furrows of the ice the smaller particles of dirt principally rest, and the direction of the furrows, which always corresponds with that of the blue veins, is thus rendered so manifest, that a practised observer can at any moment pronounce upon the direction of the lamination from the mere inspection of the surface of a glacier. But besides these narrow grooves, larger patches of discoloration are sometimes observed, which take the form of curves sufficient in width to cover hundreds or thousands of the smaller ones. To an eye placed at a sufficient height above a glacier on which they exist, their general arrangement and direction are distinctly visible. To these Professor Forbes has given the name of "Dirt-Bands," and the discovery of them, leading as it did to his theories of glacial motion, and of the veined structure of glacial ice, is to be regarded as one of the most important of his observations.

On the evening of the 24th of July he walked up the hill of Charmoz to a height of

about 1000 feet above the level of the glacier, and, favoured by the peculiar light of the hour, observed "a series of nearly hyperbolic brownish bands on the glacier, the curves pointing downwards and the two branches mingling indiscriminately with the moraines." The cause of these bands was the next point to be considered, and his examination of them satisfied him "that the particles of earth and sand and disintegrated rock, which the winds and avalanches and water-runs spread over the entire breadth of the ice, formed a *lodgement* in those portions of the glacier where the ice was most porous, and that, consequently, the 'dirt-bands' were merely *indices of a peculiarly porous veined structure traversing the mass of the glacier in these directions.*"

Professor FORBES was afterwards led to regard these intervals as the marks of the annual growth of the glacier; he called the dirt-bands "annual rings*," and calculated, from their distance apart, the yearly rate of movement. In fine, the conclusion which he deduces from the dirt-bands is, that a glacier throughout its entire length is formed of alternate segments of porous and of hard ice. The dirt which falls upon the latter is washed away, as it has no hold upon the surface; that which falls upon the former remains, because the porous mass underneath gives it a lodgement. "The cause of the dazzling whiteness of the glacier des Bossons at Chamouni is the comparative absence of these layers of granular and compact ice: the whole is nearly of uniform consistence, the particles of rock scarcely find a lodgement, and the whole is washed clean by every shower†." "It must be owned, however," says Professor FORBES, "that there are several difficulties which require to be removed, as to the recurrence of these porous beds." In his fifteenth letter upon glaciers, and in reference to some interesting observations of Mr. MILWARD'S, he endeavoured to account for the difference of structure by referring it to an annual "gush" of the ice, which is produced by the difference of action in summer and winter. We are ignorant of the nature of the experiments on which this theory of the dirt-bands is founded, and would offer the following simple explanation of those which came under our own observation.

Standing at a point which commanded a view of the Rhone glacier, both above and below the cascade, we observed that the extensive ice-field above was discoloured by sand and débris, distributed without regularity. At the summit of the ice-fall the valley narrows to a gorge, and the slope downwards is for some distance precipitous. In descending, the ice is greatly shattered; in fact, the glacier is broken repeatedly at the summit of the declivity, transverse chasms being thus formed; and these, as the ice descends, are broken up into confused ridges and peaks, with intervening spaces, where the mass is ground to pieces. By this breaking up of the glacier the dirt upon its surface undergoes fresh distribution: instead of being spread uniformly over the slope, spaces are observed quite free from dirt, and other spaces covered with it, but there is no appearance of regularity in this distribution. At some places large

* "I cannot help thinking that they are the *true annual rings* of the glacier, which mark its age like those of a tree."—Appendix to Travels, p. 408.

† Travels, p. 406.

irregular patches appear, and at others elongated spaces covered with dirt. Towards the bottom of the cascade the aspect changes; but still, were the eye not instructed by what it sees lower down, the change would have no significance. When the ice has fairly escaped from the gorge, and has liberty to expand laterally in the valley below, the patches of dirt are squeezed by the push behind them, and drawn laterally into narrow stripes, which run across the glacier; and as the central portion moves more quickly than the sides, these strips of discoloration form curves which turn their convexity downwards, constituting, we suppose, the "Dirt-Bands" of Professor FORBES. On the Grindelwald glacier, where one of us, in his examination of the bands, was accompanied by Dr. HOOKER, this change in the distribution of the dirt,—the squeezing, lateral drawing act, and bending of the dirt patches below the bottom of the ice-fall,—was especially striking.

Such then appears to be the explanation of the dirt-bands in the cases where we have had an opportunity of observing them. We have not seen those described by Professor FORBES, but the conditions under which he has observed them appear to be similar. An illustration of the explanation just given is furnished by the dirt-bands observed below the "cascade" of the Talèfre. The character of this ice-fall may be inferred from the following words of Professor FORBES, and from the map which accompanies his 'Travels.' "The structure," he says, "assumed by the ice of the Talèfre is extirpated wholly by its precipitous descent to the level of the Glacier de Léchand, where it reappears, or rather is reconstructed out of the broken fragments according to a wholly different scheme." One of the results of this "scheme" would, it is presumed, be a redistribution of the dirt, and the formation of bands in the manner described. Those who consult the map will, however, see dirt-bands marked on the Glacier du Géant also, while no cascade is sketched upon it; but at page 167 of the 'Travels,' Professor FORBES, in referring to this glacier, says, "I am not able to state the exact number of dirt-bands between *the foot of the ice cascade* opposite La Noire and the corner of Trelaporte." Here we are not only informed of the existence of a cascade, but are left to infer that the dirt-bands begin to form at its base, as in the Glacier du Géant, and in those which have come under our own observation. The clean Glacier des Bossons, also, which was referred to by Professor FORBES, in one of his earliest letters, as affording no lodgement to the dirt, possesses its cascade (page 181), and here also we find (page 182) "that the peculiar phenomena of '*dirt-bands*' on a great scale are not wanting, although from the dazzling whiteness of the ice they may be very easily overlooked." We make these remarks with due reserve, not having yet seen the glaciers referred to.

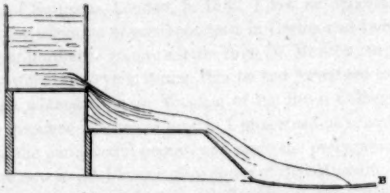
The explanation just given has been brought to the test of experiment. ABCD, fig. 10, is a wooden trough intended roughly to represent the glacier of the Rhone, the space ACEF being meant for the upper basin. Between EF and GH the trough narrows and represents the precipitous gorge down which the ice tumbles, while the wide space below represents the comparatively level valley below the fall, which is filled with the ice, and constitutes the portion of the glacier seen by travellers descending from

the Grimsel or the Furka pass. ACLM is a box with a sluice front, which can be raised so that the fine mud within the box shall flow regularly into the trough, as in the

Fig. 10.

cases already described. The disposition of the trough will be manifest from the section, fig. 11. While the mud was in slow motion downwards, a quantity of dark-coloured

Fig. 11.



sand was sifted over the space ACEF, so as to represent the débris irregularly scattered over the corresponding surface of the glacier; during the passage of the mud over the brow at EF, and down the subsequent slope, it was hacked irregularly, so as to represent the dislocation of the ice in the glacier. Along the slope this hacking produced an irregular and confused distribution of the sand; but lower down, the patches of dirt and the clean spaces between them gradually assumed grace and symmetry; they were squeezed together longitudinally and drawn out laterally, bending with the convexity downwards in consequence of the speedier flow of the central portions, until finally a system of bands was established which appeared to be an exact miniature of those exhibited by the glacier. On fig. 10 is a sketch of the bands observed upon the surface of the mud, which however falls short of the beauty and symmetry of the original. These experiments have been varied in many ways, with the same general result.

In conclusion we would remark, that our joint observations upon the glaciers of Switzerland extended over a period of a few days only. Guided by the experience of our predecessors, much was seen even in this brief period; but many points of interest first suggested themselves during the subsequent experimental investigation. While, therefore, expressing our trust that the substance of the foregoing memoir will be found in accordance with future observation, we would also express our belief in the necessity of such observation. Indeed the very introduction of the principle of regela-

tion, without which it may be doubted whether the existence of a glacier would be at all possible, opens, in itself, a new field of investigation. This and other questions, introduced in the foregoing pages, must however be discussed with strict reference to the phenomena as Nature presents them. Much might be said even now upon these subjects, but the known liability of the human mind to error when speculation is substituted for observation, renders it safer to wait for more exact knowledge, than to hazard opinions which an imperfect acquaintance with the facts must necessarily render to some extent uncertain.

Royal Institution,
January 1857.

XVI. *Description of the Fœtal Membranes and Placenta of the Elephant* (Elephas Indicus, Cuv.), *with Remarks on the value of placentary characters in the classification of the Mammalia. By Professor* Owen, *F.R.S.*

Received April 14,—Read May 14, 1857.

Seba gives a figure of the fœtus of an Elephant in tab. iii. vol. i. of his great 'Thesaurus'*; but a knowledge of the fœtal membranes of this remarkable quadruped has long been a great desideratum in both physiology and zoology.

From the period of my appointment to the Hunterian Professorship of Physiology at the Royal College of Surgeons, London, in 1836, I lost no opportunity of urging this desideratum upon the attention of correspondents in Ceylon and India; and, after some years, I had the gratification to receive a letter from Dr. Morton, an accomplished Army Medical Officer quartered in Ceylon, stating that he had forwarded to me the membranes of a fœtal Elephant, addressed to the Museum of the Royal College of Surgeons. On the arrival of this specimen in a keg of arrack, I prosecuted as complete an examination as the condition of the parts would permit, and made the preparations (Nos. 3558 c. and 3558 d.), which are now in the Physiological Series of that Museum. The results of this examination were orally communicated in the Theatre of the College, in the 17th Lecture of my Course "On the Generation and Development of the Vertebrate Animals," delivered in 1850†; and were illustrated by diagrammatic views of the chief peculiarities which the parts presented. Circumstances prevented further publication of the facts at that time; but I am now able to submit the following description of the fœtal membranes and placenta of the Elephant to the Royal Society.

The chorion of the Elephant (Plate XVI. fig. 1, *a*, *a'*, *d*), at about the middle of the period of utero-gestation, forms a transversely oblong sac, 2 feet 6 inches in long diameter, and 1 foot 4 inches in short diameter, encompassed at its middle part by an annular placenta (*b*, *b*), 2 feet 6 inches in circumference, varying from 3 to 5 inches in breadth, and from 1 to 2 inches in thickness. This placenta is partially divided by opposite constrictions into two moieties, one measuring 12 inches, the other 10 inches in length, and the extreme breadth being 5 inches in each; the connecting isthmus is 3 inches in breadth. The placenta presents the same spongy texture and vascularity as does the annular placenta of the Hyrax and of the *Carnivora*; but the capillary filaments or villosities enclosing the fœtal vessels enter into its formation in a larger proportion,

* Locupletissimi Rerum naturalium Thesauri accurata Descriptio, &c., 4 vols. fol. Amst. 1734—1765.

† "Fœtal membranes of the Hog- and Horse-tribe: fœtal membranes and placenta of the Hyrax and Elephant." Printed Synopsis, 1850.

and are of a relatively coarser character. The greater part of the outer convex surface of the placenta is smooth; the rough surface, which had been torn from the maternal or uterine placenta, exposed the fœtal capillaries, and occupied chiefly a narrow tract (*c*, *c*) near the middle line of the outer surface. A thin brown deciduous layer is continued from the borders of the placenta, for a distance varying from 1 to 3 inches, upon the outer surface of the chorion. Flattened folds of a similar substance, or false membrane, could be raised from some parts of the surface of the placenta; at other parts the substance formed irregular fibrous bands, the fibres extending in the direction of the circumference of the placental ring. The outer surface of the chorion is for the most part smooth and shining; but at each of the obtuse extremities of the sac there was a villous and vascular subcircular patch (*d*, *d*), between 2 inches and 3 inches in diameter, the villi being short and graniform, $\frac{1}{8}$th of a line in diameter, or less. Thus the chief points of attachment of the chorion to the uterus are, at the equator, by the annular placenta, and at each pole of the elongated sac, by the subcircular villous patch.

The umbilical cord (*f*), formed by one venous and two arterial trunks, and by the slender neck of the allantois, with the connecting cellular tissue and the covering of amnios, is short and somewhat flattened. It measured about 6 inches in length, before the division of the vascular trunk, and about 3 inches in circumference.

The inner surface of the amnios is roughened by brownish hemispherical granules, from one line to $\frac{1}{10}$th of a line in size, commonly about half a line; the outer surface is finely wrinkled, but smooth. The amnios is continued from the base of the umbilical cord upon the allantois, which is of considerable size, and is so interposed between the chorion and amnios, as to prevent any part of the amnios attaining the inner surface of the placenta. The amnios consists of two layers: one is the granular layer, continued upon the inner or fœtal surface of the allantois, and thence upon the umbilical cord; the other is the smooth outer layer, continued upon the outer or chorional surface of the allantois, and thence upon the inner surface of the chorion.

The allantois, into which the urachus (*g*) at the torn fœtal end of the cord rapidly expands, divides, where the amnios begins to be reflected upon it, into three sacculi: one, which is the largest, extends over the inner surface of the annular placenta, and projects a little way into the unruptured sac of the chorion (*a*); a second division or horn of the allantois extends into the opposite or ruptured sac of the chorion (*a'*), whence the fœtus had escaped; it there bends round towards the placenta, and its apex adheres at that part to the first division of the allantois: the third prolongation subdivides into two smaller cavities, each terminating in a cul-de-sac, encompassing, and closely attached to the primary divisions of the umbilical vessels. The line of adhesion of the amnios to the allantois, where it is reflected upon these cul-de-sacs, measures 3 feet 6 inches.

The umbilical vein, at the distance of 6 inches from the torn or fœtal end of the cord, divides; and, 3 inches further, one of the branches subdivides. The first division of the umbilical arteries takes place 9 inches from its torn extremity. The primary branches of the umbilical arteries and vein diverge from the umbilical cord in four divisions.

The allantois, coexpanding with the divergence of these trunks, developes its principal sac through two opposite interspaces, and the two smaller divisions at the intermediate interspaces.

It may, perhaps, aid in the comprehension of this rather complex structure, if I here cite from the rough notes which I penned on the first examination of the connexions of the allantois:—"The urachus expands in an infundibular form into the allantois. About 4 inches from the infundibular commencement of the allantois an umbilical vessel recedes from the amnios towards the chorion, carrying with it a fold of the free and thickened margin; this fold expands to 5 inches in breadth: the layers adhere together, but could be separated. Each of the other two hypogastric vessels carry similar folds of the allantois inwards, from the surface of the amnion to the chorion, and the allantois is again reflected by three broad folds diverging from one centre upon the amnios of the opposite side. The amnios becomes blended with, or closely adherent to, the allantois about 3 inches from the umbilical cord; prior or anterior to this part, the two membranes are connected by loose reticulated tissue."

I now proceed with the description, which was written after a recognition of the nature of the several parts observed in the course of the examination. The primary branches of the umbilical vessels reach, first, the borders of the placenta, and then ramify in the substance of the placenta and upon the inner surface of the chorion, being supported there, and more or less surrounded, by the continuation of the allantois which forms the so-called endochorion.

Upon the endochorionic vessels are developed a number of flattened, oval or subcircular bodies (*e*, *e*), varying in diameter from an inch or more to half a line. On separating the chorion from the allantois, these bodies were found to belong entirely to the latter membrane. To the naked eye they present a compact, structureless tissue, of a grey colour. On dissecting one of the branches of an umbilical vessel, upon which one of these bodies was developed, the vessel was found to pass on the chorionic side of the body without undergoing any apparent change in the passage, the body being developed from the allantois and from that part which forms the allantoic side of the sheath of the vessel. These bodies are most abundant near the placenta, where they are separated from each other by interspaces often less than their own diameter. They are wider apart upon the non-placental part of the chorion, especially as they approach the poles of the chorionic sac. Their number on the unruptured division of the chorionic sac was about 120. They are exclusively an allantoic development. Several small specimens occur on the free duplicatures of the allantois continued from the umbilical trunks near the placenta: in almost every case they are developed on the course of the large vessels, and are restricted, with few exceptions, to that part of the allantois which is in contact with the chorion, and which forms the endochorion of embryologists. The free surface of these peculiar bodies is smooth and polished, not villous like the cotyledons of the *Ruminantia*; from which they likewise differ essentially in projecting inwards towards the cavity of the allantois.

The most important modification in the vascular structures connecting the chorion with the uterus, in the Elephant, is their combination of two forms of the placenta, viz. the 'annular' and the 'diffused,' which forms are restricted in other Mammalia to distinct kinds of quadrupeds.

This fact appears to me to point to the inapplicability of modifications of the placenta to the characterising of primary groups of the Class *Mammalia*.

According to the annular or zonular part of the placental structure in the Elephant, it should be classed with the true *Carnivora*: according to the diffused vascular villosities at the two poles of the chorion, it would rank with the true Pachyderms of the Cuvierian System. The Elephant, however, is not the only hoofed quadruped which has an annular placenta: the Hyrax, which CUVIER so sagaciously discerned to be a kind of miniature Rhinoceros, differs from its great ally by having an annular or zonular placenta.

Professor MILNE-EDWARDS, who has revived Sir EVERARD HOME'S idea* of a system of Mammalogy on placental characters†, consistently included the Hyrax with the Seals, Dogs, Cats and other *Carnivora* having a zonular placenta‡; and a logical application of the same principles of classification would lead to a transference of the Proboscidian *Ungulata* to the same association. Those Naturalists who, with the able Professor of the Jardin des Plantes, may have placed confidence on the value of the temporary fœtal structures, as indications of natural affinity in the mammalian class, will feel surprised, perhaps, since the chorion of the Elephant developes a compact placental body, that it should not have presented either the discoid or the cotyloid form, so common in the *Rodentia*. Perhaps no existing hoofed mammal offers so many similarities of structure to the *Rodentia* as does the Elephant. CUVIER long ago pointed them out, and others have expressed their idea of this affinity, by calling the proboscidian pachyderm a 'gigantic mouse.'

But the placenta does not present the same form in all *Rodentia*: in the Guinea-pig it is more cup-shaped than plate-shaped, more cotyloid than discoid; whilst in the Hare and Rabbit it is a thin oblong mass, divided into from three to five lobes§. In the Rat, the maternal position of the placenta is cotyloid, and is applied to a small convex lobe or button-shaped process on the middle of the uterine surface of the fœtal placenta, which may thus be said to be 'pedunculate.' In the Water-vole (*Arvicola amphibia*) the fœtal placenta is small and circular, convex towards the uterus and flat towards the chorion, which is attached to the centre of the disk.

In most of the *Insectivora* the placenta is proportionally larger than in most of the

* Lectures on Comparative Anatomy, 4to, vol. iii. 1823, p. 501.

† "Considérations sur quelques Principes relatifs à la Classification Naturelle des Animaux: et plus particulièrement sur la distribution méthodique des Mammifères," Annales des Sciences Nat., sér. 3. tom. i. p. 65, 1844.

‡ Ibid. p. 96.

§ Description of Preparation, No. 3472, 'Physiological Catalogue, Museum, Royal College of Surgeons,' vol. v. p. 118, 4to, 1840.

Rodentia. In the Hedgehog, when the embryo is from half an inch to an inch in length, it is enclosed in a cup-shaped placenta, as in a nest: this is subsequently spread out and flattened by the growth of the fœtus, and converted into a thin, shallow discoid plate, with its concavity applied to the back of the embryo, and with the central part of its convex surface attached to the uterus. In the Mole the placenta is a circular disc at the early period of gestation, and subsequently becomes an oblong flat band, with its long axis parallel to that of the fœtus: the linear tract of the uterine surface to which the placenta is attached shows a fine areolar structure, penetrated by the fœtal placentary filaments, which are brought away, distinct from the maternal structure, like the fœtal cotyledon in the Cow. In the Bat (*Vespertilio noctula*), the placenta has the form of an obtuse cone. In all the foregoing insectivorous mammals the umbilical sac is large in proportion to the allantois.

In a female large frugivorous Bat (*Pteropus medius*, Temminck), preserved in spirits, which I found, on dissection, to be pregnant, the vitellicle, or umbilical sac, was a small reniform, compactly folded body, which lay in the concavity of the placenta, between it and the allantois: the placenta was subcircular, discoid, slightly concave towards the fœtus, proportionally more convex towards the uterus. The fœtal villi are long, delicate, and branched, giving a flocculent appearance to the small portion of the centre of the disk by which the fœtal placenta is attached to the womb. In a loose and general way, all the varieties of the placenta in *Rodentia* and *Insectivora*, with those in *Quadrumana*, may be termed 'discoid.' But the structure of the discoid placenta in the *Pteropus* more resembles that of the fœtal portion of the cotyledon in the Cow, than that of the cellulo-vascular spongy placenta of the *Quadrumana*; and this difference, with the more important one of the larger umbilical sac in the Bats and Insectivora, appears to me to greatly outweigh the degree of resemblance, such as it is, in the mere outward form of the placenta, in the above-cited orders of Mammalia.

Any argument in favour of the affinity of the *Cheiroptera* to the *Quadrumana*, based on that degree of resemblance, must be affected by the prevalence of the double discoid placenta in the *Quadrumana*. Since John Hunter first made known that modification* in a species of *Macacus*, which, from a comparison of the fœtus now preserved in the Museum of the Royal College of Surgeons, I believe to be the 'Wrinkled Baboon' of Shaw (*Macacus rhesus*, Desm.), Professor Breschet† has described and figured the two separate discoid placentæ in the small South American Squirrel-monkey (*Callithrix Sciureus*, Kuhl), in the Green-monkey (*Cercopithecus sabæus*, Desm.), in the Mitre-monkey (*Semnopithecus mitratus*, Is. Geoff.), and in the Long-nosed Monkey (*Nasalis larvatus*, Geoff.). Yet this well-marked modification of the cellulo-vascular placenta is not constant in the *Quadrumana*, nor even in the primary groups of the order. In the Platyrhines, *e. g.* the Howler (*Mycetes seniculus*, Kuhl) has a single placenta‡, and, amongst

* Animal Economy, 1786, 4to.

† Mémoires de l'Académie des Sciences, tom. xix. 1845.

‡ Breschet, *loc. cit.* p. 57.

the Catarhines, I have ascertained that in the Chimpanzee (*Troglodytes niger*) the placenta is single, as in the Human subject.

The true or horned Ruminants all present the much subdivided cotyledonal type of placenta; but the Camel-tribe are unquestionably much more nearly allied to the true Ruminants than to the Equine-tribe, which they resemble in the diffused vascular villosity of the chorion, which is substituted for a placenta. Thus that organ is not one of the characters which indicate the real affinities, and which guide to the true grouping, of the *Ungulata*, any more than it subserves to the formation of natural groups of the *Quadrumana*. The Hog, which has a diffused chorionic villosity, as in the Camel, but with a modification in the arrangement of the vessels, which Sir Everard Home defines by the term 'stellate structures*,' is, like the Camel, much more nearly related to the true Ruminants than to the odd-toed hoofed animals, such as the Horse, Tapir and Rhinoceros. So, likewise, the Hyrax, with an annular placenta, is more closely related to the Rhinoceros than to the Elephant or the Lion.

I would not be misunderstood as deeming the placenta to present no correlation of form with modifications of mammalian structure: the constancy of the annular type in the *Carnivora*, and that of the cotyledonal type in the true *Ruminantia*, are valid instances of the degree to which that correlation exists. But the modifications of the placenta do not form a true guide to the affinities or classification of the *Mammalia* generally. The primary groups of that class are based upon modifications of a much more important and significant organ—the brain: and of this truth Sir Everard Home himself appears to have been aware†, when he published the placentary system of classification, which has lately been revived, with modifications, by the distinguished Professor in the Jardin des Plantes‡.

The placenta of the phytiphagous Sloth (*Bradypus*), which has a complex subdivided stomach, is almost as much subdivided as in the smaller Ruminants; whilst the placenta of the Armadillo, which has a simple stomach, is a single thin oblong body, as in the Hedgehog. But the true affinities of the Sloth would be as much violated by transferring it to the *Ruminantia*, as those of the Armadillo would be by classing it with the *Insectivora*, or as those of the *Hyrax* are by its transference to the same primary group with *Carnivora*, on the score of mere accordance of placental form.

A very significant evidence of the limited correlation of any particular form of placenta with any of the known modifications of the mammalian type, is afforded by the

* See Eschricht, 'De organis quæ respirationi et nutritioni fœtus mammalium inserviunt,' 4to, 1837, p. 4; and the description of the venous plexules of the chorion, in 'Catalogue of the Physiological Series, Mus. College of Surgeons,' vol. v. p. 134, No. 3543.

† "The truth becomes obvious of there being no organ belonging to an animal, except the brain, that will bear us out in affording characters for a general classification, the structure of the other organs being varied whenever it was necessary to adapt the animal to the climate which it is to inhabit, or the food on which it is to subsist: and the brain we are not sufficiently acquainted with, to take it as our guide."—Lectures on Comparative Anatomy, vol. iii. p. 445.

‡ Milne-Edwards, *loc. cit.* Annales des Sciences Naturelles, sér. 3. vol. v. 1844.

ill-success of guesses as to the nature of the fœtal membranes, from the natural affinities indicated by any such known modification. Thus, as regards the Elephant itself, Sir Everard Home wrote:—"The mare has no placenta, only a chorion; the period" (of gestation) "is eleven months. The Elephant, whose period is twenty-two months, I have no doubt, from the appearance of the inside of the uterus, has no placenta, but a chorion*;" and the Elephant is classed accordingly with the Horse, Whale, Hog, and Camel†.

The convictions as to the limited value of the placenta, in Mammalian Classification, which were expressed and illustrated in the Hunterian Lectures for 1850, and were derived mainly from a study of the series of fœtal structures in the Hunterian Physiological Collection, described in the concluding volume of the 'Catalogue' of that Collection‡, appear to me to be strengthened by the remarkable modifications of the placental structures of the Elephant, which form the chief subject of the present communication.

Explanation of the Plate.

PLATE XVI.

Fig. 1. Chorion and placenta of the Elephant (*Elephas Indicus*), on the scale of 2 inches to a foot.

a, *a'*. The chorion; *a'*, the torn end from which the fœtus had escaped.

b. The annular placenta.

c. The tract along which the fœtal was connected with the maternal portion.

d, *d*. The superficial villosity at the poles of the chorion.

e. The allantoic corpuscles, seen through the exochorion.

f. Umbilical cord.

g. The urachus, before expanding into the allantois.

Fig. 2. A portion of the villous surface of the chorion, from the part marked *d* in fig. 1, of the natural size.

Fig. 3. A portion of the allantois, viewed from its amniotic surface, showing the corpuscles of the natural size.

* *Tom. cit.* p. 452.

† 'Order 7,' *tom. cit.* p. 503.

‡ "Physiological Series," vol. v. 4to, 1840, pp. 112—170.

XVII. *Photo-chemical Researches.*—Part I. *Measurement of the Chemical Action of Light.* *By* ROBERT BUNSEN, *Professor of Chemistry at the University of Heidelberg, and* HENRY ENFIELD ROSCOE, *B.A., Ph.D., Professor of Chemistry at Owens College, Manchester. Communicated by Professor* STOKES, *Sec. R.S.*

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THE first and only attempt which has been made to refer the chemical action of light to a standard measure, is to be found in the researches of DRAPER. The description of the instrument and mode of observation employed by him was published in 1843, in volume xxiii. p. 401 of the 'London, Edinburgh and Dublin Philosophical Magazine.'

Even with this instrument, which as we shall show is in many respects very defective, DRAPER has succeeded in establishing experimentally some of the most important relations of the chemical action of light.

In these experiments DRAPER collected hydrogen, evolved by electrolysis, over hydrochloric acid saturated with chlorine, and to this hydrogen he admitted so much chlorine, either by diffusion from the saturated acid or by electrolysis, that the mixture consisted of nearly equal volumes of the two gases, and entirely, or almost entirely, disappeared on exposure to light. The alteration in the volume of the gaseous mixture arising from the absorption of the hydrochloric acid formed by the action of the light, was read off on a scale, and being, within certain limits, proportional to the time of exposure, served as a measure of the chemical rays.

This method affords, however, no means for correctly determining the point at which the gas collected over the acid consists of equal volumes of its constituent parts; still more, however, was the accuracy of DRAPER'S instrument impaired by not considering certain conditions, which must be fulfilled, if exact photo-chemical measurements with chlorine and hydrogen are required. The first and most essential condition, as influencing the regularity of the measurements, relates to the perfectly constant composition of the gas.

When two gases, V and V_1, having respectively the coefficients of absorption α and α_1, are mixed together in such proportions that the unit of volume contains v volumes of the one gas and v_1 volumes of the other, and when such a mixture passes through a liquid, it rapidly changes its composition, according to a complicated law, until the moment arrives when the gases absorbed by the liquid are contained in the relation of αv to $\alpha_1 v_1$. If the gaseous mixture be collected over the liquid before the relation $\frac{\alpha v}{\alpha_1 v_1}$ has been established, an exchange between the dissolved and free gases must occur, and hence the composition of the gaseous mixture must constantly alter, the variations being

of a greater or less amount according as the volumes of the solvent and the free and absorbed gases increase or diminish. In the method employed by DRAPER for the preparation of his sensitive gas, the conditions necessary for obtaining this relation, $\frac{\alpha v}{a_1 v_1}$, are not even approximately attained, so that from this cause alone, an exact agreement between the indications of the Tithonometer cannot be possible.

Another source of more considerable error is to be found in the alterations of pressure to which the liquid and the gaseous mixture are subjected during the whole course of the observations. This variation of pressure causes a certain quantity of gas to escape from solution in the liquid, and as the atmosphere in the liquid has a different composition from that above it, the relation between the chlorine and hydrogen will be altered, and all regularity in the indications of the instrument impaired.

As we were unable, from the foregoing reasons, to employ the mode of observation used by DRAPER, we were compelled to seek for some new method, free from the errors above alluded to.

With this object we have been obliged to examine all the circumstances which could possibly affect the combining power of chlorine and hydrogen, in a series of experiments which have extended over a long space of time, and in which difficulties of no ordinary kind have had to be overcome. For this purpose we required a mixture composed of equal volumes of chemically pure chlorine and hydrogen gases. Amongst the various methods which presented themselves for the preparation of such a mixture, that by electrolysis of aqueous hydrochloric acid appeared at first sight the least likely to give the wished-for result, as DRAPER especially mentions in his research that the electrolysis of this acid, however long continued, never yields equal volumes of chlorine and hydrogen. A series of careful experiments has however proved, most distinctly, that DRAPER'S statement is erroneous; and we have found that the electrolytic chlorine and hydrogen mixture not only consists of exactly equal volumes of chlorine and hydrogen, but that it is perfectly free from oxygen and all oxides of chlorine which might have been formed by secondary electrolytic action.

As the possibility of executing accurate photo-chemical determinations entirely depends upon this purity of the gas, we deem it necessary here to detail the experiments made upon this subject.

When a current from three or four common-sized carbon-zinc elements is passed, by means of carbon-poles, through hydrochloric acid of about specific gravity 1·148, all the chlorine set free at the anode is first absorbed by the liquid. After a short time the evolution of chlorine becomes visible, and gradually increases, whilst the colour of the liquid becomes deeper, until a point is attained, more or less quickly according to the intensity of the current and the quantity of hydrochloric acid employed, at which a statical equilibrium between the free gases and those absorbed in the liquid ensues. From this point forwards the composition of the free and absorbed gases remains constant, provided that the temperature and pressure do not vary, and the amount of hydro-

chloric acid contained in the liquid electrolyte does not diminish below a certain amount. The composition of the absorbed gaseous mixture varies with every change of temperature, whereas the composition of the free gas remains the same for all temperatures, provided that the statical equilibrium is not disturbed by alteration of the temperature and pressure during the evolution of the gas.

After the statical equilibrium has been established, the gas invariably contains one volume of chlorine to one of hydrogen. This important fact, which is strictly conformable to the laws of gas absorption, is deduced from the following analyses made with electrolytic chlorine and hydrogen evolved both from carbon and platinum poles of different dimensions, by currents of various degrees of intensity. In order accurately to measure the volume of gas required for analysis, we allowed the dried electrolytic gas to pass through the glass cylinders represented in Plate XVII. fig. 1. These cylinders, of the same diameter, but of various lengths, are closed by surfaces of plate glass pressed upon the ground ends of the cylinders by means of the screws *a a a*, and the joint rendered perfectly air-tight by a thin roll of white wax squeezed flat between the two surfaces of glass. Each cylinder is furnished with a delivery- and exit-tube ground into the glass, and firmly fastened by cements of gutta-percha and gypsum. The gases passed through the cylinders until the whole of the atmospheric air was displaced; and after the height of the barometer and the temperature (which was kept constant during the experiment) had been observed, the caoutchouc joinings were closed by screw-clamps, and one of them directly opened under a solution of iodide of potassium. The solution rose immediately into the vessel and liberated a quantity of iodine exactly equivalent to the amount of contained chlorine. A volumetric analysis* of this liberated iodine gave us the amount of the chlorine contained in the gas. This amount, subtracted from the total volume of gas contained in the vessel, gave the volume of hydrogen.

If α signify the amount of iodine in grammes contained in one division of the burette, n the number of measures of sulphurous acid required for complete decolorization of the iodine liquid, t the number of divisions of the burette which represents one measure of sulphurous acid, and t' the number of divisions which must be added to the liquid to which n measures of sulphurous acid have been added until the blue colouring appears; and if the atomic weight of chlorine Cl=221·87, and that of iodine I=794·37, and the specific gravity of chlorine gas s=2·45307, the volume of chlorine contained in the vessel in cubic centimetres is equal to

$$C=\frac{773}{s}\cdot\frac{Cl}{I}\alpha(nt-t'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

If the heights of the barometer and thermometer on closing the vessel were respectively P and T, and if the capacity of the vessel expressed in cubic centimetres is equal to C, the total volume of gas present, reduced to 0° C. and 0^{m}·760 expressed in cub. cent., is found from the equation

$$C_1=\frac{CP}{(1+0{\cdot}00366\,T)\times 0{\cdot}76}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

* Ann. Ch. Pharm. lxxxvi. p. 265, and Quart. Journ. Chem. Soc. viii. p. 219.

The amount of chlorine, C, subtracted from the total volume of gas, C_1, gives the volume of the hydrogen contained in the gaseous mixture. The results of a series of experiments conducted according to this method are given in the following Table:—

Series of Experiments I.

	Capacity of vessel.	Temperature. T.	Barometer. P.	a.	t.	t'.	a.	C_1.	Chlorine found. C.	Chlorine calculated. $\frac{1}{4}C_1$.
Exp. 1.	34·81	11·3	0·7363	2	66·1	57·7	0·002443	32·385	16·02	16·19
Exp. 2.	34·81	11·6	0·7415	2	67·0	58·3	0·002443	32·580	16·27	16·29
Exp. 3.	190·24	11·5	0·7518	6	79·6	66·6	0·0024869	180·590	89·96	90·28
Exp. 4.	61·01	12·9	0·7393	2	71·9	15·2	0·0024869	56·672	28·15	28·34
Exp. 5.	64·27	12·9	0·7393	2	77·4	19·5	0·0024869	59·700	29·61	29·85
Exp. 6.	51·60	12·9	0·7393	2	76·7	45·7	0·0024869	47·931	23·57	23·96

The two last vertical columns show that the electrolytic chlorine and hydrogen gas invariably contains equal volumes of its constituents, and that the small variations from this proportion, which occur in these experiments, do not exceed in amount the errors of observation to which even the most accurate analytical processes are necessarily subject. These analyses show also that the gas under examination did not contain any free oxygen, for if a decomposition of the water as well as the hydrochloric acid had taken place, by which two volumes of hydrogen are disengaged for every volume of oxygen, the volume of hydrogen could not have stood to the volume of chlorine, nor to the volume of chlorine and oxygen, in the simple relation of equality; especially as the free oxygen is not estimated by the volumetric method employed. The absence of all oxides of chlorine which might have been formed by secondary action may also be deduced from the following considerations. In the electrolysis of hydrochloric acid, the liberation of every two volumes of hydrogen is accompanied by an evolution of two volumes of chlorine; in the electrolysis of water the same two volumes of hydrogen are accompanied by only one volume of oxygen. This oxygen, combined with two volumes of chlorine, gives two volumes of hypochlorous acid gas. As soon as the equilibrium between the electrolytic liquid and the evolved gas has been established, the liberated gas would contain, supposing this decomposition to occur, in six volumes, four volumes of hydrogen and two volumes of hypochlorous acid gas. These two volumes, ClO, liberate, however, just as much iodine as four volumes of chlorine. Hence the analysis would have given four instead of three volumes of chlorine, on the supposition that the whole of the chlorine combined with oxygen to form hypochlorous acid. If only small quantities of hypochlorous acid were present, a slight excess of chlorine above the normal amount would have been found, whereas experiment always gave a slight loss of chlorine. By similar reasoning, it may be proved that none of the higher oxides of chlorine can be contained as admixtures in the gas.

The perfect purity of the electrolytic gas having been thus established beyond doubt, a question of no less importance presented itself; namely, does the combination of chlo-

rine and hydrogen take place in the dark at ordinary atmospheric temperatures, or is the presence of light a necessary condition? For the purpose of determining this question, the experiments just cited, calculated according to formula (1.), were repeated with the difference, that the analyses were made after the vessels containing the gaseous mixtures had remained for some hours in the dark. We obtained,—

Series of Experiments II.

	No.	Capacity of vessel. C_1.	Temperature. T.	Barometer. P.	n.	t.	t'.	a.	Chlorine found. C.	Chlorine calculated. $\frac{1}{2}C_1$.
21 hours in dark...	I.	190·24	11·1	0·7265	5	92·0	59·9	0·0024869	87·57	87·38
	II.	113·03	11·1	0·7265	3	89·9	34·4	0·0024869	51·50	51·91
	III.	73·42	11·1	0·7265	2	39·8	26·0	0·0024869	33·62	33·72
2 hours in dark...	IV.	51·60	12·9	0·7393	2	76·7	45·7	0·0024864	23·57	23·96
	V.	61·01	12·9	0·7393	2	71·9	15·2	0·0024864	28·14	28·34

These experiments show, that, after standing from two to twenty-one hours in the dark, only from 1 to 1·5 per cent. of the total chlorine had disappeared. The numbers in the two last vertical columns prove clearly that the electrolytic chlorine and hydrogen mixture remains unaltered when light is excluded, and that if the small differences which occur do not arise altogether from errors of observation, still their influence on the photochemical determinations is so small that it may be entirely neglected.

Having thus assured ourselves, in the foregoing experiments, that the electrolysis of aqueous hydrochloric acid affords a simple and reliable method of preparing a gaseous mixture of perfectly constant composition, which is unalterable in the dark, but sensitive when exposed to light, it now only remained carefully to examine all the circumstances which could possibly modify the action of light on the mixture, so that in the determinations they may be either entirely removed or allowed for in the calculations.

As we do not deem it necessary here to enumerate the various preliminary experiments which at last led us to the wished-for result, but tried our patience for upwards of half a year, we proceed at once to the description of the instrument by means of which we have been ultimately enabled to exclude all disturbing causes from our determinations, and have succeeded in referring the chemical action of light, not only to a comparative, but, as we shall show in a later section, even to an absolute measurement.

The arrangement of the apparatus we have employed is as follows:—The glass tube, *a*, fig. 2, blown before the glass blowpipe, serves for the electrolysis of the hydrochloric acid, and contains two electrodes of carbon, which are connected with three or four common-sized zinc-carbon elements, C, by means of the platinum wires *bb* melted through the glass. Between this vessel *a* and the battery C, is placed the gyrotrope D, by means of which the resistance in the circuit can be at any time greatly increased; this is instantly effected by reversing the balance-wire, so that the current passes through the small decomposing cell *d* containing a very dilute acid. By including and excluding

this cell from the circuit, the evolution of gas in the vessel *a* may be at any moment brought to a maximum or minimum amount. The chlorine and hydrogen mixture evolved by the current passes from the vessel *a*, through an accurately ground-glass joint rendered perfectly air-tight by a layer of water contained in the cup *g*, into the washing-bulbs *w*, partly filled with water, whence it traverses the tube furnished with the glass stopcock *h*. Between this stopcock and the horizontal tube K, is placed the vessel *i*, in which the gas is exposed to the action of the light. This insolation-vessel contains from 2 to 3 cub. cent. of water, and is attached to the tubes at either end by air-tight ground-glass joints, so that when the glass stopcock is open, the gas bubbles through the water contained in the vessel *i*, then passes into the tube K and through the water contained in the vessel *l*, and finally escapes into the condenser E, in which all the chlorine is absorbed by alternate layers of charcoal and slaked lime. When the gas is allowed to pass through the whole apparatus, the liquids contained in the vessels *a*, *w*, *i* and *l* become by degrees saturated; the composition of the free gas undergoes therefore a constant alteration, which diminishes in proportion as the liquids approach the point of saturation. As soon as this point of saturation is reached, and the equilibrium of absorption established, the composition of the free gas remains perfectly constant, and is represented by one volume of chlorine to one volume of hydrogen.

The gas in different parts of the apparatus is subject to different pressures. At the surface of the carbon-poles the gas is under a pressure of the columns of liquid in *a*, *w*, *i* and *l*; at *f*, of the columns in *w*, *i* and *l*; at *h* to the pressure of *i* and *l*; and lastly, in the insolation-vessel and in the observation-tube *ss*, to the pressure of the column of liquid in *l*. Any alteration in the pressure at any one of these parts of the apparatus causes an alteration in the composition of the free gas, and necessitates a fresh saturation, in order to regain the required equilibrium. Hence the pressure must be kept constant, not only whilst the observations are being made, as the nature of the apparatus requires, but also during the time which elapses between each set of observations. For this purpose the following arrangement was made:—As soon as the volume of gas to be employed for the observations has been shut off by closing the stopcock *h*, by which the pressure in no part of the apparatus is altered, the current of electricity is weakened by reversing the balance-wire of the gyrotrope, and thus the evolution of gas in the decomposing cell *a* is rendered very slight. This feeble current of gas, which merely serves to keep the liquids in *a* and *w* in a state of equilibrium, cannot escape by the stopcock, but finds its way through a small manometer, *m*, fused on to the tube *f* and containing water, and thence by means of a caoutchouc connector into the tube *p*, which dips under water in the bottle F. This bottle serves as a regulator of the pressure, which may at pleasure be diminished or increased by raising or depressing the tube *p* in the liquid. From this bottle, F, the gas passes into the second charcoal and lime condenser, G.

In order to protect the gas contained in the insolation-vessel from alterations of volume ensuing from external heat, either from the source of light or from the body of the observer, we have employed several arrangements of screens, which we shall more fully

describe when treating of the testing of the instrument. For the sake of simplicity, in figure 2 only a part of these screens has been represented; namely, the screen L, the convex lens M, and the cylinder N, closed by plates of glass and filled with water. In all the observations, in addition to these arrangements, the double metallic screen, fig. 4, afterwards described, was employed.

The following is the method adopted for filling the apparatus. The washing-bulbs *w*, the insolation-vessel *i*, and the small vessel *l*, having been supplied with the proper amount of water, the evolution-tube *a* is filled with hydrochloric acid of about sp. gr. 1·148, and the neck carefully placed upon the ground end of the tube *f*. The lower end of the tube *a* dips in mercury, which acts as a spring, pressing the tube against the end of the washing-bulbs *f*; the upper part of the tube is surrounded by water, to prevent the platinum wires fused into the glass from being heated by the passage of the electricity. A current from three or four elements is now led through the acid by means of the gyrotrope; the mixed gases are rapidly evolved, and can pass in two directions, according to the depth to which the tube *p* dips under the level of the water in the bottle F. If the lower end of this tube is placed at such a depth in the liquid that the sum of the pressures *rr*, *vv* is larger than the sum of the pressures of the columns of liquid at *w*, *i* and *l*, the gas will pass through *f*, *w*, *h*, *i*, *l* into the condenser E, supposing, of course, that the stopcock *h* is open. If, on the contrary, the tube *p* be drawn up through the caoutchouc cap *t*, in which it moves air-tight, until the sum of the pressures of the columns *rr* and *vv* is less than that of *w*, *i* and *l*, the gas takes the direction *m*, *vv*, *p*, *rr*, and issues into the condenser G. The tube *p* is usually fixed in such a position that the slightest displacement is sufficient to cause the gas to pass in either direction. During the evolution of the gas and saturation of the liquids, the equilibrium between the free and absorbed gases and the complete expulsion of the air, is effected by allowing the gas to pass through the apparatus in both directions. The apparatus being thus filled with the sensitive gas, and the stopcock *h* shut, the observations are made by measuring the diminution of the volume of enclosed gas on the scale of the horizontal tube *ss* of accurately determined capacity.

It being necessary, as we shall hereafter show, in order completely to saturate the 10 or 12 grammes of water contained in the whole apparatus, to evolve not less than from 6 to 10 litres of the electrolytic gas, we found that the platinum wires in the evolution-tube were so quickly acted upon by the chlorine, that they had to be renewed after two or three days' use. This continual reparation of the apparatus was the more inconvenient, because, in the original saturation of the water, the gas has to be led through the apparatus for a period of from three to six days. After many fruitless attempts, we have at last overcome this difficulty by the following arrangement of the carbon-poles. Small carbon points or plates are cut off from the cylinder of a carbon-zinc element which has been long in use, and after they have been boiled in aqua regia, they are heated to whiteness in a stream of dry chlorine until all sublimation of chloride of iron or other volatile chlorine compounds has completely ceased. Platinum wires, whose ends

have been sharpened and hardened by hammering, are then firmly fixed into small holes bored in the carbon-poles by a needle, and over the platinum wire is placed a thread of glass drawn out before the blowpipe. This thread is melted in the blowpipe flame at the point at which it touches the carbon, and pressed down over the carbon and wire so as to form a button of glass, as is seen in fig. 3, Plate XVIII. The whole wire and glass thread is then slowly drawn through the flame, so that the glass is completely melted upon the platinum wire, forming an enamel throughout its whole length, and completely protecting it from the action of the chlorine. In order to preserve the ends of the wires which are fixed into the carbon, from the action of the acid, it is sufficient to impregnate the upper end of the carbons with a little white wax. If the precaution be adopted of fusing the wires into the upper end of the tube, as is seen in fig. 2, so that the necessarily unprotected part of the wire does not come in contact with the liquid, no destruction of the electrodes, or of the wires need be feared, even when the instrument is in continual use. Another very important precaution which must be attended to regarding the electrodes, is that the carbon-poles must never be allowed to come in contact with the free gas, for carbon prepared in the manner described, acts upon the mixture of chlorine and hydrogen exactly as spongy platinum acts upon the mixture of oxygen and hydrogen. In the course of our experiments we have had many opportunities of observing this catalytic action of the carbon, and more than once has this action proceeded so rapidly that inflammation of the gaseous mixture has occurred, and the apparatus was shattered with a sharp explosion.

The varying stand of the water in the horizontal tube *ss* can be conveniently observed by the light of a distant candle or lamp, which exerts no perceptible action on the sensitive gas.

The following conditions, necessary in order to obtain exact photo-chemical measurements with a mixture of chlorine and hydrogen, are completely fulfilled in the apparatus just described.

(1) The gas which is exposed to the action of the light consists of exactly equal volumes of chlorine and hydrogen.

(2) It does not contain any foreign impurities.

(3) Throughout the whole apparatus it does not come in contact with caoutchouc or other organic matter which could alter its composition.

(4) The change of pressure during the whole series of observations is quite imperceptible, owing to the volume of water in the vessel *l* being much larger than that in the observation-tube *ss*.

(5) The statical equilibrium between the absorbed and free gases can be perfectly established.

(6) The layer of water in the insolation-vessel *i* does not alter its position during the experiment, so that the hydrochloric acid formed by the action of the light is always absorbed under precisely the same conditions.

(7) The vessel *i* is covered outside with black varnish, reaching as high as the level of

the water, so that no alteration in the free gas can ensue from the combination of the chlorine and hydrogen dissolved in the water.

(8) The volume of uninsolated gas which enters the vessel *i* from the horizontal tube *ss* during the observations, is very small compared with the total volume of gas exposed to the light. In most of the experiments it amounted to about $\frac{1}{1000}$ of the total volume for every division on the observation-tube.

(9) The disturbing action of radiant heat is fully eliminated.

Having thus explained the arrangement of our instrument, we now pass to a description of the experiments instituted for the purpose of testing its efficacy.

The first action of light on the electrolytic chlorine and hydrogen is accompanied by a phenomenon of a very peculiar nature, which we have named Photo-chemical Induction, to the full consideration of which a special section of this investigation has been devoted. On exposing the sensitive gas to a constant source of light, the chemical action does not at once commence to its full extent; to begin with, it is either not perceptible or of very slight amount, but gradually increases until a permanent maximum has been reached. The following experiment shows this gradual increase of the original action.

Series of Experiments III.

Time in minutes.	Corrected observation.	Action every minute.
0	100·0	
1	100·5	0·5
2	102·1	1·6
3	102·6	0·5
4	103·2	0·6
5	105·3	2·1
6	119·9	14·6
7	139·1	29·2
8	170·2	31·1
9	200·6	30·4

In the first minute the action was only 0·5, and after the lapse of eight minutes it had reached 31·1, at which it remained constant. The enlargement of the combining power of chlorine and hydrogen is also observed, when the amount of light falling on the gas is suddenly increased. This peculiar action of the light determines the method which must be followed in the measurement of the chemical rays. The observations can never be made use of until the successive differences have become constant. In order to form an estimate of the trustworthiness of an experiment from the regularity of the differences, and to eliminate as much as possible the slight variations in the working of the instrument, it is almost always sufficient to read off the stand of the liquid every minute, and to take the mean of ten or twelve such observations. The following experiment, calculated in this manner, gave the mean values as below:—

Series of Experiments IV.

Time.	Corrected observation.	Action of the light in one minute.	Mean action from three observations.
0	102·1		
1	104·2	2·1	
2	107·3	3·1	
3	113·5	6·2	
4	125·0	11·5	
5	135·4	10·4	
6	145·9	10·5	
7	159·3	13·4	
8	172·5	13·2	13·23
9	185·6	13·1	
10	198·8	13·2	
11	211·9	13·1	13·50
12	226·1	14·2	
13	237·2	11·1	
14	251·4	14·2	13·35
15	266·1	14·7	

The mean of the three values in the fourth column is 13·36, giving the following differences:—

Found.	Mean.	Difference.
13·23 . . .	13·36 . . .	+0·13
13·50 . . .	13·36 . . .	−0·14
13·35 . . .	13·36 . . .	+0·01

From these numbers, it is seen that by the combination of such a series of observations, the small irregularities, arising chiefly from capillary resistance in the tube *ss*, are satisfactorily eliminated.

In order to ascertain the limits of accuracy to which the indications of our instrument reached, it was necessary to determine, first, the length of time during which the electrolysis of a given volume of hydrochloric acid yields a gas of uniform composition; and secondly, the circumstances which influence the establishment of the statical equilibrium between the free and absorbed gases in the liquids contained in the apparatus. For this purpose we required a source of light which should remain constant for a considerable length of time; this we obtained by employing a coal-gas flame of constant dimensions issuing from Scorr's annular burner. A millimetre scale was etched upon the glass chimney of the lamp by hydrofluoric acid, and in order to avoid the errors arising from parallax, the height of the flame was read off on the scale through a telescope placed at the same height.

The amount of the chemical rays emitted by any flame depends upon its dimensions and its luminous intensity. We eliminated all sources of error arising from variations in the dimensions of the flame, by cutting off the greater part of the light with a screen, and thus employing only a small portion of the most intensely luminous rays as the source of light. In order that the intensity of the flame should remain as much as possible constant during our experiments, we determined the height of the flame of the

Scott's burner at which an alteration in the dimensions of the flame produced the least variation in the luminous intensity. In the following experiments, conducted for this purpose, the intensity of the light from the most luminous parts of flames of different dimensions was measured by allowing the various amounts of light to fall through the same opening upon the sensitive gas in the instrument. The first horizontal division in the following Table contains the heights of the various flames measured by the millimetre scale on the chimney; the remaining divisions give the amount of chemical action produced by radiation from equal surfaces of the various flames.

Series of Experiments V.

Height of the flame.	82 millims.	95 millims.	110 millims.	133 millims.
Observed intensity of the chemical rays	21·0	18·77	15·84	9·0
	22·0	19·17	17·51	9·0
	19·0	17·61	16·59	9·0
	21·0	19·23	17·20	9·0
Mean intensity	20·75	18·69	16·78	9·0

These experiments give the following variation in the intensity of the chemical rays for alteration in the heights of different-sized flames:—

Height of flame.	Intensity of chemical rays.	Difference of height.	Difference of intensity.
130 mm.	9·00		
110 „	16·78	20·0 mm.	7·78
95 „	18·69	15·0 „	1·91
82 „	20·75	13·0 „	2·06

Hence it is seen that when the flame is from 95 to 110 millims. in height, alterations in the size of the luminous cone are accompanied by the smallest alterations in the intensity of the chemical rays. For this reason, in all the experiments in which the annular burner was employed, we placed the flame at a height of about 100 millimetres. As the limit of height within which the flame could be kept constant was about 6 millimetres, and as according to the above experiments a variation of 15 millims. in the height of the flame causes an alteration of $\frac{1}{10}$ of the total intensity, the errors in the determinations arising from variations in the size of the flame could not amount to more than 4 per cent., supposing that no other errors occurred from differences in the composition of the coal-gas employed. The accurate agreement of a large number of photochemical measurements made at different times with the Heidelberg coal-gas, for the preparation of which the process remained unaltered during the period of our experiments, proved, as we shall show, that no appreciable difference in the intensity of the chemical rays occurred in the flame of coal-gas prepared at various times. The next point to determine was the influence which the strength of the electrolysed hydrochloric acid, as well as the intensity of the current employed to decompose it, exerted upon the composition of the sensitive gas; and the length of time which is necessary to lead the gas

through the apparatus before all the air is expelled, and the requisite equilibrium between the absorbed and free gases established.

In order to answer these questions, we led a current of gas for several days through our instrument, freshly filled, observing from time to time the amount of action effected by the constant source of light. We obtained the following results, which were similarly observed each time the apparatus was refilled.

Series of Experiments VI.

Experiment 1.				Experiment 2.			
Duration of the saturation.	Action for each minute from twelve observations.	Duration of the saturation.	Action for each minute from twelve observations.	Duration of the saturation.	Action for one minute from twelve observations.	Duration of the saturation.	Action for one minute from twelve observations.
First day.		Fourth day.		First day.		Third day.	
h m		h m		h m		m	
1 9	0·00	30	9·65	1 5	0·00	5	9·90
15	2·49	1 1	18·39	1 35	0·00	10	10·10
15	3·06	15	23·06	33	2·08	30	11·52
Third day.		5	22·26	5	2·08	15	12·83
18	5·57	27	23·22	Second day.		20	12·92
16	5·21	10	25·63	30	5·93	15	13·64
25	5·36	1 8	19·66	20	6·41	10	13·60
39	5·01	15	17·28	15	7·40		
31	4·86	1 8	14·60	1 0	8·43		
39	5·14	Fifth day.		10	8·39		
1 19	3·32	15	26·49	35	6·60		
19	2·47	2	25·32				
29	2·12	2	27·19				
20	2·43	2	26·95				
		2	26·08				
		2	25·87				
		2	27·43				

From the first of these experiments, in which, as in all the later ones, hydrochloric acid of spec. grav. 1·148 was used, it is seen that after the gas had passed one hour through the apparatus the action was 0·0; after two hours' saturation the action rose to 5·57, at which amount it remained stationary after the gas had again passed through for $2\frac{1}{2}$ hours; on further saturation the action diminished to about 2·43. The difference between the weight of the decomposing cell before and after the experiment, showed that 3755 cub. cent. of the chlorine and hydrogen mixture had passed through the vessel *i*, of about 7 cub. cent. capacity and containing 1·8 cub. cent. of water, and that the quantity of hydrochloric acid contained in the electrolyte had diminished from 30 to 20·5 per cent.

The experiment, which thus far had occupied two days, was extended over the following days, the diluted acid being replaced by concentrated acid. After the gas had passed through the instrument for half an hour the action increased to 9·65, and after saturation for another quarter of an hour the action amounted to 23·06, at which number it remained constant for several successive hours. On further saturation the action diminished to 14·6. The volume of gas which had passed through the insolation-vessel

amounted to 2544 cub. cent., and the quantity of anhydrous acid contained in the electrolytic liquid had decreased from 30 to 21·4 per cent. On continuing the experiment on the fifth day with fresh hydrochloric acid, the action attained a maximum of 26·49 after the gas had been led through for fifteen minutes. This maximum action remained constant in all subsequent experiments with renewed quantities of acid, and in each case a short saturation sufficed to bring the gas to the maximum degree of sensibility, in which the action was 26·49.

The second series of experiments gives, as may be seen, the same general results as the one just described.

From these and a large number of similar experiments which we have made in the course of the testing of the instrument, the following conclusions may be drawn:—

(1) When the quantity of hydrochloric acid contained in the liquid electrolyte has decreased from 30 to about 23 per cent., the evolved gases are no longer fitted for comparative photo-chemical measurements.

(2) In order to attain the first constant action, at least 2000 cub. cent. of gas must be passed through the small vessel of 7 cub. cent. capacity, containing 1·8 cub. cent. of water.

(3) By continued saturation the action increases up to a certain point at which a permanently constant maximum is reached, which in our apparatus was not attained until more than 6000 cub. cent. of the chlorine and hydrogen mixture had passed through the insolation-vessel.

(4) The attainment of the necessary equilibrium in the apparatus is much accelerated by allowing the water to remain in contact with the gases during the night. Were the saturation continued without intermission, a much larger quantity of gas would have to be led through the apparatus before the maximum action is attained, than was required in the foregoing experiments. All the attempts which we have as yet made to shorten this most tedious period of preparation necessary for the execution of photo-chemical measurements with chlorine and hydrogen have proved fruitless.

Long experience has shown that the apparatus is most conveniently filled by the following manipulation. The operation is commenced in the morning by filling the evolution-tube *a*, fig. 2, with about 55 cub. cent. of hydrochloric acid of spec. grav. 1·148, and leading through the apparatus the gas evolved from this cell at such a rate that about two bubbles pass through the washing-bulbs and insolation-vessel every second of time. In the afternoon, the acid, which can be reduced by 7 per cent. of its original volume, is changed, and the operation repeated, with the important exception that the gas evolved during the first hour, containing air and excess of hydrogen, is not allowed to pass through the apparatus, but escapes through the manometer by the tubes *m*, *vv*, *p* into the condenser G. The instrument, protected from all exposure to light, is then allowed to stand during the night with the stopcock closed, and the operations are commenced in the morning anew, care being always taken to allow the impure gas to escape during the first hour of evolution. During the intervals of saturation, the photo-chemical

sensibility of the gaseous mixture is determined by observing the action produced by a gas-flame of constant dimensions.

As soon as a maximum action has been reached, which generally occurs in from three to six, sometimes even from eight to nine days, according to circumstances, the instrument can be used for comparative measurements extending over a period of several months, and requires only a short preparation to enable it to give correct results throughout the day. All that is necessary for this purpose, is to pass the gas evolved from fresh hydrochloric acid, for an hour through the manometer m, and afterwards the glass stopcock h is opened, and the gas allowed to pass through the apparatus for from fifteen minutes to an hour, after which the maximum action has generally been attained. As soon as the stopcock is shut, in order that the observations may be begun, the balance-wire of the gyrotrope is reversed, so that the hydrochloric acid is constantly decomposed by a very feeble current. If this precaution be neglected, the equilibrium of the gases dissolved in the acid is soon destroyed, owing to a contact-combination of the chlorine and hydrogen taking place on the surfaces of the carbon-poles. Above all, access of air in the washing-bulbs and in the insolation-vessel must be most carefully avoided. If the insolation-vessel be removed from the apparatus for merely a few moments, a saturation lasting from two to six days is necessary before the maximum degree of sensibility is again reached. The amount of foreign gas which is sufficient to render the indications of the instrument incorrect, cannot approach to a billionth part of the total volume of gas. We shall have occasion to return to this remarkable fact in a later part of the research; but we may here remark, that when the maximum action has been reached, the explosive nature of the gas is so great, that a thin glass bulb of the size of a pigeon's egg filled with the gas, immediately explodes on exposure to the diffuse light of an overclouded sky at an open window. On taking the apparatus to pieces, portions of the gas have several times exploded on exposure to evening light, even after the sun had disappeared below the horizon.

Having thus determined the conditions which must be fulfilled in order to obtain comparative results, we proceed to the examination of the influence which the heat radiated from the source of light, and the heat of combustion evolved by the combination of the gases, exert upon the indications of the instrument.

The relation between the volumes of the insolation-vessel used in these experiments, and the volume occupied by one division on the observation-tube, was as 7430 to 1. Hence, if this vessel were heated from 0° C. to 0°·0366 C., the whole mass of gas would be increased by a volume represented by one division of the tube. Our instrument is therefore not only a photometer, but also a very sensitive air-thermometer, and all experiments with it must be conducted in a space within which the changes of temperature are not large enough to cause any perceptible error.

This condition was fulfilled by experimenting in a room of which the outer wall was at no time of day exposed to the direct sunlight, and the only window completely closed from without.

In order to eliminate the more serious errors arising from the radiant heat of the source of light, we placed, as we previously mentioned, a glass cylinder, N, Plate XVII. fig. 2, closed by plates of glass and filled with water, between the source of light and the insolation-vessel *i*. In addition to this water-cylinder, the insolation-vessel was further protected from access of radiant heat by the double metallic screen, fig. 4, placed between the insolation-vessel and the water-cylinder N, fig. 2. The apertures in the middle of each sheet of metal were closed by perfectly clear thin plates of mica, behind which was placed the insolation-vessel *i* contained in the blackened cap *c* and covered by the lid *d*. Lateral slits in the cap, with others corresponding in the lid, allowed the tubes *oo* to pass out, and currents of air were excluded by stopping the lateral slits with black wool. As it is important to observe the height of the flame whilst the absorption is being read off on the scale, a small hole, *l*, was made in the lid *d*, through which the point of the flame could be easily seen.

By this arrangement all errors arising from radiant heat were completely excluded, for it was found that if the vessel *i* were filled with a gas which is not sensitive to light, the stand of the water in the observation-tube remained constant, whether the gas were exposed to the light or not.

It now only remained to determine the effect produced on the indications by the heat evolved from the combination of the gases. A phenomenon of great importance here presented itself. If the insolation-vessel be suddenly darkened during the normal working of the apparatus, it is found that the action does not instantly cease, but that the volume s_0 of gas read off at the moment of darkening t_0 diminishes after the lapse of the times $t_1 t_2 \ldots t_n$ to the volumes $s_1 s_2 \ldots s_n$, so that for the times $(t_1 - t_0)(t_2 - t_1)$, &c., after the light had been cut off, a corresponding decrease of volume $(s_0 - s_1)(s_1 - s_2)$ is found. The time during which this decrease of volume takes place is a very short one, as may be seen from the following experiments, made at 20°·9 C. and 0^m·7497 Barom.

Series of Experiments VII.

Experiment 1.					Experiment 2.					Experiment 3.				
	Time in seconds. t.	Observations on the scale.* s.	$t_1 - t_0$.	$s_0 - s_1$.		Time in seconds. t.	Observations on the scale. s.	$t_1 - t_0$.	$s_0 - s_1$.		Time in seconds. t.	Observations on the scale. s.	$t_1 - t_0$.	$s_0 - s_1$.
	"		"			"		"			"		"	
Insolated	0	241·0			Insolated	0	293·5			Insolated	0	339·5		
	60	253·5	60	12·5		60	307·5	60	14·0		30	322·0	30	17·5
	120	265·5	60	12·0		120	321·5	60	14·0		60	306·0	30	16·0
Darkened	180	278·0	60	12·5				10	1·5	Darkened	90	301·0	30	5·0
	190	279·5	10	1·5	Darkened	130	323·0	10	0·3		120	300·0	30	1·0
	200	279·7	10	0·2		140	323·3	10	0·1		150	300·0	30	0·0
	210	279·8	10	0·1		150	323·4	10	0·0					
	220	279·8	10	0·0		160	323·4	10	0·0					
	230	279·8	10	0·0		170	323·4							
	240	279·8	10	0·0										

* The increasing numbers on the divided arbitrary scale of the instrument correspond to a diminution of the volume of the total gas.

Experiment 1 shows that in the first 10″ after the withdrawal of light, the volume of gas diminished 1·5 volume, in the following 10″ 0·2, in the next 10″ 0·1 volume, and that then no further diminution was perceptible. In experiment 2 the diminution of volume amounted in the first 10″ to 1·5, in the following 10″ to 0·3 volume, and in the next 10″ to 0·1 volume, after which the gas preserved its constant volume. In experiment 3 the volume diminished after the first 30″ by 5·0 volume, after the second 30″ by 1·0 volume, and then the water in the tube remained stationary. This subsequent action, after removal of light, may arise from several causes.

In the first place, the slow combustion of the chlorine and hydrogen produces a known amount of heat, and the increased degree of temperature W_o, arising therefrom, becomes constant when the amount of heat lost by radiation and conduction is equal to the amount added by combustion of the gases. The gas in the insolation-vessel is therefore always warmer than the surrounding medium by a temperature $(W_o - W_s)$ where W_s represents that of the air. When the insolation, and with it the supply of heat, ceases, the gas cools down from W_o to W_s, and the contraction ensuing from this diminution of temperature accounts for at least a portion of the action observed after withdrawal of the light. This subsequent contraction may, in the second place, be caused by the non-instantaneous absorption, by the water, of the hydrochloric acid formed by the slow combustion of the chlorine and hydrogen. In this case, as with the heat, an equilibrium between the amount of hydrochloric acid formed and that absorbed by the water, will ensue as soon as the quantity of acid absorbed is equal to that produced, and a constant amount of hydrochloric acid will remain in the gaseous mixture. With the exclusion of light the supply of hydrochloric acid ceases, and this constant amount of acid is absorbed by the water, causing a corresponding diminution in the volume of the gaseous mixture. Lastly, it is possible that the combination of the chlorine and hydrogen does not instantly cease with the exclusion of light, but that by insolation the gases are brought into a condition of increased affinity, which continues for a certain time in the dark. As these three conditions might exert a great influence upon the action of our instrument, we thought it necessary to determine their relative importance.

The three questions to be solved were,—

1st. What is the influence exerted by the slow combustion of the chlorine and hydrogen upon the action of the instrument?

2ndly. To what extent can the hydrochloric acid formed by insolation, but not absorbed by the water, cause an irregularity in the indications?

3rdly. Is the presence of light necessary for the combination of the chlorine and hydrogen, or can this combination occur for a short time after exclusion of light?

The following considerations prepare the way for the solution of these questions.

When a sensitive mixture of chlorine and hydrogen is exposed to the action of light, the temperature of the gas gradually rises up to a point at which it remains stationary, owing to the amount of heat absorbed being equal to that evolved from the combustion of the gases. As the amount of heat evolved by the combination of the chlorine can be calculated from experiment, we only require to determine the amount of heat absorbed

during the same space of time in order to obtain the value of $(W_0 - W_a)$, or the difference of temperature between the gas and the surrounding medium. If the volume of gas burnt in the unit of time immediately before exclusion of light, measured in cubic centimetres at the temperature W_0 and pressure P, be called R, the weight in grammes of the chlorine combined with hydrogen in the unit of time is found from the formula

$$\frac{s_1}{773} \cdot \frac{P}{0{\cdot}76\,(1+0{\cdot}00366\,W_0)} \frac{R}{2} = A,$$

where s_1 is the specific gravity of chlorine. If, therefore, the amount of heat evolved by the combustion of 1 grm. of chlorine with hydrogen, measured by the rise of temperature of 1 grm. of water, be called C, the amount of heat evolved from combustion of the chlorine in a unit of time expressed in units of heat is AC.

The quantity of heat absorbed can be calculated when the cooling-velocity of the insolation-vessel for a degree Centigrade is known. In order to determine this value, which we may call a, we exchanged our insolation-vessel for another of the same dimensions, in the interior of which a fine platinum wire was fixed by fusion through the glass. When a weak current of electricity was passed through the wire, the sensitive gas being protected from the action of the light, an immediate expansion by heat occurred, which, as soon as the current was interrupted, gave place to a corresponding contraction. As the weight of the insolation-vessel was at least 1000 times greater than that of the gas contained in it, the amount of heat communicated from the gas to the vessel may be regarded as infinitely small.

The observations for the determination of a were thus conducted. First, the constant position w_a of the water in the tube was noted; then, after the gas had been expanded to the position w_0 by means of the current passing through the platinum wire, the electricity was cut off and the time observed, by means of a pendulum vibrating half seconds, during which the gas diminished in volume from w_0 to w_1. According to this method of observation, it was only necessary to determine the interval of time during which the volume of gas diminished a constant amount; hence results of sufficient accuracy were obtained, notwithstanding the rapidity at which the gas cooled down.

The determination gave the following numbers:—

Series of Experiments VIII.

	Exp. 1.	Exp. 2.	Exp. 3.	Exp. 4.	Exp. 5.
w_0	320	355	427	420	427
w_1	310	340	420	410	420
w_a	300	330	410	400	410
Time $t_0 - t_1$ in which the change $w_0 - w_1$ occurred	1″·1	0″·9	1″·1	1″·2	1″·1

The figures in the first three horizontal divisions may be supposed to represent temperatures measured on a relative but arbitrary scale.

The excess of temperature of the gas over that of the surrounding medium was therefore, for the time $t_0 t_1 \ldots$, $w_0 - w_n = u_0$, $w_1 - w_n = u_1 \ldots$; and if a represents the cooling velocity for the thermometric unit and this difference of temperature, we have

$$\frac{du}{dt} = -au.$$

The integral of this differential equation is log nat $u = -at +$ const., and according to definition of u_0, u_1, t_0 and t_1, log nat $u_0 = -at_0 +$ const., and log nat $u_1 = -at_1 +$ const. Hence

$$a = \frac{\text{log nat } u_0 - \text{log nat } u_1}{t_1 - t_0}.$$

The values of a, calculated from the experiments according to this formula, are as follow:

	a.	Difference.
Exp. 1 . . .	0·630	−0·047
Exp. 2 . . .	0·567	−0·110
Exp. 3 . . .	0·806	+0·129
Exp. 4 . . .	0·578	−0·099
Exp. 5 . . .	0·806	+0·129
Mean . .	0·677	

Although the difference between each value of a and the mean is considerable, yet in the calculation of the temperature this variation has very little influence, a difference of 0·11 in the value of a causing an error of only a few hundredths of a centesimal degree.

If the difference of temperature between the gas and the surrounding medium at the moment of exclusion of light is $(W_0 - W_n)$, measured in centesimal degrees, the cooling-velocity immediately after darkening is $a\,(W_0 - W_n)$. If, further, the volume in cub. cent. of the gaseous mixture at the moment of closing is called V_0, its specific gravity s, the pressure to which it is subject P, its temperature W_0, and its specific heat by constant pressure compared with water c; and if the weight of the platinum in the vessel is p, and its specific heat c_1, the decrease of temperature in the unit of time during the cooling, expressed in units of heat representing the amount required to raise one cub. cent. of water one degree Centigrade, is found from the expression

$$\left\{\frac{P}{0{\cdot}76\,(1+0{\cdot}00366\,W_0)}\cdot\frac{sc}{773}\,V_0 + c_1 p\right\} a\,(W_0 - W_n).$$

As the amount of heat evolved by combustion is equal to that lost by radiation, we have

$$\frac{s_1}{773}\cdot\frac{P}{0{\cdot}76\,(1+0{\cdot}00366\,W_0)}\,\frac{RC}{2} = \left\{\frac{sc}{773}\cdot\frac{P}{0{\cdot}76\,(1+0{\cdot}00366\,W_0)}\,V_0 + c_1 p\right\} a\,(W_0 - W_n).$$

As the values of W_0 and W_n differ only by a few hundredths of a degree Centigrade, it is possible to replace the value $(1+0{\cdot}00366\,W_n)$ for $(1+0{\cdot}00366\,W_0)$ without exceeding the limits of the errors of observation. Hence if $A = \frac{s_1 PC}{2.773.0{\cdot}76\,(1+0{\cdot}00366\,W_n)}$, and $B = \frac{scP}{773.0{\cdot}76\,(1+0{\cdot}00366\,W_n)}$, we have

$$A.R=(B.V_0+c_1p)\,a\,(W_0-W_n);\text{ and hence}$$

$$W_0-W_n=\frac{A.R}{a\,(BV_0+c_1p)}.$$

In order to be able to employ the experiments communicated in Series VII., page 369, for the calculation of (W_0-W_n), the volumes read off on the divided scale were reduced to cub. cent., and placed in the following Tables:—

Series of Experiments IX.

Experiment 1.			Experiment 2.			Experiment 3.		
Time.	Volume in cub. cent.	Diminution of volume in one second.	Time.	Volume in cub. cent.	Diminution of volume in one second.	Time.	Volume in cub. cent.	Diminution of volume in one second.
Insolated 0′ 0″	4·20926		Insolated 0′ 0″	4·20255		Insolated 0′ 0″	4·20929	
1 0	4·19983	0·000157	1 0	4·19199	0·000176	0 30	4·19660	0·000423
2 0	4·19078	0·000151	Darkened 2 0	4·18143	0·000176	Darkened 1 0	4·18453	0·000402
Darkened 3 0	4·18136	0·000157	2 10	4·18036	0·000113	1 30	4·18075	0·000126
3 10	4·18023	0·000113	2 20	4·18008	0·000022	2 0	4·18000	0·000025
3 20	4·18008	0·000015	2 30	4·18000	0·000008	2 30	4·18000	0·000000
3 30	4·18000	0·000008	2 40	4·18000	0·000000			
3 40	4·18000	0·000000	2 50	4·18000	0·000000			
3 50	4·18000	0·000000						

The constants required for this calculation are,—

$s_1=2{\cdot}4531$ $\quad a = 0{\cdot}677$

$s = 1{\cdot}2612$ $\quad C = 669{\cdot}9$

$c_1=0{\cdot}0324$ $\quad P = 0{\cdot}7497$

$c = 0{\cdot}2114$ $\quad W_n=20^{\circ}{\cdot}9$ C.

$p = 0{\cdot}0120$

The values of A and B, and from these (W_0-W_n), calculated from these data, are found in the following Table, in which the variables V_0 and R are placed in the first and second horizontal divisions:—

	Exp. 1.	Exp. 2.	Exp. 3.
V_0	4·18136	4·18146	4·1845
R.....	0·000157	0·000155	0·0004128
A	0·96859	0·96859	0·96859
B.....	0·0003144	0·0003144	0·0003144
(W_0-W_n)...	$0^{\circ}{\cdot}13$	$0^{\circ}{\cdot}13$	$0^{\circ}{\cdot}35$

These results prove that the photo-chemical combustion of the chlorine and hydrogen mixture may be increased from 12 or 14 divisions in the minute up to 32 or 35, whilst the increase of temperature accompanying this increased action amounts only to $0^{\circ}{\cdot}2$ C. As various later experiments have shown that differences of several degrees in temperature do not affect the combining power of the chlorine and hydrogen mixture to any appreciable extent, we may answer the first question as follows:—

(1) That the heat evolved in the insolation-vessel from the photo-chemical combustion of the chlorine and hydrogen exerts no perceptible influence upon the combining power of these gases, or on the indications of our instrument; and

(2) That in the first few seconds after a change from one amount of light to another, irregularities in the indications of the instrument will be observed, but that these irregularities do not extend over a period longer than a few seconds, after which time a constant temperature in the insolation-vessel is attained.

Having thus calculated from the heat of combustion and from the cooling-velocity, the constant temperature which the gaseous mixture attains by exposure to various amounts of light, it is easy to solve the two other questions stated at the commencement of the inquiry. From the preceding considerations we have determined the value of $(W_o - W_\infty)$, or the excess of temperature of the gas above the surrounding medium. Hence the decrease of volume Δ, which the gas suffers in cooling from W_o to W_∞, is known. If this calculated decrease of volume Δ is equal to that actually observed, Δ_1, the whole of the phenomenon is explained by the contraction from diminution of temperature. If, however, the calculated decrease Δ is not so large as the observed decrease Δ_1, a certain diminution of volume $\Delta_1 - \Delta$ must be caused by absorption of the hydrochloric acid, or by a chemical action taking place after removal of light. From the following Table, in which the observed and calculated contractions are expressed as per-centage of the total volume, it is seen that the first supposition is the true one, as the values of $\Delta_1 - \Delta$ are extremely small, and all negative.

	Δ_1.	Δ.	$\Delta_1 - \Delta$.
Exp. 1.....	0·033	0·042	−0·009
Exp. 2.....	0·034	0·042	−0·008
Exp. 3.....	0·108	0·119	−0·011

Hence we may conclude,—

(1) That the amount of hydrochloric acid formed during the insolation is absorbed almost as quickly as it is produced, and therefore the insolated gas can contain only a trace of this substance; and

(2) That the chemical action ceases instantly on exclusion of the light, proving the absence of any perceptible amount of after-action.

The light emitted from a Scott's annular burner, although it gives, as we have shown, satisfactory results, and may serve for many photometric observations, is, however, not sufficiently constant to serve for more accurate measurements. We therefore endeavoured to find means to obtain a flame of coal-gas of constant dimensions and luminous intensity.

Flames of coal-gas issuing under the usual pressure of several lines of water, and burning free in the air, vary and flicker far too much to allow them to be made use of

for photo-chemical measurements. These variations of the flame can be best prevented by allowing the gas to issue from a small air-chamber, and by surrounding the flame with a screen so that all lateral currents of air are completely excluded.

We have found that by means of the arrangement represented in Plate XVIII. fig. 5, a flame of coal gas could be obtained of extremely constant dimensions and luminous power. AA_1, fig. 5, is a blackened tin box in the interior of which the gas-flame l burns. The bottom of the box is perforated to admit of an upward current of air, and the sliding front of the box is furnished with a screen of water, a, contained between two parallel plates of glass. The gas streams from the delivery-tube b into the small air-chamber c, and issues from the platinum burner with a pressure of about 1 millim. of water, as is seen by the small manometer g attached to the air-chamber. The dimensions of the flame are kept constant by regulating the supply of gas, so that the highest point of the flame always touches the thin platinum wire attached to a moveable ring fitting on to the divided tube h, upon which the height of the flame can be read off. In order to place the flame at any wished-for distance from the insolation-vessel, the tube b carrying the gas-burner can be moved through a holder, k, at the back of the box, and the distance read off on an etched millimetre scale. When the distance of the insolation-vessel from the flame has been once measured for a known position of the flame, a simple observation on the divided tube b gives the required distance. When the box was in use the top was covered on each side of the flame, leaving only a space of about two inches open. The dimensions of the flame burning in the box thus adjusted, when observed through the opening of the screen, l, fig. 4, were seen to remain perfectly constant, so much so that the light appeared to be emitted from a solid body.

We have employed this flame, which for the sake of brevity we shall call the standard flame, in the first place for the purpose of determining the constancy of the indications of the instrument at different times and under various circumstances. For this purpose, the chemical intensity of the light proceeding from standard gas-flames of the same heights was repeatedly measured at various intervals during sixteen days in the month of June 1856. In all these measurements, each one of which is the mean of ten independent observations, the height of the standard flame was 42·2 millims., and the distance between the flame and insolation-vessel was 216·0 millims.

Series of Experiments X.

Time of the observations.	Action in one minute.	Deviation from mean value.
11th June ...	14·00	+0·01
12th " ...	14·26	+0·35
13th " ...	13·80	−0·11
19th " ...	13·83	−0·08
21st " ...	13·88	−0·03
26th " ...	13·71	−0·19
Mean value ...	13·91	

In order more fully to test our instrument, we made, some time later, other determinations which are of interest in a different point of view. It is well known that supporters are still found for the idea that photo-chemical phenomena are not to be ascribed to the action of the common luminous rays of a particular refractive index, but to a peculiar agent differing entirely from light, and possibly subject to other laws. In order to free our investigations from any objections which might be made from this point of view, we have endeavoured to avoid as much as possible applying the general laws of light to the chemically active rays without previous experimental proof. On this account we thought it advisable, whilst testing the reliability of the instrument, to prove that the action of the chemical rays diminishes inversely as the square of the distance from the point from which the light emanates. These experiments were conducted by determining the amount of chemically active rays which fall upon the insolation-vessel from the standard flame placed at various known distances in the blackened box. The first vertical column of the following Table contains the distance between the standard flame and the middle point of the insolation-vessel; the second column the reciprocals of the squares of these distances; the third the chemical action, W, measured by the instrument; and the fourth the amount of action, W_1, calculated according to the distance by means of the formula

$$\frac{\Sigma W}{\Sigma \frac{1}{r^2}} \cdot \frac{1}{r^2} = W_1.$$

Series of Experiments XI.

Experiment 1.						Experiment 2.					
	r.	$\frac{1}{r^2}$	W.	W_1.	Difference.		r.	$\frac{1}{r^2}$	W.	W_1.	Difference.
1	0·3900	6·575	3·98	4·11	−0·13	1	0·3900	6·575	4·43	4·32	+0·11
2	0·3310	9·127	5·56	5·70	−0·14	2	0·3315	9·108	6·07	6·00	+0·07
3	0·2495	16·064	10·33	10·05	+0·28	3	0·2150	21·634	14·07	14·24	−0·17

In these last experiments the standard flame was adjusted at exactly the same height as was the case in Series X., when the distance from the flame to the insolation-vessel was $0^m{\cdot}216 = r_1$. These series of determinations may therefore be compared with one another by means of the formula $W_s = \frac{W r^2}{r_1^2}$, from which the following values are obtained for the corresponding action at the distance $0^m{\cdot}216$, as employed in Series X.

(1) 14·44,
(2) 14·28,
(3) 14·11.

The last number agrees very nearly with the mean action 13·91 of Series X., the difference being scarcely larger than that found among the single determinations of Series X.

In order to have still more striking proof of the accuracy of the indications of our instrument, we have compared the results obtained from the standard flame of coal-gas in June with those of the same-sized flame in September, when all the liquids contained in the apparatus had been renewed. The standard flame, of the same size as previously employed, was placed at a distance of $0^m{\cdot}1365$ from the insolation-vessel, and the instrument was furnished with a new observation-tube, whose degrees were 3·40 times larger than that employed in Series X. An action of 10·17 degrees per minute was obtained as a mean of 96 well-agreeing observations: this 10·17, reduced to the former scale, represented 34·58 degrees, which gives for the distance $0^m{\cdot}216$ the action 13·80, which differs from the June mean 13·91 by only 0·11.

The exact agreement of these numbers at such distant periods of time is the more remarkable, as it could not have been expected that the coal-gas generated by a common manufacturing process should possess so constant a composition that it could be used during the space of several months as a fixed unit for the measurement of photochemical action. In order to form an idea in how far the intensity of the chemical rays emanating from a flame is dependent upon the composition of the gases undergoing combustion, it was necessary to determine the variation in the composition of the gas employed in the experiments.

Although we have not analysed the gas employed in each experiment, we are still able to give the composition of the gas employed by us from a series of analyses made for another purpose by Dr. LANDOLT at various times during the course of our investigations.

Four samples of Heidelberg coal-gas, collected by LANDOLT on the spot and at the time at which our experiments were conducted, gave, on analysis, the following results:—

	I.	II.	III.	IV.
Hydrogen	39·30	41·04	44·00	41·37
Light carburetted hydrogen	40·56	40·71	38·40	38·30
Carbonic oxide	4·95	7·64	5·73	5·56
Elayl	4·04	5·10	4·13	5·00
Ditetryl	3·15	2·18	3·14	4·34
Nitrogen	8·00	2·75	4·23	5·43
Oxygen	0·00	0·00	0·00	0·00
Carbonic acid	0·00	0·58	0·37	0·00
	100·00	100·00	100·00	100·00

Hence it is seen that the composition of the Heidelberg coal-gas varies but little during a considerable period, but still sufficiently to render the exact agreement of the photochemical measurements very remarkable.

In order to determine which of the ingredients of the flame principally influences the amount of the chemically active rays, we have made the following experiments with a gas-lamp, which has been introduced by one of us into the laboratory to supersede the ordinary wire-gauze flame, and is particularly well suited to produce a flame of various

degrees of luminosity and of varying form and colour. The principle upon which the lamp is constructed, is simply that the gas is allowed to issue under circumstances in which it mixes with such a quantity of air, that the mixture does not quite reach the point at which it is combustible without addition of oxygen. This condition is fulfilled by the following simple arrangement:—*a*, fig. 6 (Plate XVIII.), is a common triangular burner, rising inside the cylindrical cavity *b* to the surface of the cube *cc*. This cylindrical opening, *bb*, which is 15 millims. in depth and 10 millims. in diameter, communicates with the outer air by four holes, *dd*, 7 millims. in diameter. After screwing on the tube *e*, which is 76 millims. long and 8·5 millims. in diameter, the coal-gas is allowed to issue from the burner *a*. On issuing, the gas causes a current of air to enter the tube by the holes *dd*, and the mixture burns at the upper end of the tube with an almost colourless flame perfectly free from smoke. On closing the openings *dd*, the original luminous smoky flame of coal-gas appears.

In order to obtain a large and steady brightly-coloured flame, a small hollow cylinder of porous coke, saturated with a concentrated solution of the salt which colours the flame, is placed on the upper end of the tube *e*. These small cylinders of carbon are best prepared from the carbon employed as the negative pole in a zinc-carbon battery. In order to obtain the purest and brightest-coloured flame, it is necessary to be very careful to select pure salts, and to free the coke cylinder as much as possible from all foreign ingredients by ignition in a stream of chlorine, and boiling out in aqua-regia.

In the first place, the openings *dd* were closed, and the action of the luminous flame was found to be 21·5 in the minute. As soon as the holes *dd* were opened, and the illuminating power of the flame vanished, all perceptible chemical action also ceased. As the heat evolved in the non-luminous is greater than in the luminous flame, and as the two flames only differ by the one containing incandescent particles of carbon which are wanting in the other, we must conclude that the chemical action is chiefly effected by the rays proceeding from the incandescent carbon liberated during the combustion. This carbon is produced from the elayl and ditetryl, the first of which varied in the samples of gas analysed from 4 to 5 per cent., and the second from 2 to 4 per cent. The exact photochemical measurements made with different portions of this gas appear, therefore, to point out that the amount of carbon set free in the combustion of coal-gas depends rather upon the dimensions and form of the flame than upon the small variations in the composition of the gas. After the incandescent carbon, the carbonic oxide in the flame appears to effect the most chemical action, and although the small amount of 6 or 7 per cent. of carbonic oxide in the coal-gas is insufficient to cause any perceptible action, still the flame of pure carbonic oxide produces much more action than that of hydrogen or light carburetted hydrogen. The smallest trace of foreign bodies in the flame can exert a great influence on the chemical action. The red, violet, yellow and green flames produced by the chlorides of lithium, strontium, potassium, sodium and barium, which can be seen to great advantage by the arrangement just described, give no more chemical action than the colourless flame. The fine green flame produced by chloride of copper,

and the pale flame produced by chloride of antimony are, on the contrary, so rich in chemically active rays, that the action begins as soon as the saturated coke is placed on the tube; and in one case it reached the amount of twenty-seven divisions in one minute.

At the end of this Part we have to consider the important question regarding the influence which the changes in atmospheric temperature exert on the indications of the instrument. For this purpose we have conducted four series of observations on the 17th, 18th, 19th, and 20th of September, 1856, in a room the temperature of which was kept constant within half a degree CELSIUS, during the filling and observation with the apparatus. The volume represented by one degree on the observation-tube was 3·4 times as large as in Series X. The distance between the flame, of the same height as in the former experiments, and the insolation-vessel was $0^m{\cdot}1365$. In order to compare the results thus obtained with those of Series X., it was only necessary to multiply the observations with the factor $\frac{3{\cdot}4\,.\,0{\cdot}1365^2}{0{\cdot}216^2}=0{\cdot}1358$.

The observed values are placed in the following Table. The first vertical column contains the constant temperatures at which the apparatus was filled and the observations made; the second, the action produced by the constant standard flame every minute, expressed in degrees of the instrument (a mean of eight actual observations); the third, the mean of these actions, and the fourth their differences.

Series of Experiments XII.

I.	II.	III.	IV.
18°·0 C.	14·24 14·54	14·39	
19·4	13·85 13·85	13·85	+0·54
25·8	14·23 14·73	14·56	−0·71
26·3	14·73 14·11 14·42	14·26	+0·30

From the irregularity of the signs of the differences in the fourth column, we may conclude that the indications of the instrument remain unaffected by changes of temperature between 18° C. and 27° C., any alteration which takes place falling within the limits of observational errors.

As the barometric pressure varied in the foregoing experiments from $0^m{\cdot}746$ to $0^m{\cdot}760$ without the differences between the indications becoming perceptibly larger, the influence of the changes in the atmospheric pressure can be overlooked, except in cases in which greater accuracy, as in the reduction to absolute measure, is required. In a later Part, when treating of the relation of the chemical action to the density of the chlo-

rine and hydrogen mixture, we shall describe experiments in which the influence of the atmospheric pressure is corrected.

In the next Part we shall consider the relations of Photo-chemical Induction, under which term we class a number of phenomena which open a new and most interesting field of investigation, and are connected in an intimate manner with catalytic actions.

XVIII. *Photo-chemical Researches.*—Part II. *Phenomena of Photo-chemical Induction.* *By* Professor Bunsen, *Professor of Chemistry at the University of Heidelberg, and* Henry Enfield Roscoe, *B.A., Ph.D., Professor of Chemistry at Owens College, Manchester. Communicated by Professor* Stokes, *Sec. R.S.*

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Affinity, or the force which causes the particles of different bodies to unite to form a chemical compound, is, both as regards its nature and quantity, a definite magnitude, which, like all other forces and like matter itself, can neither be created nor destroyed. It is therefore a badly chosen form of speech to say, that under certain circumstances a body attains an affinity which under other circumstances it loses. This expression can only be understood to mean, that at one time the bodies are enabled to follow the attraction of their chemical forces, whilst at another they are prevented from so doing by other forces acting in an opposite direction. These opposite attractions, which must be overcome in order that the chemical combination should take place, may be presented to the mind under the image of a resistance similar to that occurring in friction, in the passage of electricity through conductors, in the distribution of magnetism in steel, or in the conduction of heat. We overcome this resistance when we quicken the formation of a precipitate by agitation, or when, by increase of temperature, by catalytic action, or by insolation, we cause a chemical action to occur.

Such a resistance to combination presupposes a certain combining power, the magnitude of which is measured by the mass of the body chemically combined in the unit of time under the influence of the unit of force.

The act by which the resistance to combination is diminished, and the combining power thus brought into greater activity, we call Chemical Induction; and we specify this as photo-chemical, thermo-chemical, electro-chemical, and idio-chemical, according as light, heat, electricity, or pure chemical action is the active agent concerned in overcoming the resistance.

The mode of action of affinity, when separated from the influence of these resistances to combination, that is, the law which regulates this force when all disturbing causes are removed, is entirely unknown. Although the determination of this law, solving the the most important problem of the science, appears in the present state of our knowledge still to lie in the far distance, it is certainly not premature to search for facts which may serve as starting-points in this field of investigation. The interesting relations in which photo-chemical induction stands to this question have, therefore, attracted our special attention.

The fact that the action of light upon a mixture of chlorine and hydrogen does not take place instantaneously, but that a considerable continuation of the exposure is necessary before an action occurs, was noticed by DRAPER in 1843. From this observation he concluded that chlorine, on exposure to light, undergoes an allotropic modification, permanent for several weeks, which differs from ordinary chlorine by possessing a greater combining power.

We shall show that the true cause of this phenomenon is totally a different one; that it cannot be explained by the supposition of an allotropic modification of chlorine, but that the peculiar action which we have called photo-chemical induction here comes into play.

If the chemical rays emanating from a constant source of light be allowed to fall upon a mixture of chlorine and hydrogen which has stood for some time in the dark, no appreciable quantity of hydrochloric acid is formed during the first moments. After some time has elapsed, a small action is observed which very gradually increases, until after a considerable space of time a permanently constant maximum action is obtained.

The following experiments give an example of this gradual increase of the action by constant insolation. The first vertical division contains the times of insolation in minutes; the others give the volumes of hydrochloric acid formed by exposure during the times contained in the first division. In Experiments 3, 4, 5 and 6, lamplight was employed; in Experiments 1 and 2 diffuse sunlight from the zenith was used.

Series of Experiments I.

I.	Diffuse sunlight.		Lamplight.			
	II.	III.	IV.	V.	VI.	VII.
Time in minutes.	Exp. 1.	Exp. 2.	Exp. 3.	Exp. 4.	Exp. 5.	Exp. 6.
1	0·0	0·0	0·0	0·0	6·25	4·13
2	0·0	0 0	1·58	0·0	4·71	10·42
3	0·9	0·0	0·50	0 0	6·26	10·42
4	1·0	0·21	0·00	0·0	5·20	13·53
5	1·3	0·60	0·50	0·0	5·21	11·07
6	2·0	1·01	2·09	1·04	10·43	10·83
7	2·2	0·41	14·58	3·13	11·46	15·63
8	1·7	1·61	29·18	19·78	10·42	14·37
9	3·0	1·70	31·11*	29·19	13·33	20·24
10	5·2	2·73	30·36	32·68*	16·20	25·30
11	5·8*	3·43	32·38	33·40	20·24	26·44
12	5·7	6·34	.....	.....	25·30	45·74
13	.....	7·67*	.....	.....	32·98	53·00
14	.....	.....	.....	.....	38·00	53·00
15	.....	.....	.....	.....	47·00*	55·00*
16	.....	.....	.....	.....	47·0	55·00
17	.....	.....	.....	.....	46·0	55·00
18	.....	.....	.....	.....	47·0	

* In this and all the following experiments, the point at which the maximum action is reached is denoted by an asterisk. The mean of the numbers below this asterisk is the maximum action for the particular experiment.

The first action was observed in Experiment 4 after 6 minutes; in Experiment 2 after 4 minutes; in Experiment 1 after 3 minutes; and in Experiments 5 and 6 a considerable action occurred during the first minute. The maximum action was obtained in Experiments 5 and 6 after 15 minutes, in Experiment 1 after 11 minutes, and in Experiments 3 and 4 after 9 minutes.

From these experiments we may conclude, that the time which elapses from the first insolation until the first traces of the photo-chemical induction become visible, and until the maximum action is attained, is according to circumstances extremely different.

One of the chief conditions modifying the induction relates to the volume of the gas exposed to the light. The influence of the mass of the gas on the induction is easily shown by exposing the chlorine and hydrogen mixture in cylinders of equal surface, but unequal length, to the same amount of light under precisely similar conditions. For this purpose we employed the glass cylinders already described in our former communication, and represented by fig. 1, Part I. The relative lengths of the cylinders No. 1, No. 2, and No. 3 were respectively 2·6, 1·5 and 1·0.

The experiments were conducted in the following manner:—Having proved by various experiments, which we here omit, that the gas could be preserved in these cylinders for twenty-four hours by exclusion of light without undergoing any appreciable change, we connected the cylinders together by caoutchouc* joinings, as represented in fig. 1, Part I., and at the end of the system we placed the tube r of known capacity. After the dried electrolytic gas entering at the tube c had passed through the system for a considerable time in the dark, all the caoutchouc joinings were closed by suitable screw-clamps, any alteration of temperature during the closing being carefully avoided, and the temperature T and the barometric pressure P were noted. In order to be convinced that the gas was pure, we analysed the contents of the tube r according to the iodometric method.

If the amount of chlorine found from the formula given in Part I.,

$$c=\frac{773}{s}\cdot\frac{\mathrm{Cl}}{\mathrm{I}}\alpha(nt-t_1)$$

agreed with the amount calculated according to the formula $c_1=\frac{\mathrm{CP}}{(1+0{\cdot}00366\,\mathrm{T})\times 0{\cdot}76}$ from the capacity of the tube, we could rely on the purity of the gas in the cylinders. The cylinders thus filled were employed in the following experiments.

Experiment I.—Cylinders 1, 2, and 3 were placed close together and exposed in exactly the same position for two hours to the diffuse light from a cloudless sky. The temperature on closing the cylinder was T=12°·9 C., and the atmospheric pressure P=0·7533; the capacity C of the tube r, fig. 1, was 34·81 cub. cent.; of cylinder 1, 190·24 cub. cent.; of cylinder 2, 113·03 cub. cent.; and of cylinder 3, 73·42 cub. cent. Analysis gave,—

* For these joinings common sheet caoutchouc must be employed, the chloride of sulphur formed from the vulcanized tubing rendering the surface of the glass plates dim.

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Series of Experiments II. A.

	Tube r.	Cylinder 1.	Cylinder 2.	Cylinder 3.
α	0·002487	0·002487	0·002487	0·002487
n	1	6	4	1
t	78·6	70·0	70·0	70·0
t_1	3·9	6·8	41·0	28·0

The calculation from these numbers shows—

	Exp. 1.	Exp. 2.	Exp. 3.
	Cylinder 1.	Cylinder 2.	Cylinder 3.
Chlorine mixture at 0° and $0^m{\cdot}760$ before insolation	180·1	107·0	69·5
Ditto, after insolation	180·4	104·4	18·3
Hydrochloric acid in 100 vols. gas	0·0	2·4	73·7

If the gas consisted of equal volumes of chlorine and hydrogen, the tube r, whose capacity at T° and P pressure was found to be C, must, according to calculation, have contained 16·4 cub. cent. of chlorine at 0° and $0^m{\cdot}76$; as the amount of chlorine found by analysis was 16·3 cub. cent., it may be concluded that the gas was pure. The great loss of 73·7 per cent. in cylinder 3 could not arise from leakage, as a mixture of chlorine and hydrogen preserved in the dark in this cylinder showed on analysis no perceptible change.

Experiment II. The tube r, with cylinders 1 and 2, were exposed to the diffuse light of a cloudless sky for three hours, and gave the following elements for calculation:— $T=12°{\cdot}9$, $P=0^m{\cdot}7521$.

Series of Experiments II. B.

	Tube r.	Cylinder 1.	Cylinder 2.
α	0·002487	0·002487	0·002487
n	1	5	1
t	79·9	79·0	79·7
t_1	5·2	32·0	48·5

These values give—

	Exp. 1.	Exp. 2.
	Cylinder 1.	Cylinder 2.
Mixture at 0° and $0^m{\cdot}76$ before insolation.	179·8	106·8
Ditto, after insolation	160·3	13·2
Hydrochloric acid formed in 100 gas ...	10·8	87·6

As the analysis of the tube r gave 16·3 chlorine to 16·5 hydrogen, we may conclude that in this experiment also the gas had the normal composition.

In order to determine more accurately the influence which the volume of gas exerts upon the duration of the induction, we employed an arrangement similar in principle to the instrument represented in fig. 2 of Part I., but having a peculiar insolation-vessel

(which we shall fully describe in the following communication), by means of which columns of sensitive gas of various lengths could be exposed to the action of a constant source of light. In the first experiment the depth of the insolated column of gas was 154·0 millims., and in the second 81·0 millims.

The first vertical column of the following Table gives the times of observation from the beginning of the insolation, the second and third contain the corresponding chemical action for each depth of gas, measured by the quantity of hydrochloric acid formed every thirty seconds.

Series of Experiments III.

Time in minutes.	Exp. 1.	Exp. 2.
	154·0 millims.	81·0 millims.
0·0	0·0	0·0
0·5	0·9	1·0
1·0	1·8	2·4
1·5	3·9	5·1
2·0	5·5	6·7
2·5	6·4	8·2*
3·0	7·7	8·0
3·5	8·8	8·2
4·0	?	8·6
4·5	?	8·0
5·0	10·1*	8·2
5·5	10·3	8·6
6·0	11·4	8·0
6·5	10·4	
7·0	11·2	
7·5	10·5	
Mean action	10·7	8·2

The mean maximum action in the long cylinder was 10·7, in the shorter 8·2. In the longer column the induction was completed in four and a half minutes, in the column which was only half as long, in two minutes. It is also seen that the induction increases more rapidly in the smaller than in the larger volume of gas. Hence we may conclude that the photo-chemical induction is retarded when, other circumstances remaining the same, the mass of insolated gas is increased. The two curves, Plate XIX. fig. I., clearly show this relation; the abscissæ denote the times of observation, and the ordinates the corresponding action measured by the amount of hydrochloric acid formed in thirty seconds. Curve No. 1 represents the increased action of the column 154 millims., and curve No. 2 of column 81 millims.

The decrease of the rate of induction with the mass of the gas appears at first sight inexplicable. In the first experiment the insolation was conducted under exactly the same conditions, in cylinders of unequal length, but of equal diameter. The uppermost layer of gas in both cylinders was therefore exposed to the light under precisely the same circumstances, but in the larger cylinder a second layer of gas existed below, which was not the case in the smaller cylinder, and still the smaller cylinder gave a greater amount of action than the larger. Series of Experiments II. A., Exp. 1, showed that the amount

of action in the cylinder 83 millims. long was not perceptible, whilst Experiment 2 of the same series showed that in the cylinder 32·0 millims. long, the action amounted to 73 per cent. The full explanation of this singular phenomenon will be given in experiments which we shall communicate at the end of this, and in the next Part of this research. We shall there show that the action of light rapidly decreases with the depth of the insolated gas, and that the combining power of the chlorine and hydrogen mixture diminishes in the dark, or on exposure to a smaller amount of light. If the particles of the induced gas are brought by diffusion to a depth in which the intensity of the light is much decreased, they must therefore lose a portion of their combining power. On this account the layer of insolated gas employed for experiments on photo-chemical induction should be as thin as possible. For this purpose we have employed the insolation-vessel represented in Plate XVII. fig. 2. This flattened vessel was made by pressing a bulb blown before the blowpipe, whilst still red-hot, between two wet pieces of wood, so that the thickness of insolated gas did not exceed 3 millims., and all differences in the insolation in the various depths could be disregarded.

The relation which the intensity of the light stands to the induction next engaged our attention. The source of light employed for the experiments on this subject was the flame from the annular burner previously described. The rays from the flame passed first through a large plano-convex lens by which they were concentrated, and afterwards through a screen of water 63·0 millims. in thickness, contained between two parallel plates of glass, by which, as preliminary experiment showed, all disturbing action of heat was perfectly excluded. The lens was furnished with an arrangement of screens, by means of which any desired segment of the lens could be left open, and the amount of light thus regulated.

The following experiments made with a chlorine and hydrogen mixture which had attained the constant maximum action, show the influence of the intensity of the light on the photo-chemical induction.

Series of Experiments IV.

Experiment 1.				Experiment 2.				Experiment 3.				Experiment 4.			
Amount of light = 1·0.				Amount of light = 1·78.				Amount of light = 2·45.				Amount of light = 4·17.			
Time in minutes.	Action in 1 minute.			Time in minutes.	Action in 1 minute.			Time in minutes.	Action in 1 minute.			Time in minutes.	Action in 1 minute.		
a.	*b.*	*c.*	*d.*	*a.*	*b.*	*c.*	*d.*	*a.*	*b.*	*c.*	*d.*	*a.*	*b.*		
0				0				0				0			
1	0·0	0·0	0·0	1	4·2	2·1	3·1	1	?	5·2	2·6	1	65·6		
2	0·0	0·0	0·0	2	7·3	10·8	9·0	2	36·5	65·6	51·0	2	122·1		
3	0·0	0·0	0·0	3	44·8	33·1	38·9	3	71·4	80·1	75·7	3	128·0		
4	0·0	0·0	0·0	4	51·0	53·7	52·3	4	76·4	78·0	77·2				
5	0·0	0·0	0·0	5	54·3	56·2	55·2								
6	2·1	2·2	2·2												
7	14·6	3·1	8·9												
8	29·2	21·8	25·5												
9	31·4	29·2	30·3												
10	30·1	32·7	31·4												

In each of the four experiments with varying amount of light, the first column, *a*, gives the length of exposure expressed in minutes, the second and third columns, *b* and *c*, the amount of action expressed by the volume of hydrochloric acid formed each minute, and the fourth column, *d*, the mean of *b* and *c*.

The experiment in which the amount of light was equal to 1·0, proves that the action began to show itself after the gas had been exposed for five minutes; with the amount of light 1·78 the action began in the first minute; with the amount of light 2·45 a considerable action was attained during the first minute, and when the intensity of the light was 4·17, more than half the amount of the maximum action was attained in the first minute. When the intensity of the light was equal to 1·00, the maximum action was reached after nine minutes' exposure, with intensity 1·78 after five minutes, with intensity 2·45 after from three to four minutes, and with intensity 4·17 after three minutes. The differences between the observations show that the photo-chemical induction does not vary proportionally with the times of exposure. With intensity 1, the action increased in the first minute 2·1 divisions, in the following minute 12·5, in the third 14·6, and in the fourth 2·2; whilst with intensity 1·78 it increased 2·1 in the first minute, 8·7 in the second, 22·3 in the third, 20·6 in the fourth, and 2·5 in the fifth, and in a similar way for the other intensities.

These experiments show,—

(1) That the time of exposure necessary to effect the first action of the photo-chemical induction decreases with the increase of the light, and in a greater ratio than the increase of light.

(2) That the time which elapses until the maximum action is attained also decreases with increase of light, but in a much less ratio than the increase of light.

(3) That the increase of the induction proceeds at first in an expanding series, attains a maximum, and again converges until the constant action is attained.

The curves, Plate XIX. fig. II., of which the abscissæ represent the duration of the insolation, and the ordinates the corresponding action expressed in volumes of hydrochloric acid formed, are obtained from the mean *d* of the two series of observations. They show the law which regulates the decrease of the resistance to combination of the chlorine and hydrogen mixture with increasing amounts of light.

The results of these experiments suggested the question, Is this condition of increased combining power, into which the mixture of chlorine and hydrogen passes by insolation, permanent, or is it merely confined to the space of time during which the gas is exposed to the light?

The following was the method employed for determining this important point. We filled the apparatus (Plate XVII. fig. 2) with uninsolated sensitive gas, and observed the increase of the induction during every 30″, until the maximum action was attained. The insolation-vessel was then darkened for one minute, and after lapse of this time the action was again observed until the maximum was attained. These observations were repeated several times, each period of darkening being longer than the preceding.

The first vertical column of the following Table contains the times of observation, beginning at the point at which the light was admitted; the following columns contain the corresponding action of the light expressed in volumes of hydrochloric acid formed in the unit of time. The second vertical column gives the action for gas which has not been before exposed to the light, the following divisions for gas which was allowed to stand in the dark for thirty, two, four, eight, and sixteen minutes after the induction had been fully attained.

Series of Experiments V. Observation A.

	No. 6.	No. 7.	No. 1.	No. 2.	No. 3.	No. 4.	No. 5.
Time in minutes.	Non-induced gas.	Induced 30′ in dark.	Ditto, 2′ in dark.	Ditto, 4′ in dark.	Ditto, 8′ in dark.	Ditto, 16′ in dark.	Mean of Nos. 6 and 7.
0							
0·5	0·0	0·0	1·2	0·5	0·5	0·0	0·0
1·0	0·0	0·0	4·5	2·5	1·5	0·0	0·0
1·5	0·0	0·5	5·0	5·0	?	1·0	0·3
2·0	0·0	0·2	5·3	5·3	5·0	3·0	0·1
2·5	1·0	0·8	5·7	5·2	5·6	4·4	0·9
3·0	2·0	2·1	4·5	5·2	5·6	5·8	2·1
3·5	3·5	4·3	5·5	5·0	5·8	6·2	3·9
4·0	4·5	4·3	5·5	6·0	5·0	5·1	4·4
4·5	4·0	5·2	5·0	5·0	6·0	5·9	4·6
5·0	5·0	6·5	5·0	5·0	...	5·0	5·8
5·5	5·0	5·0	...	...	...	...	5·0

Observation B.

With an insolation-vessel eight times smaller than that employed in observation A.

	No. 6.	No. 7.	No. 1.	No. 2.	No. 3.	No. 4.	No. 5.
Time in minutes.	Non-induced gas.	Induced 30′ in dark.	Ditto, 2′ in dark.	Ditto, 4′ in dark.	Ditto, 8′ in dark.	Ditto, 16′ in dark.	Mean of Nos. 6 and 7.
0							
0·5	0·0	0·0	5·1	3·1	1·0	?	0·0
1·0	0·0	0·0	6·6	6·1	4·2	2·2	0·0
1·5	0·0	0·0	6·3	6·1	6·1	2·5	0·0
2·0	2·7	1·0	6·3	6·1	6·6	5·0	1·9
2·5	5·2	3·0	...	...	6·1	6·3	4·1
3·0	5·2	3·5	...	...	...	6·0	4·4
3·5	6·2	5·5	...	...	...	...	5·9
4·0	6·3	6·0	...	...	...	...	6·2

The last columns in both sets of observations contain the mean action of the gas which had not been at all exposed to light, and that which by standing in the dark had completely regained its original properties.

From these experiments we may conclude,—

(1) That the resistance to combination overcome by the influence of the light is soon restored when the gas is allowed to stand in the dark.

(2) That whether the induced condition be completely or only partially removed by exclusion of light, it always returns by insolation according to the same law, and that the

increase of the induction on exposure gradually becomes greater until a maximum is attained, after which the rapidity considerably diminishes.

(3) That the increase of the induction from exposure to light takes place much more rapidly than the diminution of the same on darkening.

The curves Nos. 1, 2, 3, 4 and 5, fig. III., Plate XX., from observations A, represent the law according to which the restoration of the induction proceeds when it has gradually diminished by increasing duration of the period of darkening. The abscissæ show the times of insolation, and the ordinates the amount of action in these times measured by the volume of hydrochloric acid formed in one minute. The curve No. 1 represents the increase of the induction when the gas brought to the maximum action was darkened for two minutes; and the following curves, Nos. 2, 3, 4 and 5, when the gas was darkened for four, eight, sixteen, and thirty minutes respectively. Observations B of the same series of experiments give an indication of so short a duration that the form of the curves is not rendered very striking.

The maximum increase of the action is shown in all these curves by the sudden change of direction which the curve takes. The ascending curves in Plate XX. fig. IV., A and B, represent the increase of the chemical action with the insolation, and the descending curves the diminution on withdrawal of the light from series of observations A and B. The horizontal line represents, as in the figs. I., II., III., the maximum action, which, as we shall hereafter show, is proportional to the time of exposure.

We have explained the fact, that the mixture of chlorine and hydrogen does not combine in the dark to form hydrochloric acid, by the supposition of the existence of a certain resistance to combination which is overcome when the gaseous mixture is exposed to light. This resistance to combination can be considerably increased by a variety of circumstances. The presence of a very small quantity of foreign gas in the normal mixture of chlorine and hydrogen is sufficient to cause this resistance to be increased to a very great extent.

Experiments made for the purpose of determining the influence of such impurities on the sensibility of the gas are subject to a great difficulty, owing to the fact, that in order to avoid all the disturbing phenomena of absorption and diffusion, it is not possible merely to add the foreign gas to the normal mixture in the insolation-vessel, but it is necessary to allow the impurity as well as the chlorine and hydrogen to pass for hours through the liquids of the apparatus until the absorptiometric equilibrium is completely established. It is perfectly useless to collect the electrolytic gas in a large vessel, then to mix it with the impurity, and afterwards to allow the mixture to pass through the apparatus, for there is no possible means of collecting this pure mixture of chlorine and hydrogen in large quantities and again forcing it out in a given direction. We have overcome this difficulty by evolving the gas, which was to serve as the impurity, electrolytically with the same current which gave us our normal gas, so that the foreign gas was allowed to pass through the whole apparatus in company with the chlorine and hydrogen. By separating a secondary current, which was led through two separate

porous cells filled with hydrochloric acid or water, it is possible to liberate any desired volume of chlorine, oxygen, or hydrogen, the amounts of which, uninfluenced by any variation in the intensity of the current, always stand in a constant relation to the volume of chlorine and hydrogen evolved by the primary current. In this manner we have procured mixtures of gases of constant composition which we have used in the following experiments.

The observations, which were carried on through a very considerable space of time, were only made use of, when, by repeatedly leading the gas through the apparatus, we had convinced ourselves that the maximum action of the induction had been reached. The mixture of gas containing excess of hydrogen, used in the first experiment, was easily analysed by collecting a portion of the mixture, allowing the chlorine and hydrogen to combine in the light, and measuring the residual hydrogen. It contained only $\frac{8}{1000}$ more hydrogen than the normal electrolytic gas. The first vertical division of the following Table gives the times of observation, the other divisions give the actions corresponding to the times, effected at a temperature of 16°·3 C. and a barometric pressure of 0^{m}·755, and measured by the volumes of hydrochloric acid formed in one minute.

Series of Experiments VI.

Time of observation.	Normal gas.	Normal gas with $\frac{8}{1000}$ hydrogen.	Ditto, after further saturation.	Normal gas.
0				
1	15·6	8·3	7·3	17·7
2	24·0	12·5	13·5	28·1
3	27·1	14·6	13·6	30·2
4	29·9	14·6	16·6	37·9
5	38·5	15·8	12·5	43·5*
6	38·5	15·4	13·6	?
7	44·7*	18·2*	12·5	43·0
8	43·3	15·2	12·2	42·0
9	.....	16·2	14·2	44·0
10	.....	17·2	14·2	43·0
11	.....	16·2	15·2	
12	.....	17·7	16·2*	
13	.....	.....	16·2	
14	.....	.....	15·7	

The mean induction-maximum of the normal gas was at the beginning of the experiment 43·85, and at the end of the same 43·10; the photo-chemical composition of the normal gas had therefore not changed during the course of the experiment. This maximum action 43·47 of the normal gas, diminished by addition of $\frac{8}{1000}$ of hydrogen to 16·78 in the first experiment and to 16·03 in the second, or in mean to 16·40. The gaseous mixture was expanded by this addition of hydrogen from 1 to 1·003; this expansion would be effected by a change of temperature from 16°·3 to 17°·1=t, as is seen from the equation

$$\frac{(1+0{\cdot}00366\times 16{\cdot}3)}{(1+0{\cdot}00366\times t)}=\frac{1}{1{\cdot}003}.$$

As we have shown that expansion from so small an increase of temperature exerts no perceptible influence on the combining power of the chlorine mixture, the force which has diminished this combining power from 43·47 to 16·4 must be a chemical force proceeding from the particles of the hydrogen.

In these experiments we have to do with the purest form of the phenomena which are classed under the name of catalysis, freed from all foreign disturbing causes. We thus arrive on the special field of catalytic phenomena to which the chemical action of light is most closely related. In order fully to understand these relations, we must first form a clear idea of the nature of these contact-actions.

All chemists agree that the phenomena of affinity depend upon the specific attractions which exist between the particles of bodies of different natures. These attractions must necessarily exist when the particles are prevented from following them to form a chemical compound. Let us suppose the particles a and b so brought together that a chemical attraction is exerted between them; and let us suppose a third body, c, brought into the sphere of attraction of the other two; this third body will then also exert an attraction upon a and b. The attraction between a and b will not remain the same as it originally was, but it will be the resultant of all the forces originating in a, b and c. It is thus easily seen, that the attractions, which tend either to effect or support a chemical combination between two bodies, must be altered in the sphere of action of a third body; and that the presence of a third body may therefore, according to circumstances, effect or prevent the formation of a chemical compound. From this point of view the explanation of contact phenomena offers no difficulties. The point which has in general been thought to be the most inexplicable, is the fact that the smallest given quantity of one substance is able to combine or decompose the largest given quantity of another substance, without thereby losing any of its decomposing or combining power; as if an amount of effect could be produced without the expenditure of an equivalent amount of force. We need, however, only consider a little more closely the conditions under which the phenomena of contact occur, in order to assure ourselves that these phenomena do not violate this first law of mechanics, but that they are merely common and simple effects of chemical affinity. If we suppose a number of chemically combined particles placed in the neighbourhood of a contact-substance, the attraction of the foreign contact-body is added to that which holds the chemically combined substances together, and the resultant of all these forces may be such that a decomposition of the chemically combined particles can ensue. When the decomposition has been effected, and the contact-substance is still surrounded by the separated particles, all further action must cease. If, however, these separated particles are removed, and undecomposed molecules arranged in their stead, the phenomenon just described will be repeated *ad infinitum* as often as new particles are brought in place of the old ones. This removal and replacement of the molecules undergoing catalysis, and the expenditure of force connected therewith, furnishes, however, the equivalent of force for that exerted by the contact-substance. For as the substance produced by the catalytic action either sinks to the ground on account of its greater specific gravity, or is vaporized by assumption of heat, escapes by

effervescence, or diffuses through the liquid by virtue of molecular attractions and thus makes room for new, not yet decomposed molecules, a certain amount of force is expended which corresponds to that exerted by the catalytic action. From this point of view these catalytic phenomena are not to be regarded as manifestations of a peculiar force, but as actions which must always occur as necessary consequences of affinity, when a mass in the state of chemical action undergoes an essential alteration of its constituent parts.

The quantitative estimation of the relations which exist in these phenomena of contact between the mass of the substance, the time and other modifying conditions, has as yet not been possible, owing to the absence of any case in which these relations were exhibited in their simplest forms. Our method of photo-chemical measurement points out a direction which promises to afford interesting results concerning these quantitative relations.

As we intend at a future time to consider these relations more fully, we shall here merely enter into the subject as far as is necessary for the due understanding of the phenomena of induction. The following conclusions may be drawn from the foregoing experiments with excess of hydrogen:—

(1) That the maximum of the induction of the normal chlorine mixture is reduced from 100 to 37·8 by the presence of $\frac{3}{1000}$ of hydrogen.

(2) That this diminution of the action does not arise from the dilution caused by the addition of the hydrogen, but from a catalytic action proceeding from its particles.

Still more remarkable than in the case of the hydrogen, is the effect produced by the admixture of other foreign gases on the combining power of the normal chlorine and hydrogen. This is most distinctly seen for oxygen gas in the following Table:—

Series of Experiments VII.

Temp. 15°·4 C. Barometric pressure $=0^{m}\cdot7550$.

I.	II.	III.	IV.	V.	VI.
Time of observation in minutes.	Normal gas, alone.	Ditto, with $\frac{1}{1000}$ of oxygen.	Ditto, after longer saturation.	Ditto, with $\frac{1}{1000}$ of oxygen.	Normal gas, alone.
0					
1	5·2	1·0	4·0	1·0*	15·6
2	7·3	2·0	5·0	2·0	24·0
3	9·4	4·0	3·0	1·0	27·1
4	11·5	3·0	3·0	1·0	29·9
5	13·5	3·0	5·0	1·0	38·5
6	15·6	4·0	6·0	1·0	38·5
7	20·3	4·0	3·0	2·0	44·7
8	24·3	5·0*	3·0	1·0	?
9	30·4	5·0	3·0	1·0	47·0*
10	36·9	4·0	4·0*	1·0	45·0
11	40·0	4·0	5·0	1·0	47·0
12	48·0	5·0	6·0	1·0	
13	53·0*	5·0	5·0	1·0	
14	53·0	5·0	4·0	2·0	
15	.....	5·0	5·0	2·0	
16	.....	5·0			

The first column, as before, contains the times of the observations in minutes, the following ones the volumes of hydrochloric acid formed by insolation in one minute. The mean action of the normal gas before and after the experiment amounted to 48·8. On addition of $\frac{5}{1000}$ of oxygen the mean action diminished to 4·7, and by addition of $\frac{1}{1000}$ of oxygen it sank to 1·3. These experiments show,—

(1) That the combining power of the chlorine and hydrogen mixture undergoes a much greater diminution in the presence of oxygen than of hydrogen; so much so, that the maximum induction of the normal gas in presence of $\frac{5}{1000}$ of oxygen diminished from 100 to 9·7, and in presence of $\frac{13}{1000}$ from 100 to 2·7.

(2) That the diminished maximum of induction effected by the catalytic action of the oxygen is attained by exposure to light in a much less time than the larger maximum of the normal gas.

The following experiments show the catalytic action which an excess of chlorine exerts upon the normal mixture. The somewhat considerable irregularities seen in a few of the determinations arise from capillary resistances in the observation-tube, and therefore disappear when the mean of a number of observations is taken.

Series of Experiments VIII.

Time of observation.	Normal gas, before experiment.	Ditto, after experiment.	Ditto, with $\frac{10}{1000}$ of chlorine.	Ditto, with $\frac{75}{1000}$ of chlorine.	Ditto, with $\frac{180}{1000}$ of chlorine.
0					
1	8·3	3·1	2·1	2·1	0·0
2	20·8	14·6	6·7	2·6	0·0
3	26·1	55·9	14·8	10·9	0·0
4	35·3	56·0	26·0	21·9	15·6
5	43·5	57·3*	34·0	25·9	13·5
6	50·2	.....	32·9	30·2	18·8
7	60·0*	.....	37·4*	26·0	19·8
8	61·0	.....	36·0	27·6	23·4*
9	?	.....	39·5	29·1*	25·3
10	60·0	.....	35·0	31·5	?
11	.....	.....	35·0	30·5	25·9
12	.....	.....	35·0	30·5	24·0
13	.....	.....	.....	26·5	23·0
14	.....	.....	.....	29·5	24·0

The results, obtained from the experiment before us, show—

(1) That the catalytic action which chlorine exerts on the photo-chemical induction is very much less than that effected by oxygen or hydrogen, and that the mean maximum action diminishes when $\frac{10}{1000}$ chlorine is present from 100 to 60·2, and when $\frac{75}{1000}$ chlorine is present from 100 to 50·3, and when $\frac{180}{1000}$ chlorine is present from 100 to 41·3.

It appeared to us of great interest to determine the action which the presence of hydrochloric acid exerts on the normal chlorine mixture. If the action were as considerable as was found with the gases just mentioned, the indications of the instrument would not be constant when (as is often required) various-sized insolation-vessels are

employed. For although the hydrochloric acid formed is absorbed very quickly by the water, still a certain amount must always be mixed with the normal gas during exposure to light, and this amount must vary with the size and shape of the insolation-vessel. Fortunately, however, the catalytic action of the hydrochloric acid upon the mixture of chlorine and hydrogen is, as the following experiments show, so small, that it may be entirely disregarded.

The experiments were carried out as follows:—In the first place, the insolation-vessel was filled with pure water and the maximum action observed; hydrochloric acid of spec. grav. 1·148 was then placed instead of the water, and the observations not made use of until, after repeated saturation, the maximum action was attained. In order to determine the amount of hydrochloric acid contained in the gaseous mixture at the temperature 16°·4 of the experiment, a litre of air was slowly led, first, through a small quantity of hydrochloric acid of the above strength, and next through a bulb apparatus containing an alkaline solution free from chlorine. The solution of alkali acidified with nitric acid and precipitated by a salt of silver gave 0·0086 grm. chloride of silver, which is equivalent to 1·3 cub. cent. hydrochloric acid gas in one litre of gas at 16°·4 and $0^{m}\cdot7516$. The normal gaseous mixture contained therefore 1·3 pro 1000 hydrochloric acid. The photochemical measurements with this mixture gave the following results:—

Series of Experiments IX.

Time of observation, in minutes.	Gas with pure water.	Ditto, saturated again.	Gas with 1·3 pro mille hydrochloric acid.
0			
1	10·4*	10·4*	
2	12·0	9·9	12·5*
3	12·0	11·0	13·5
4	12·5	10·4	9·4
5	11·5	11·5	8·3
6	8·3	11·5	9·4
7	10·3	.....	9·4
8	10·1	.....	12·3
9	.....	.....	12·1
10	.....	.....	12·1
11	.....	.....	10·1
12	.....	.....	11·1

The mean of the first series of observations is 10·9, of the second 10·8, and of the third 10·9. This exact agreement, which was verified by other experiments, shows—

(1) That an amount of 0·13 per cent. of hydrochloric acid contained in the normal gas exerts no perceptible action on the induction.

The complicated law which, as we have seen, regulates the increase of the induction of the chlorine and hydrogen mixture under the influence of light, renders it possible that the presence of uninsolated gas may have precisely the same action on the insolated gaseous mixture as the foreign gases have been shown to exert. The following experi-

ments verify this supposition. The determinations were so conducted that a small quantity of uninsolated gas, measured on the scale, was allowed to mix with the insolated gas, which had been brought to the induction-maximum, and the time observed which elapsed before the maximum action was again reached.

Series of Experiments X.

Time of observation in minutes.	Induced gas.	Ditto, with $\frac{6{\cdot}2}{1000}$ non-insolated gas.	Ditto, with $\frac{6{\cdot}4}{1000}$ non-insolated gas.
0·5	7·5*	4·0	4·0
1·0	6·5	4·0	6·0
1·5	7·0	4·5	6·0
2·0	8·0	3·0	5·0
2·5	7·0	5·0	8·0*
3·0	.....	7·0*	7·5
3·5	.....	7·5	7·5
4·0	.....	7·5	8·0
4·5	.....	7·5	7·5
5·0	.....	7·5	

These experiments show that the non-insolated gas acts upon the insolated in the same manner as a contact-substance, an addition of 0·6 per cent. of the first producing a diminution of the maximum from 100 to 55·6*.

The curves, Plate XXI. fig. V., represent the relation in which the induction of the chlorine and hydrogen stands to the mass and nature of the foreign admixtures. The abscissæ express the volumes of the foreign gas to 1000 volumes of the normal mixture, and the ordinates the action effected by equal amounts of light under the influence of these admixtures.

If the effect produced by the expansion of the gas from the addition of the foreign body were eliminated, and if the action were reduced to the unit of intensity of light, we might construct curves of the values thus found, which must express the relation existing between the catalytic action and the substance and mass of the contact-body. Our experiments are, however, as yet not extended enough to enable us here to enter more fully into this new field of research.

Our experiments have led us to another very singular relation, which is of great interest as playing an important part in the whole of the phenomena of photo-chemical induction. The normal gas from which all light has been excluded, needs, as we have shown, a certain length of exposure before the maximum action is attained. The length of exposure is not altered when the pure gas is allowed to stand for some time in the dark before insolation. If, however, the gas contains such minute traces of impurities that the maximum action is not thereby lessened, we still find evidences of the presence of these admixtures in the fact, that on standing in the dark the time required for the attainment of the maximum induction is considerably shortened. This phenomenon

* The mean of the second vertical column =100.

is best observed when gas has been led through the apparatus until the traces of gaseous admixture (only to be estimated as millionths of the whole mass) have become so small that their absence would not diminish the resistance to combination of the induced gas. In order to establish this fact we have made many experiments, one of which we subjoin; the insolation-vessel contained only about 3 cub. cent. of gas, and many litres of the chlorine and hydrogen mixture had passed for two days through the apparatus. The first horizontal division of the following Table contains the number of the experiments according to the order in which they were made; the second division contains the times during which the newly-evolved gases stood in the dark previous to insolation; and the third division gives the length of time which the gas was led through the instrument between each set of observations. The numbers in the corresponding vertical columns represent the photo-chemical action for the times found in the first vertical column.

Series of Experiments XI.

	No. 1.	No. 2.	No. 3.	No. 4.	No. 5.	No. 6.	No. 7.	No. 8.	No. 9.	No. 10.
	0′	30′	0′	30′	0′	30′	0′	30′	0′	40′
	4′	4′	4′	?	20′	†	2′	5′	4′	
1	?	2·1	3·1	5·2	4·3	1·0	10·0	2·1	6·3	2·1
2	6·3	9·4	8·3	15·6	16·7	21·9	19·3	21·8	20·8	12·2
3	7·3	20·8	10·4	22·9	19·8	28·1*	22·9	27·0*	21·9	21·9
4	10·4	21·9	11·5	23·3	21·9	27·8	25·5	27·3	25·0	24·3
5	12·6	24·0	15·6	25·3	23·9	28·3	26·1*	27·3	28·5*	26·7*
6	14·6	26·3*	17·7	27·3*	26·3*	29·4	26·8	29·3	28·3	25·8
7	?	27·3	19·8	26·9	25·3	.....	28·8	26·8	28·5	26·3
8	19·2	26·3	22·3	27·0	27·3	.....	28·0			
9	23·3	27·9	26·3*	.....	29·8					
10	24·3	.....	26·3							
11	24·3	.....	26·9							
12	28·0*									
13	28·0									
14	28·0									

These figures show,—

(1) That the resistance to combination effected by the slightly impure gas diminishes when the gas is allowed to stand in the dark.

(2) That the diminution of the resistance ensuing spontaneously in the dark becomes less as the purity of the gas increases by longer saturation.

(3) That this diminution of the resistance is observed after the gas is so nearly pure that on further saturation no increased maximum induction is perceptible.

The experiments give the following maxima of induction arranged in the order in which the observations were made:—

† Evolution-vessel changed, and gas led through for some time.

	Maxima.	Difference from Mean.
No. 1 . . .	28·0	+0·6
No. 2 . . .	27·0	−0·4
No. 3 . . .	26·8	−0·9
No. 4 . . .	27·1	−0·3
No. 5 . . .	27·2	−0·2
No. 6 . . .	28·4	+1·0
No. 7 . . .	27·4	0·0
No. 8 . . .	27·5	+0·1
No. 9 . . .	28·1	+0·7
No. 10 . . .	26·3	−1·1
Mean. .	27·4	

The curves (Plate XXII. fig. VI.) clearly show the diminution of the resistance to combination on darkening. The abscissæ denote the duration of the insolation, the ordinates the chemical action corresponding to these exposures, measured by the volume of hydrochloric acid formed in one minute. The curves with odd numbers proceeding from the bottom upwards, represent the progression of the induction for gas rendered continually more pure by longer saturation; those with even numbers show the increase of the induction for the same gas, which was allowed to stand for some time in the dark. It is seen that the lower curves representing the action of the freshly prepared gas, Nos. 1, 3, 5, 7 and 9, continually rise with increased purity of the gas, and approach the curves which show the constant action of the gas which has stood in the dark, until, in Nos. 9 and 10, the two curves differ only by an amount which falls within the observational errors.

The perfectly pure chlorine and hydrogen mixture acts therefore very differently from that which contains even the most minute trace of impurity. The resistance to combination of the first gas does not undergo any change when left to itself in the dark; in the second case this resistance is diminished under similar circumstances, resembling a wire, which, having been lengthened by a suspended weight, returns gradually to its original dimensions on removing the weight.

There is thus a normal resistance to combination w belonging to the pure gaseous mixture, and this resistance does not alter by standing, but only on exposure to light; and there is also an increased resistance to combination $w+n$ (belonging to an impure gas), which returns to the normal resistance w by simply standing in the dark. It appeared possible that this increased resistance might depend upon the nature of the decomposing poles; experiment, however, showed that the same phenomenon was observed whether poles of carbon or platinum were employed.

An explanation of the laws of photo-chemical induction derived from the experiments already communicated, may easily be found in the assumption that the chlorine or hydrogen, or both gases, undergo upon exposure to light an allotropic modification, in which they possess more than the ordinary combining power, or in other words, that both

these gases (or one of them), similar to ozonized and common oxygen, can, under certain circumstances, be in an active, and under others, in a passive condition*. If this hypothesis be true, each gas must undergo this peculiar modification when separately exposed to the action of light. The following experiments show, however, that this is not the case.

The gases, prepared as usual by electrolysis, were evolved in the dark from two separate porous cells, and each gas allowed to pass through a thin glass tube 3 feet long, which was so bent that its whole length could be placed outside the window-shutter, whilst the two ends were retained in the darkened room. The two gases passed separately through this bent tube into the daylight, and then both traversed a communicating tube which led them into the instrument. By means of these tubes we were able to expose each one of the separated gases to the direct or diffuse sunlight, and to examine the effect produced on the sensibility of the resulting mixture. To begin with, the tubes reaching beyond the shutter were carefully protected from the action of light, and the mixture led through the apparatus until the gas had attained the sensibility requisite for the observations. The increase of the induction was then determined in two successive experiments, in the first of which the separate gases were kept in the dark, and in the second of which they were previously insolated. In the first of the following experiments diffuse daylight was used for the insolation of the separated gases, and in the second direct sunlight. Division I. contains the times of observation, Division II. the action for each minute corresponding to these times, *a* for the non-insolated gas, and *b* for the gas which had been exposed to light immediately before.

Series of Experiments XII.

1. Experiment with diffuse daylight.			2. Experiment with direct sunlight.		
I.	II.		I.	II.	
	a.	*b.*		*a.*	*b.*
1	5·5	8·5	1	0·5	0·5
2	6·0	8·5	2	13·5	0·5
3	6·5	9·0	3	18·0	19·0
4	11·5	10·0	4	16·0	35·0
5	32·0	12·0	5	29·0	54·0
6	73·0	27·0	6	38·0	50·0
7	.....	75·0	7	55·0	

Although the intensity of the light to which the gases were separately exposed was so great that, had the gases been mixed, their combining power would have been increased to such an extent that explosive combination would immediately have occurred, still the experiment showed no greater difference between the duration of the induction of the insolated and non-insolated gas than the errors of observation rendered unavoidable. The experiments, therefore, show that the light does not effect a permanent modification, either of the chlorine or of the hydrogen, but that the combination produced by

* See DRAPER, Philosophical Magazine, 1857, vol. xiv. page 161.

the light must entirely depend upon photo-chemical action affecting only the increasing attractions of the chemically active molecules.

We have often had occasion, in the course of this research, to show that the curves which represent the increase of the chemical action for a constant source of light have all a maximum of the increasing action, which is rendered visible by a flexure in the curve. It appeared of interest to examine whether this property of the induction-curves arises from the general mode of action of affinity, or whether the light plays a necessary part therein. We have therefore carried out some experiments upon idio-chemical induction; that is, upon reactions which occur altogether under the influence of chemical attractions, without the assistance of heat, light or other outward forces. For this purpose a very dilute aqueous solution of bromine with tartaric acid was employed. This mixture, when allowed to stand in the dark at a constant temperature, undergoes a very slow decomposition, the free bromine gradually decomposing the organic acid and forming hydrobromic acid. The quantity of bromine contained in equal volumes of this solution was determined at various times after mixing $T_0\ T_1 \ldots T_n$ by the iodometric method. The amount of bromine thus found, B_1, subtracted from that originally contained, B_0, gave the amount of free bromine ($B_0 - B_1$) which had entered into combination from the beginning of the experiment up to the time of the analysis. If α represent the weight of iodine contained in one burette division of the standard solution, n the number of measures of sulphurous acid required to reduce the free iodine, t the number of burette degrees necessary to decompose one measure of sulphurous acid, and t^1 the number of burette degrees required to neutralize the excess of sulphurous acid, and Br and I the atomic weights of bromine and iodine, the amount of bromine, B, contained in a volume of the liquid under examination is found from the equation

$$B = \frac{\alpha(nt - t^1)Br}{I}.$$

Having thus determined the quantity of bromine present in the same volume of liquid for the times $T_0\ T_1 \ldots T_n$, the quantity of bromine which has entered into combination from one volume of the liquid in the times $T_1\ T_2$, &c., is found from the equations $\frac{B_0 - B_1}{B_0}$, $\frac{B_0 - B_2}{B_0}$, &c. The liquids employed in Experiment 1, Series XIII. contained, to 1 gramme of bromine, 1·139 grms. crystallized tartaric acid and 648·9 grms. water; in Experiment 2, to 1 grm. bromine, 1·654 grm. tartaric acid and 705·5 grms. water; in Experiment 3, to 1 grm. bromine, 6·6166 grms. tartaric acid and 1410·2 grms. water. The value of α in all the analyses was 0·0022593. The atomic weights of bromine and iodine employed in the calculations were 500·00 and 794·37. The following arrangement of the experimental results gives, in the first vertical column, the duration of the spontaneous decomposition in hours, $T_0\ T_1$, &c.; the second, third and fourth columns, the elements of analysis; and the fifth, the values $B_0\ B_1 \ldots$; and the last, the values of $\frac{B_0 - B_1}{B_0} \ldots$, or the loss of bromine for the successive times $T_1\ T_2 \ldots T_n$.

Series of Experiments XIII.

Experiment 1.						Experiment 2.						Experiment 3.					
T.	n.	t_1.	t.	B.	$\frac{B_0-B_1}{B_0}$.	T.	n.	t_1.	t.	B.	$\frac{B_0-B_1}{B_0}$.	T.	n.	t_1.	t.	B.	$\frac{B_0-B_1}{B_0}$.
0	2	30·2	53·4	0·10893	0·00000	0	2	10·9	42·1	0·10424	0·00000	0	1	5·0	48·6	0·05347	0·00000
2	1	17·2	92·6	0·10722	0·01551	17	2	45·0	59·0	0·10381	0·00413	17	1	22·5	59·0	0·05190	0·02935
18	2	29·7	52·3	0·10651	0·02222	27	2	30·1	50·0	0·099403	0·04643	27	1	18·3	50·0	0·04508	0·15695
20	2	19·1	46·7	0·10566	0·03002	102·5	1	33·3	69·1	0·05091	0·51164	76	1	18·4	35·5	0·02432	0·54517
26	2	15·0	44·0	0·10381	0·04700	125·0	1	29·1	60·7	0·04493	0·56901	102·5	1	63·4	69·1	0·00811	0·84833
42·5	2	40·9	56·5	0·10253	0·05854	150·0	1	35·8	61·2	0·03612	0·65354	125	1	58·3	60·7	0·00341	0·93623
65·5	2	39·7	45·6	0·07323	0·32773	198·0	1	52·4	71·2	0·02673	0·47362	150	1	60·1	60·1	0·00000	
88·5	1	16·4	59·9	0·06185	0·43121												
136·5	1	35·5	68·6	0·04707	0·56789												
163·0	1	50·5	79·5	0·04124	0·62135												

The curves Nos. 1, 2 and 3, fig. VII., Plate XXI., corresponding to the Experiments 1, 2 and 3, represent the increase of the hydrobromic acid produced in the times T_0 T_1, &c. drawn on the abscissæ. It is seen that these curves have also a point of contrary flexure, so that in the idio-chemical as well as the photo-chemical induction, a phase in the decomposition is found in which a maximum in the increase of the action occurs. The cause of this maximum increase appears therefore not to lie in any peculiarity which the light may possess, but rather in the mode of action of affinity itself.

The laws of photo-chemical induction which we have developed in this Part, explain most completely many of the singular phenomena which lie at the foundation of the photographic processes. Without entering into detail concerning the relations which these laws bear to the general processes of the photographer, we will now merely consider one phenomenon which has been so difficult to explain, that it has been found necessary to assume the existence of certain rays endowed with very peculiar properties, to which the name of "rayons continuateurs" has been given. This phenomenon was first established by Edmond Becquerel from the following observations. If one-half of an iodized plate of silver, or a sensitive photographic paper, be protected from the action of light, whilst the other half is exposed to a constant source of light for such a time that no alteration, either perceptible to the eye or capable of being developed by photographic preparations, is effected on the sensitive surface, the plate possesses the property, when exposed to a uniform but very slight amount of light, to blacken on the insolated half, whilst the part not previously exposed remains unaffected. If, in the first short insolation, every part of the plate was not exposed to the same intensity of light, the blackening effected by the following uniform exposure corresponds to this intensity, so accurately indeed, that a daguerreotype picture commenced by this slight exposure may be developed by subsequent uniform insolation, almost as perfectly as if each amount of light represented on the picture, and not a constant amount, had acted for the whole time upon the plate. The explanation of this fact does not require the assumption of a new class of rays, which cannot commence, but only continue, the photo-chemical

action. The phenomenon is in reality a simple consequence of the laws of photo-chemical induction, and proves that these relations, which we have examined only in the case of chlorine and hydrogen, occur in a slightly modified form in other photo-chemical processes. If the chlorine and hydrogen mixture were a solid substance which could be fixed upon paper without diminishing its sensibility to light, this paper must exhibit exactly the same singular property which BECQUEREL first observed with the daguerreotype plate, or other photo-chemically sensitive surface. In order to prove this assertion, we will imagine that such a solid chlorine and hydrogen surface was as sensitive as the gas in Series of Experiments IV. *b* of this Part. We will further imagine that half of this sensitive surface was darkened, and the other half insolated for five minutes with the luminous intensity 1, as in Experiment 1. The action on both halves of the plate after the five minutes' insolation is therefore, as the numbers of the experiment show, still 0. The plate has undergone therefore no visible alteration, nor could a difference be effected by any reagent. If the whole surface be now exposed to a uniform insolation, the half previously exposed will act quite differently from the half which remained in the dark. The former will undergo an alteration in the first five minutes, which is represented by the amounts of hydrochloric acid formed, namely,

$$2{\cdot}1+14{\cdot}6+29{\cdot}2+31{\cdot}4+30{\cdot}1=107{\cdot}4,$$

whilst the non-insolated portion of the surface exposed during the same space of time to the same amount of light does not show, according to Experiment 1. *b*, Series IV., any alteration in its chemical properties. If the formation of hydrochloric acid were accompanied by a deepening of colour, the half of the chlorine and hydrogen plate, which was not previously exposed, would still appear white, whilst the other half would be of a dark colour, thus acting exactly like the sensitive surface of a daguerreotype plate.

Let us next consider the second case, in which an image commenced by lights of various degrees of intensity is developed by a subsequent uniform exposure in depths of colouring proportional to the various original intensities. We again imagine a solid layer of chlorine and hydrogen which is as sensitive as the chlorine and hydrogen mixture employed in Series of Experiments IV., and suppose that at the point p_1, p_2, p_3, p_4 of the surface the various intensities of the light 1, 1·78, 2·45, 4·17 were acting. According then to Experiments 1 *c*, 2 *c*, and 3 *c*, the amount of chemical action measured by the hydrochloric acid formed in the minute is at p_1 0, at p_2 2·1, at p_3 5·2, and at p_4 65·6. If these numbers are divided by their respective amounts of light, we obtain the quantity of action which is effected by the unit amount of light at the point p_1, p_2, p_3, p_4; these are 0, 1·18, 2·12 and 15·7. These numbers express the combining power, or what is the same thing, the sensibility of the plate at the points p_1, p_2, p_3, p_4. After an exposure of two minutes the sensibility of the plate at the same points would be 0, 6·07, 26·8 and 29·3, and after three minutes' exposure 0, 18·6, 32·7 and 54·7. It is thus seen that the sensibility of the plate is altered by the first exposure to light, and in such a manner that the greater luminous intensity corresponds to the greatest sensibility. If the varying

amounts of sensibility which the surface of a plate has reached by the first exposure could be represented by depths of colouring, the various shades would represent the picture itself. With a subsequent uniform insolation the chemical action must therefore proceed proportionally to the various amounts of light in the original picture, and if the chemical action is made visible by a change of colour, an actual picture must be produced.

Having determined in this part of our investigation the most important phenomena of photo-chemical induction, we shall in the next Part consider the laws which regulate the chemical action of light after the induction is completed.

XIX. *On the Immediate Principles of Human Excrements in the Healthy State.*

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Communicated by H. Bence Jones, *M.D., F.R.S.*

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In June 1854 I had the honour of communicating to the Royal Society an account of some investigations I had undertaken respecting the composition of the excrements of man and animals; since that time I have continued my researches on human excrements, and obtained further results which form the subject of the present paper. I have been most ably seconded in this work by my assistant, Mr. Frederick Dupré, Ph.D., and have derived much valuable aid from his thorough knowledge of chemical and physical science.

The method of investigation employed in this instance is similar to that which had been adopted on the former occasion; alcohol and ether were again the principal means employed for conducting the analysis, chemical decompositions being thereby avoided and the constituents of excrements consequently obtained under the form of *Immediate Principles*. It will also be observed that in this case, for the purpose of extracting excretine, the alcoholic solution of excrements was in many instances not mixed with milk of lime, which simplified materially the operations and increased the interest of the investigation.

The results obtained are as follows:—

1. *Margarate of lime*, *phosphate of lime*, and *margarate of magnesia* were discovered to be three immediate principles of human evacuations.

2. I found a new and simple method for obtaining excretine, and its chemical formula has been established.

3. The fact that vegetable food increases the presence of margaric acid in excrements has been confirmed.

4. The existence of a comparatively large quantity of *cholesterine* in the spleen, which I had mentioned as probable, has been confirmed *.

When healthy human evacuations are exhausted with boiling alcohol an extract is obtained, which, on cooling, leaves a deposit. This deposit was examined as follows:—The dark alcoholic solution being decanted, the residue was thrown upon a filter and washed thoroughly, first with cold and then with boiling alcohol. The boiling alcoholic solution, on cooling, yielded more or less of a peculiar substance, in the form of a light granular precipitate, which has been alluded to in my previous communication. The

* I introduce here this result, having alluded to the subject in my previous communication.

substance insoluble in boiling alcohol and remaining on the filter was now submitted to examination; the original alcoholic extract having filtered nearly quite clear through the calico, it was evident that this deposit could consist but in small quantity of insoluble substances, which might have found their way through the meshes of the cloth. The deposit, removed from the filter into a capsule, was boiled with a solution of caustic potash, when the greatest part of it dissolved; an attempt to filter the fluid having failed, I decomposed the alkaline solution with hydrochloric acid, which induced the formation of a precipitate. The precipitate, collected on a filter, was washed with water until the washings ceased to exhibit an acid reaction and induce a cloudiness in a solution of nitrate of silver. Boiling alcohol and ether dissolved the substance on the filter, it crystallized on cooling from its solution in alcohol, and by spontaneous evaporation from its solution in ether; by means of repeated crystallizations in a mixture of alcohol and ether, the substance in question was obtained perfectly colourless and pure. When heated on a platina spatula, it fused and then burnt with a flame; a carbonaceous residue was left, which, on the further application of heat, disappeared completely, leaving no ashes on the spatula. When viewed with the microscope, the crystals had the peculiar stellate structure of margaric acid; the substance, when fused and allowed to cool, assumed a crystallized nature; it was found to fuse at 60° Centigrade. The above characters are those of *margaric acid**, so that this fatty acid is constantly present in healthy human evacuations†.

It was now of much importance to ascertain in which form the margaric acid found in the above circumstances existed as a constituent of human fæces, or in other words, to determine the Immediate Principle containing the fatty acid in question; but previous to entering on this subject, I beg to relate an experiment instituted for the purpose of detecting the best method to be employed to prepare the alcoholic extract of excrements. It was noticed that the quantity of the deposit yielded by the extract, on cooling, varied according to the manipulations adopted to prepare the solution; consequently a healthy and fresh evacuation was triturated in a mortar, in order that every part of it should be equally moistened, and the mass was divided into three quantities, weighing each 60 grammes. Each portion was boiled four times successively with two ounces of alcohol; the first part being treated with alcohol in an open tin capsule and strained through a clean laboratory towel, the second being boiled in a long-necked flask and strained through the same towel, and the third being treated in a similar flask and strained through muslin folded in two. The fluid passed very freely through the muslin, and there appeared to be a greater deposit on cooling in the alcoholic extract of the third portion than in that of the two others; about two hours afterwards the alcoholic extracts were filtered separately through filtering-paper, and a clear dark-coloured solu-

* Notwithstanding the fusing-point and crystalline structure of this fatty acid, it might have been mixed with a small proportion of stearic acid and traces of oleic acid. Stearic and margaric acid have the same chemical properties.

† I had shown in my previous communication that margaric acid was occasionally present in fæces.

tion was thus obtained. The residue left on the three filters, after being washed with cold alcohol and subsequently removed to capsules with the platina knife, was boiled in alcohol, the fluid being afterwards rapidly strained through calico. On cooling, a very small quantity of deposit occurred in the solutions No. 1 and No. 2; solution No. 3 yielded rather more: the fluids were subsequently evaporated to dryness and the residue boiled in a little alcohol, when, on cooling, an increase of the deposit occurred; No. 3 yielded decidedly more than No. 1 and No. 2. Having proceeded so far with the experiment, it was necessary to examine the part of the original deposit insoluble in boiling alcohol. The three quantities were boiled separately in a solution of caustic potash till they were nearly dissolved, and then an excess of hydrochloric acid was added. There occurred immediately an abundant precipitate; when the fluid had become cold, the precipitate was collected on a filter and thoroughly washed with water. Each filter being dried, first by a temperature not exceeding 55° C., then under the air-pump, over sulphuric acid, they were next treated separately with ether until no more was dissolved; by spontaneous evaporation the ethereal solution deposited a crystalline substance*, which was washed with water, again collected on a filter, and dissolved in ether; the operation being repeated a third time for No. 1, No. 2, and No. 3, the solutions were finally evaporated to dryness, and each residue was dried and weighed: in the three cases the substance was colourless, or nearly so.

The weight of the fatty acid was in

No. 1	. . . . .	0·047 gramme.
No. 2	. . . . .	0·056 „
No. 3	. . . . .	0·083 „

From these experiments I conclude that the amount of margaric acid derived from the original deposit varies according to the mode of manipulation, and that the maximum quantity is obtained by preparing the alcoholic extract in a glass flask, and straining it through muslin; in which case little of the alcohol is lost by evaporation, and the fluid has not time to cool previously to its being strained.

It is evident that the fatty acid contained in the deposit yielded by the alcoholic extract of excrements on cooling, did not occur therein as a free substance, for if in the free state, it must have been dissolved by the boiling alcohol and ether washings. For the purpose of determining with which substance it was combined, the acid filtrate from the deposit, after it had been decomposed by potash and hydrochloric acid, was submitted to a qualitative and quantitative analysis; the ashes from the burnt deposit were also analysed. The inorganic substances present were found to be *phosphoric acid* and *lime*, and in such a proportion, as will be seen from the result of the quantitative analysis, that there was more lime than required to combine with the phosphoric acid, the excess of this substance being just in the proportion necessary to convert the

* In that operation the ethereal solution was exposed to the air long enough to ensure the crystallization of the whole of the fatty acid.

margaric acid into a neutral margarate of lime. The quantitative analysis of the deposit was effected by boiling it with potash, and decomposing the soap by means of hydrochloric acid, for the purpose of precipitating the margaric acid. The whole being thrown on a filter, the precipitate was washed until the washings had ceased to exhibit an acid reaction and produce a cloudiness in a solution of nitrate of silver; the fatty acid was dried on filtering-paper, and finally dissolved in ether; the ethereal extract being evaporated to dryness in a weighed beaker, the weight of the residue could be easily ascertained. The phosphoric acid and lime were determined in the filtrate from the margaric acid precipitate, the phosphoric acid with perchloride of iron and sulphate of magnesia, and the lime with oxalic acid. The following Table shows the result of the analysis, together with the quantity of margaric acid necessary to combine with the amount of lime, exceeding that which could possibly have been combined originally with the phosphoric acid: these numbers were obtained by calculation, according to the formula for the neutral margarates,—

$$C_{34}H_{33}O_3 + MO.$$

	Found.			Calculated.
	Lime.	Phosphoric acid.	Margaric acid.	Margaric acid.
1.	0·0021	0·015	0·083	0·074
2.	0·040	0·031	0·230	0·112
3.	0·058	0·042*	0·205	0·196
4.	0·031	0·023*	0·100	0·102

These results show that the lime, phosphoric acid and margaric acid found in the deposit under examination, existed as phosphate of lime and margarate of lime, and consequently assumed this form in excrements†.

From the difficulty of extracting fatty acids perfectly pure and colourless, without experiencing more or less loss of substance, when they are mixed up with animal colouring-matters and other impurities, it is next to impossible to obtain mathematically exact results in such analysis. In analyses No. 1 and No. 3, the quantities of margaric acid found were slightly above the proportion determined by calculation, which resulted from the presence of impurities; these could not have been removed without the loss of more or less of the acid. In analysis No. 2 the margaric acid was highly coloured and impure, which accounts for its proportion being considerably above what it ought to have been. In analysis No. 4, the margaric acid obtained being comparatively pure, the result was very conclusive.

The alcoholic extract of excrements filtered from the deposit just described, four or

* In analyses 3 and 4, the quantity of phosphoric acid was obtained by calculation.

† It is very remarkable that these salts should have dissolved in the boiling alcoholic extract of fæces, and lost their property of dissolving in boiling alcohol, after having been collected on a filter; which circumstance is probably owing to the acid reaction of the alcoholic extract. The same observation applies to the margarate of magnesia.

five hours after the extract had been prepared, yielded a dark-coloured fluid. When this solution was allowed to stand undisturbed in a flask for twenty-four hours, another substance subsided and collected at the bottom of the flask; a cold night favouring in a remarkable degree the formation of this second deposit. After decanting the clear fluid and filtering the residue, a nearly white and amorphous substance remained on the filter; boiling alcohol and ether removed from it every trace of colouring matter or other impurities, the deposit being insoluble in these fluids. A sample of the substance being burnt on the platina knife, it ignited, charred, and finally left a residue of white ashes, whilst in the case of the first deposit it was found impossible, even with the blowpipe, to obtain the ashes otherwise than dark coloured. The inorganic residue submitted to analysis, proved to consist entirely of *magnesia**: in order to determine the nature of the organic substance, combined with magnesia, the compound was boiled with potash, when after a short time it dissolved, leaving in the fluid but a light precipitate; the addition of hydrochloric acid to the mixture induced the separation of a white solid substance, which floated on the liquid; this substance was collected on a filter, thoroughly washed with water, and then dissolved in a little boiling alcohol, when on cooling it solidified, assuming the crystallized form peculiar to margaric acid; its fusing-point was found to be 56° C., and on cooling, the fused mass again crystallized; in short, there could be no doubt that this substance was *margaric acid*; the deposit under examination being nearly colourless, I had no difficulty in obtaining the fatty acid in a perfectly white and pure condition. Though a qualitative analysis of the deposit showed that it contained no other substance than margaric acid and magnesia, still it was important to determine if they were present in such proportion as to form the neutral margarate of magnesia,—

$$C_{34}H_{33}O_2+MgO.$$

For this purpose, a sample of the deposit, purified by repeated washings with boiling alcohol and ether, was dried in the water-bath until it ceased to lose weight; it was then ascertained to weigh 0·088 gramme; after it had been burnt in a platina capsule, the weight of ashes left amounted to 0·007; the result of the analysis may be given as follows:—

On 0·088 gramme.

	Found.	Calculated according to formula.
Margaric acid	0·081	0·081
Magnesia	0·007	0·007
	0·088	0·088

which shows that the compound can be no other than *margarate of magnesia*.

In my previous communication I stated that I had noticed an increase of margaric

* The carbonate of magnesia resulting from the incineration of a compound of this substance with an organic acid, had been decomposed by the blowpipe flame, with the evolution of carbonic acid.

acid in fæces after a vegetable diet, a fact previously ascertained by LEHMAN from microscopical examination. It was now important to determine, by actual experiment, whether a purely vegetable diet would, according to my former method of investigation, determine the presence of an excess of margaric acid in the precipitate obtained by mixing the alcoholic extract of fæces with milk of lime. I consequently submitted myself to a vegetable diet, consisting of bread, potatoes, water-cresses, salad, other green vegetables, wine, beer and tea; vegetable soup was prepared for me without meat, gravy, or butter; I avoided milk, eggs, &c. Having confined myself exclusively to this vegetable diet for four days, I examined the second evacuation passed since the beginning of the experiment; the lime precipitate in the clear alcohol extract was collected on a filter, and dissolved in hydrochloric acid; not only did the substance insoluble in hydrochloric acid yield margaric acid, but it contained at least two or three times as much of this substance as I had ever obtained by the method in question. I was prevented from continuing the investigation from its injurious effects upon my health, but this experiment sufficed to show the influence of a vegetable diet upon excrements.

I now beg to give an account of some further investigations into the nature and composition of the substance which I proposed to call *Excretine*.

This constituent of human fæces I first obtained by adding milk of lime to the alcoholic extract of fæces, collecting on a filter the precipitate thus obtained, and subsequently treating it with ether, when the solution yielded by spontaneous evaporation very impure crystals of excretine, which were purified by repeated crystallizations. This method unfortunately yielded so small a quantity of the substance in question that I failed to collect enough to determine its chemical composition; it was therefore found necessary to postpone this investigation. Until the month of December 1855, my endeavours to extract excretine by a more satisfactory process had failed, when on coming to my laboratory one morning after a very cold night, on which occasion the temperature of the atmosphere had fallen several degrees under the freezing-point, I observed that an alcoholic extract of fæces prepared the day before, and left in an open beaker, had lost its fluid nature and assumed a thick or gelatinous consistence; a sample of the mass submitted to the microscope was discovered to be full of slender silky crystals, which proved by further examination to consist of excretine; the thick nature of the fluid was owing entirely to the presence of these crystals. It became therefore evident that the easiest way of extracting the Immediate Principle in question was by reducing the temperature of the alcoholic extract of fæces; and I took advantage of the cold nights of last winter to prepare a quantity of this interesting substance*.

The first crop of crystals being collected on a filter, the alcoholic filtrate yielded a second quantity on the same day or next morning; in very cold weather sometimes even

* Having failed to obtain excretine by means of artificial cold, I could not prepare enough of it last winter to complete its analysis; the cold weather of this season has lasted such a short time that it was necessary to adopt a modification of the first method; and thus I was compelled to delay communicating my results, for which circumstance I beg to offer my sincere apologies.

a third crop of excretine could be obtained; but as soon as the thermometer rose above the freezing-point, the alcoholic extract ceased to deposit crystals, and those which had formed were rapidly dissolved. It was also observed that the crystallization occurred more rapidly when the alcoholic solution was left in an open vessel; which is accounted for by the circumstance that the evaporation of the alcohol increased the proportion of water in the alcoholic fluid, and consequently the excretine acquired a favourable condition for assuming the solid form; the cold temperature moreover reduced its solubility in alcohol, and it was therefore induced to crystallize. The amount of crystallized excretine was apparently proportional to the intensity of the cold. Each crop of crystals was collected upon the same filter, and allowed afterwards to become comparatively dry by exposure to air in the funnel. The filter was next washed with ether or boiling alcohol, which dissolved out the whole of the crystals, and the solution by standing for a night yielded an abundant crop of nearly pure excretine; more or less of the impurities which had precipitated with the excretine were left on the filter, being insoluble in ether or in boiling alcohol. For the purpose of obtaining excretine in a perfectly pure and colourless condition, the impure substance resulting from the spontaneous evaporation of the ether solution, or from the exposure of the alcoholic extract to cold, was separated from its mother-liquor (the latter being decanted), then washed in a capsule with a little cold alcohol, which dissolved the colouring matter and a very small proportion of excretine; the cold coloured alcohol was decanted, and after repeating this operation four or five times, and collecting the washings, the excretine was dissolved in a little hot alcohol; on cooling, it was obtained colourless and beautifully crystallized; if the crystals were not yet pure, the above operation was repeated. The washings yielded another crop of crystals, which were purified by the same process; and the operation was repeated on the successive series of washings until the coloured alcoholic solution contained but very little excretine; by this method much less of the substance was lost than by repeated crystallizations in ether, and drying the impure crystals on filtering-paper.

In order to submit this new substance to analysis, it was necessary to prepare it on a large scale, and the cold weather not lasting long enough for the purpose of allowing me to extract it in sufficient quantity, I was obliged to adopt the following method, which is a modification of that described in my first paper. The clear alcoholic extract of eight or nine evacuations was mixed with lime-water and then filtered. I washed the precipitate with water, and then dried it perfectly on a water-bath; this substance being transferred to a flask, was now agitated with a mixture of alcohol and ether, and filtered after having stood for some hours. The residue left on the filter was washed with ether, dried in the open air, or in a retort (with the view of recovering the alcohol and ether), and again treated with a mixture of alcohol and ether. The solutions were but little coloured, and after twenty-four hours yielded a beautiful crop of excretine. The mother-liquor, separated by filtration or decantation, deposited some hours afterwards a second crop of crystals. The substance was purified by repeated solutions in alcohol and ether,

subsequently filtered, if necessary, through animal charcoal. This method is attended with no difficulty, and enabled me to obtain at once from nine evacuations, as much as 0·815, or not quite one gramme of excretine; its only objection is the expense resulting from the unavoidable use of large quantities of alcohol and ether. It is very troublesome, and I may say next to impossible, to wash the lime precipitate to such an extent as to remove from it every trace of excretine, and consequently the above number must be far short of the real proportion of the substance contained in nine evacuations: I am at present engaged with investigations respecting this subject.

Among the properties of excretine there is one on which I shall add a few words, as it appears to me possessed of especial interest. This immediate principle of human fæces is not subject to spontaneous decomposition, even when freely exposed to the atmosphere, and its presence can be detected in excrements long after they have been voided, a property which might be serviceable in a medico-legal point of view. Moreover, although I have not yet been able to find out a test for the purpose of showing the presence of excretine in complex mixtures, still the method for obtaining it, which consists in the addition of milk of lime to the alcoholic extracts of fæces, will suffice to show the presence of very minute quantities of this organic substance even in mixtures of the most complex description. Thus I had no difficulty in obtaining a sample of pure excretine from the contents of my laboratory water-closet, consisting of fæces, urine, paper, dust and other filth. Between two or three gallons of the above nearly fluid mixture were evaporated to dryness in a water-bath, and the dry residue was exhausted with boiling alcohol in the usual way. A small quantity of excretine was deposited in the alcoholic extract on cooling; but on adding lime-water to the solution, collecting upon a filter the precipitate thus induced, drying it and washing it with ether, a comparatively large amount of excretine was obtained.

Having prepared a sufficient quantity of colourless and pure excretine, partly by the action of cold upon the alcoholic extract of fæces, and partly by the other method, this immediate principle was submitted to chemical analysis. The qualitative examination confirmed the existence of sulphur in excretine, and consequently in the combustions it was necessary to interpose a tube full of peroxide of lead between the chloride-of-calcium tube and the potash apparatus. The presence of nitrogen in excretine was very carefully looked for by burning the substance with potassium and testing the fumes evolved when a sample of the crystals was heated with lime in a test-tube. I repeated these experiments several times with Dr. Dupré, but failed to obtain indications of the slightest traces of nitrogen; it is therefore obvious that excretine contains no nitrogen*. The absence of water of crystallization was determined by fusing in the water-bath a weighed quantity of the crystallized substance, and then weighing it again, when it was not found to have lost any weight; the experiment was repeated twice with the same

* In my first communication I stated that nitrogen was present in excretine; this error resulted from having but a small quantity of substance to operate upon, which, moreover, might possibly not have been perfectly pure.

result. Excretine burnt with carbonate of potash and nitrate of potash, yielded no phosphorus.

The quantitative analysis was first attempted by combustions with oxide of copper, but it was soon found out that from the large proportion of carbon present, it was necessary to employ a more powerful oxidizing compound, and consequently the formula was obtained from combustions with chromate of lead. The proportion of sulphur was determined in the usual way, by burning the substance with carbonate of baryta and nitrate of potash.

I. 0·145 gramme of crystallized excretine yielded,—

Sulphate of Baryta	0·032,	
corresponding to Sulphur . .	0·004; or in 100 parts of excretine .	2·758

II. 0·107 gramme of crystallized excretine yielded,—

Sulphate of Baryta	0·027,	
corresponding to Sulphur . .	0·003; or in 100 parts of excretine .	2·803
Average per-centage		2·780

The combustions with oxide of copper yielded,—

I. For 0·071 gramme of excretine,

					For 100 parts.
Carbonic acid .	0·211	Carbon . .	0·057	Carbon . .	80·281
Water . . .	0·087	Hydrogen .	0·009	Hydrogen .	12·676

II. For 0·1115 gramme of excretine,

					For 100 parts.
Carbonic acid .	0·318	Carbon . .	0·0898	Carbon . .	79·81
Water . . .	0·121	Hydrogen .	0·0134	Hydrogen .	12·01

In these analyses* the carbon was not far from correct, as the gaseous products (sulphurous and sulphuric acid) formed at the expense of the sulphur of excretine had made up for the loss depending on the incomplete combustion of the substance; that the excretine was not perfectly burnt is shown by the amount of hydrogen obtained, which is too low.

The combustions with chromate of lead yielded,—

I. For 0·1458 gramme of crystallized excretine,

					For 100 parts.
Carbonic acid .	0·430	Carbon . .	0·117	Carbon . .	80·412
Water . . .	0·182	Hydrogen .	0·020	Hydrogen .	13·746

II. For 0·1355 gramme of crystallized excretine,

					For 100 parts.
Carbonic acid .	0·400	Carbon . .	0·109	Carbon . .	80·442
Water . . .	0·167	Hydrogen .	0·018	Hydrogen .	13·284

* In the analysis with oxide of copper the tube containing peroxide of lead had not been used.

Consequently these analyses yielded, for 100 parts of excretine,

	I.	II.	Average result.
Carbon	80·412	80·442	80·427
Hydrogen	13·746	13·284	13·515
Sulphur	2·780	2·780	2·780
Oxygen	3·062	3·494	3·278

No substance having been found as yet to combine with excretine, its atomic composition was calculated from the assumption that one equivalent contained one equivalent of sulphur, and the following formula was obtained:—

78 equivalents of Carbon	468
78 equivalents of Hydrogen	78
1 equivalent of Sulphur	16
2 equivalents of Oxygen	16
Atomic weight of Excretine	578

	Found.	Calculated.
Carbon	80·427	80·969
Hydrogen	13·515	13·495
Sulphur	2·780	2·768
Oxygen	3·278	2·768
	100·000	100·000

It is now evident that excretine differs widely from cholesterine, though it should appear at first sight closely allied to it.

Most of the properties of excretine have been described in my former communication: I am at present engaged with investigations respecting this subject, more especially for the purpose of establishing the physiological relations of the substance in question. To the present paper is annexed a Plate, XXIII., showing the various forms which excretine may assume; when viewed by transmitted light, the tufts of crystals exhibit a most beautiful silky appearance, more especially if obtained perfectly colourless. It is easy to prepare large-sized crystals by dissolving in alcohol a specimen of dry excretine; the presence of water prevents the formation of the sharply defined prisms. It is impossible to separate the whole of the colouring matter from excretine when extracted by the lime method, unless its solution be filtered several times through animal charcoal; the solution in alcohol appears to be discoloured by charcoal more readily than that in ether. I hope to be soon able to communicate some further details respecting the crystallography of this substance.

In my former communication the existence of cholesterine in the spleen was reported as probable*: I have now convinced myself that this gland contains the substance in question. The subject being one of very great importance in a physiological point of

* This observation occurred accidentally whilst I was examining the spleen to discover whether it contained excretine.

view, the investigation was resumed, and another sample of the substance prepared; it crystallized readily, assuming the characteristic tabular form of cholesterine; the crystals were soluble in ether and hot alcohol, being again deposited by the alcoholic solution when cold. When boiled in a strong solution of potash, this substance did not dissolve; but the alkaline fluid being filtered, and the filter washed with boiling alcohol, the crystals again appeared in the liquid on cooling. Burnt in contact with potassium and then treated with a proto- and per-salt of iron and hydrochloric acid, no blue colour was obtained, consequently nitrogen was absent; when burnt with soda-lime, the fumes evolved did not produce a blue stain on red test-paper, confirming the above result. Its fusing-point was 139° Cent. There can be now no doubt but that the spleen always contains a comparatively large quantity of cholesterine, which is evidently independent from that which might exist in the blood retained by this organ after death. Is it that the spleen secretes cholesterine? This can only be determined by actual experiment; but it is very remarkable that a part of the blood which enters the liver should come directly from an organ containing large quantities of a substance which is known to enter into the composition of the bile.

Description of the Plate.

PLATE XXIII.

This Plate shows the microscopical appearance of excretine; the crystals are seen to consist of acicular four-sided prisms, grouped usually in round masses, the needles radiating from the centre to the periphery.

XX. *A Memoir on Curves of the Third Order.* By A. CAYLEY, *Esq.*, *F.R.S.*

Received October 30,—Read December 11, 1856.

A CURVE of the third order, or cubic curve, is the locus represented by an equation such as $U=(*)(x, y, z)^3=0$; and it appears by my "Third Memoir on Quantics," that it is proper to consider, in connexion with the curve of the third order $U=0$, and its Hessian $HU=0$ (which is also a curve of the third order), two curves of the third class, viz. the curves represented by the equations $PU=0$ and $QU=0$. These equations, I say, represent curves of the third class; in fact, PU and QU are contravariants of U, and therefore, when the variables x, y, z of U are considered as point coordinates, the variables ξ, η, ζ of PU and QU must be considered as line coordinates, and the curves will be curves of the third class. I propose (in analogy with the form of the word Hessian) to call the two curves in question the Pippian and Quippian respectively. A geometrical definition of the Pippian was readily found; the curve is in fact STEINER'S curve R_3 mentioned in the memoir "Allgemeine Eigenschaften der algebraischen Curven," *Crelle*, t. xlvii. pp. 1–6, in the particular case of a basis-curve of the third order; and I also found that the Pippian might be considered as occurring implicitly in my "Mémoire sur les Courbes du Troisième Ordre," *Liouville*, t. ix. p. 285, and "Nouvelles Remarques sur les Courbes du Troisième Ordre," *Liouville*, t. x. p. 102. As regards the Quippian, I have not succeeded in obtaining a satisfactory geometrical definition; but the search after it led to a variety of theorems, relating chiefly to the first-mentioned curve, and the results of the investigation are contained in the present memoir. Some of these results are due to Mr. SALMON, with whom I was in correspondence on the subject. The character of the results makes it difficult to develope them in a systematic order; but the results are given in such connexion one with another as I have been able to present them in. Considering the object of the memoir to be the establishment of a distinct geometrical theory of the Pippian, the leading results will be found summed up in the nine different definitions or modes of generation of the Pippian, given in the concluding number. In the course of the memoir I give some further developments relating to the theory in the memoirs in *Liouville* above referred to, showing its relation to the Pippian, and the analogy with theorems of HESSE in relation to the Hessian.

Article No. 1.—*Definitions, &c.*

1. It may be convenient to premise as follows:—Considering, in connexion with a curve of the third order or cubic, *a point*, we have—

(*a*) The *first or conic polar* of the point.

(*b*) The *second or line polar* of the point.

The meaning of these terms is well known, and they require no explanation.

Next, considering, in connexion with the cubic, *a line*—

(*c*) The first or conic polars of each point of the line meet in four points, which are the *four poles* of the line.

(*d*) The second or line polars of each point of the line envelope a conic, which is the *lineo-polar envelope* of the line.

And reciprocally considering, in connexion with a curve of the third class, *a line*, we have—

(*e*) The *first or conic pole* of the line.

(*f*) The *second or point pole* of the line.

And considering, in connexion with the curve of the third class, *a point*—

(*g*) The first or conic poles of each line through the point touch four lines, which are the *four polars* of the point.

(*h*) The second or point poles of each line through the point generate a conic, which is the *point pole locus* of the point.

But I shall not have occasion in the present memoir to speak of these reciprocal figures, except indeed the first or conic pole of the line.

The term *conjugate poles* of a cubic is used to denote two points, such that the first or conic polar of either of them, with respect to the cubic, is a pair of lines passing through the other of them. Reciprocally, the term *conjugate polars* of a curve of the third class denotes two lines, such that the first or conic pole of either of them, with respect to the curve of the third class, is a pair of points lying in the other of them.

The expression, a *syzygetic cubic*, used in reference to two cubics, denotes a curve of the third order passing through the points of intersection of the two cubics; but in the present memoir the expression is in general used in reference to a single cubic, to denote a curve of the third order passing through the points of intersection of the cubic and its Hessian. As regards curves of the third class, I use in the memoir the full expression, a curve of the third class syzygetically connected with two given curves of the third class.

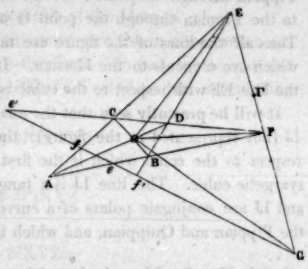
It is a well-known theorem, that if at the points of intersection of a given line with a given cubic tangents are drawn to the cubic, these tangents again meet the cubic in three points which lie in a line; such line is in the present memoir termed the *satellite line* of the given line, and the point of intersection of the two lines is termed the *satellite point* of the given line; the given line in reference to its satellite line or point is termed the *primary line*.

In particular, if the primary line be a tangent of the cubic, the satellite line coincides with the primary line, and the satellite point is the point of simple intersection of the primary line and the cubic.

Article No. 2.—*Group of Theorems relating to the Conjugate Poles of a Cubic.*

2. The theorems which I have first to mention relate to or originate out of the theory of the conjugate poles of a cubic, and may be conveniently connected together and explained by means of the accompanying figure.

The point E is a point of the Hessian; this being so, its first or conic polar, with respect to the cubic, will be a pair of lines passing through a point F of the Hessian; and not only so, but the first or conic polar of the point F, with respect to the cubic will be a pair of lines passing through E. The pair of lines through F are represented in the figure by FBA, FDC, and the pair of lines through E are represented by ECA, EDC, and the lines of the one pair meet the lines of the other pair in the points A, B, C, D. The point O, which is the intersection of the lines AD, BC, is a point of the Hessian, and joining EO, FO, these lines are tangents to the Hessian at the points E, F, that is, the points E, F are *corresponding points* of the Hessian, in the sense that the tangents to the Hessian at these points meet in a point of the Hessian. The two points E, F are, according to a preceding definition, conjugate poles of the cubic.



The line EF meets the Hessian in a third point G, and the points G, O are conjugate poles of the cubic. The first or conic polar of G, with respect to the cubic, is the pair of lines AOD, BOC meeting in O. The first or conic polar of O, with respect to the cubic, is the pair of lines GEF and G$f'efe'$ meeting in G. The four poles of the line EO, with respect to the cubic, are the points of intersection of the first or conic polars of the two points E and O, that is, the four poles in question are the points F, F, e, e'. Similarly, the four poles of the line FO, with respect to the cubic, are the points E, E, f, f'.

The line EF, that is, any line joining two conjugate poles of the cubic, is a tangent to the Pippian, and the point of contact Γ is the harmonic with respect to the points E, F (which are points on the Hessian) of G, the third point of intersection with the Hessian. Conversely, any tangent of the Pippian meets the Hessian in three points, two of which are conjugate poles of the cubic, and the point of contact is the harmonic, with respect to these two points, of the third point of intersection with the Hessian.

The line GO in the figure is of course also a tangent of the Pippian, and moreover the lines FBA, FDC (that is, the pair of lines which are the first or conic polar of E) and the lines ECA, EDB (that is, the pair of lines which are the first or conic polar of F) are also tangents to the Pippian. The point E represents *any* point of the Hessian, and the three tangents through E to the Pippian are the line EFG and the lines ECA, EDB: the line EFG is the line joining E with the conjugate pole F, and the lines ECA, EDB are the first or conic polar of this conjugate pole F with respect to the cubic. The figure shows that the line EO (the tangent to the Hessian at the point E) and the beforementioned three lines (the tangents through E to the Pippian), are harmonically related, viz. the line EO the tangent of the Hessian, and the line EF one of the tangents to the Pippian, are harmonics with respect to the other two tangents to the Pippian. It is

obvious that the tangents to the Pippian through the point F are in like manner the line GFE, and the pair of lines FBA, FBC, and that these lines are harmonically related to FO the tangent at F of the Hessian. And similarly, the tangents to the Pippian through the point O are the line GO and the lines AOD, BOC, and the tangents to the Pippian through the point G are the line GO and the lines GFE and $Gf'ef\acute{e}$. Thus all the lines of the figure are tangents to the Pippian except the lines EO, FO, which are tangents to the Hessian. It may be added, that the lineo-polar envelope of the line EF with respect to the cubic is the pair of lines OE, OF.

It will be presently seen that the analytical theory leads to the consideration of a line IJ (not represented in the figure): the line in question is the polar of E (or F) with respect to the conic which is the first or conic polar of F (or E) with respect to *any* syzygetic cubic. The line IJ is a tangent of the Pippian, and moreover the lines EF and IJ are conjugate polars of a curve of the third class syzygetically connected with the Pippian and Quippian, and which is moreover such that its Hessian is the Pippian.

Article Nos. 3 to 19.—*Analytical investigations, comprising the proof of the Theorems, Article No.* 2.

3. The analytical theory possesses considerable interest. Take as the equation of the cubic,

$$U=x^3+y^3+z^3+6lxyz=0.$$

Then the equation of the Hessian is

$$HU=l^2(x^3+y^3+z^3)-(1+2l^3)xyz=0;$$

and the equation of the Pippian in line coordinates (that is, the equation which expresses that $\xi x+\eta y+\zeta z=0$ is a tangent of the curve) is

$$PU=-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\xi\eta\zeta=0.$$

The equation of the Quippian in line coordinates is

$$QU=(1-10l^3)(\xi^3+\eta^3+\zeta^3)-6l^2(5+4l^3)\xi\eta\zeta=0;$$

and the values of the two invariants of the cubic form are

$$S=-l+l^4,$$

$$T=1-20l^3-8l^6,$$

values which give identically,

$$T^2-64S^3=(1+8l^3)^3;$$

the last-mentioned function being in fact the discriminant.

4. Suppose now that (X, Y, Z) are the coordinates of the point E, and (X', Y', Z') the coordinates of the point F; then the equations which express that these points are conjugate poles of the cubic, are

$$XX'+l(YZ'+Y'Z)=0,$$

$$YY'+l(ZX'+Z'X)=0,$$

$$ZZ'+l(XY'+X'Y)=0;$$

and by eliminating from these equations, first (X', Y', Z'), and then (X, Y, Z), we find

$$+P(X^3+Y^3+Z^3)-(1+2P)XYZ=0,$$
$$+P(X'^3+Y'^3+Z'^3)-(1+2P)X'Y'Z'=0,$$

which shows that the points E, F are each of them points of the Hessian.

5. I may notice, in passing, that the preceding equations give rise to a somewhat singular *unsymmetrical* quadratic transformation of a cubic form. In fact, the second and third equations give $X' : Y' : Z' = YZ - \frac{1}{P}X^2 : \frac{P}{l}XY - \frac{l}{P}Z^2 : PZX - lY^2$. And substituting these values for X', Y', Z' in the form

$$+P(X'^3+Y'^3+Z'^3)-(1+2P)X'Y'Z',$$

the result must contain as a factor

$$+P(X^3+Y^3+Z^3)-(1+2P)XYZ;$$

the other factor is easily found to be

$$-\frac{l}{P}\left(P(X^3+Y^3+Z^3)+3lXYZ\right).$$

Several of the formulæ given in the sequel conduct in like manner to unsymmetrical transformations of a cubic form.

6. I remark also, that the last-mentioned system of equations gives, *symmetrically*,

$$X'^2 : Y'^2 : Z'^2 : Y'Z' : Z'X' : X'Y'$$
$$= YZ - \frac{1}{P}X^2 : ZX - \frac{P}{l}Y^2 : XY - \frac{l}{P}Z^2 : \frac{P}{l}YZ - \frac{l}{P}X^2 : \frac{l}{P}ZX - lY^2 : \frac{P}{l}XY - \frac{l}{P}Z^2;$$

and it is, I think, worth showing how, by means of these relations, we pass from the equation between X', Y', Z' to that between X, Y, Z. In fact, representing, for shortness, the foregoing relations by

$$X'^2 : Y'^2 : Z'^2 : Y'Z' : Z'X' : X'Y' = A : B : C : F : G : H,$$

we may write

$$X' = AF = GH, \quad Y' = BG = HF, \quad Z' = CH = FG, \quad ABC = FGH;$$

and thence

$$X'^3 = AF.G^2H^2, \quad Y'^3 = BG.H^2F^2, \quad Z'^3 = CH.F^2G^2, \quad X'Y'Z' = F^4G^4H^4;$$

hence

$$+P(X'^3+Y'^3+Z'^3)-(1+2P)X'Y'Z'$$
$$= FGH\{+P(AGH+BHF+CFG)-(1+2P)FGH\}.$$

But we have

$$+P(AGH+BHF+CFG) = -(2P^2+P^4)(X^3+Y^3+Z^3)XYZ+(P^4+2P)(Y^2Z^2+Z^2X^2+X^2Y^2)$$
$$-(1+2P)FGH = (P^3+2P)(X^3+Y^3+Z^3)XYZ+(P^4+2P)(Y^2Z^2+Z^2X^2+X^2Y^2)$$
$$+P(1-P)(1+2P)X^2Y^2Z^2;$$

and thence

$$+P(AGH+BHF+CFG)-(1+2P)FGH$$
$$= -P(1-P)\{P(X^3+Y^3+Z^3)XYZ-(1+2P)X^2Y^2Z^2\};$$

and finally,

$$+P(X'^3+Y'^3+Z'^3)-(1+2P)X'Y'Z'$$
$$= P(-l+l^4)(lYZ-X^3)(lZX-Y^3)(lXY-Z^3)XYZ$$
$$\times\{P(X^3+Y^3+Z^3)-(1+2P)XYZ\}.$$

We have also, identically,

$$ABC-FGH=\frac{1}{l}(-l+l^3)XYZ\{l^2(X^3+Y^3+Z^3)-(1+2l^3)XYZ\},$$

which agrees with the relation $ABC-FGH=0$.

7. Before going further, it will be convenient to investigate certain relations which exist between the quantities (X, Y, Z), (X', Y', Z'), connected as before by the equations

$$\begin{aligned} XX'+l(YZ'+Y'Z)&=0,\\ YY'+l(ZX'+Z'X)&=0,\\ ZZ'+l(XY'+X'Y)&=0, \end{aligned}$$

and the quantities

$$\begin{aligned} \xi&=YZ'-Y'Z, & \alpha&=XX'=-\tfrac{1}{l}(YZ'+Y'Z),\\ \eta&=ZX'-Z'X, & \beta&=YY'=-\tfrac{1}{l}(ZX'+Z'X),\\ \zeta&=XY'-X'Y, & \gamma&=ZZ'=-\tfrac{1}{l}(XY'+X'Y). \end{aligned}$$

We have identically,

$$2XX'(YZ'-Y'Z)+(XY'+X'Y)(ZX'-Z'X)+(ZX'+Z'X)(XY'-X'Y)=0;$$

or expressing in terms of ξ, η, ζ, α, β, γ the quantities which enter into this equation, and forming the analogous equations, we have

$$\begin{aligned} 2l\alpha\xi-\gamma\eta-\beta\zeta&=0 \qquad (A)\\ -\gamma\xi+2l\beta\eta-\alpha\zeta&=0\\ -\beta\xi-\alpha\eta+2l\gamma\zeta&=0. \end{aligned}$$

We have also

$$X^2Y'Z'-X'^2YZ=\tfrac{1}{3}\{-(XY'+X'Y)(ZX'-Z'X)+(ZX'+Z'X)(XY'-X'Y)\},$$

and thence in like manner,

$$\begin{aligned} X^2Y'Z'-X'^2YZ&=\frac{1}{2l}(\gamma\eta-\beta\zeta) \qquad (B)\\ Y^2Z'X'-Y'^2ZX&=\frac{1}{2l}(\alpha\zeta-\gamma\xi)\\ Z^2X'Y'-X'^2YZ&=\frac{1}{2l}(\beta\xi-\alpha\eta). \end{aligned}$$

Again, we have

$$(YZ'-Y'Z)^2=(YZ'+Y'Z)^2-4YY'ZZ'$$

$$(ZX'-Z'X)(XY'-X'Y)=-(ZX'+Z'X)(XY'+X'Y)+2XX'(YZ'+Y'Z);$$

and thence

$$\begin{aligned} \xi^2&=\frac{1}{l^2}\alpha^2-4\beta\gamma \qquad (C)\\ \eta^2&=\frac{1}{l^2}\beta^2-4\gamma\alpha\\ \zeta^2&=\frac{1}{l^2}\gamma^2-4\alpha\beta \end{aligned}$$

$$\eta\zeta = -\frac{2}{l}\alpha^2 - \frac{1}{l^2}\beta\gamma$$

$$\zeta\xi = -\frac{2}{l}\beta^2 - \frac{1}{l^2}\gamma\alpha$$

$$\xi\eta = -\frac{2}{l}\gamma^2 - \frac{1}{l^2}\alpha\beta;$$

and conversely,

$$\frac{1}{l^2}(1+8l^3)\alpha^2 = \xi^2 - 4l^2\eta\zeta \qquad \text{(D)}$$

$$\frac{1}{l^2}(1+8l^3)\beta^2 = \eta^2 - 4l^2\zeta\xi$$

$$\frac{1}{l^2}(1+8l^3)\gamma^2 = \zeta^2 - 4l^2\xi\eta$$

$$-\frac{1}{l^2}(1+8l^3)\beta\gamma = 2l\xi^2 + \eta\zeta$$

$$-\frac{1}{l^2}(1+8l^3)\gamma\alpha = 2l\eta^2 + \zeta\xi$$

$$-\frac{1}{l^2}(1+8l^3)\alpha\beta = 2l\zeta^2 + \xi\eta.$$

8. It is obvious that

$$\xi x + \eta y + \zeta z = 0$$

is the equation of the line EF joining the two conjugate poles, and it may be shown that

$$\alpha x + \beta y + \gamma z = 0$$

is the equation of the line IJ, which is the polar of E with respect to a conic which is the first or conic polar of F with respect to *any* syzygetic cubic. In fact, the equation of a syzygetic cubic will be $x^3+y^3+z^3+6\lambda xyz=0$, where λ is arbitrary, and the equation of the line in question is

$$(X\partial_x + Y\partial_y + Z\partial_z)(X'\partial_x + Y'\partial_y + Z'\partial_z)(x^3+y^3+z^3+6\lambda xyz) = 0;$$

or developing,

$$XX'x + YY'y + ZZ'z$$
$$+\lambda\{(YZ'+Y'Z)x + (ZX'+Z'X)y + (XY'+X'Y)z\} = 0;$$

and the function on the left-hand side is

$$\left(1-\frac{\lambda}{l}\right)(\alpha x + \beta y + \gamma z),$$

which proves the theorem.

9. The equations (A) by the elimination of (ξ, η, ζ), give

$$-l(\alpha^3+\beta^3+\gamma^3) + (-1+4l^3)\alpha\beta\gamma = 0,$$

which shows that the line IJ is a tangent of the Pippian: the proof of the theorem is given in this place because the relation just obtained between α, β, γ is required for the proof of some of the other theorems.

10. To find the coordinates of the point G in which the line EF joining two conjugate poles again meets the Hessian.

We may take for the coordinates of G,

$$uX+vX',\ uY+vY',\ uZ+vZ';$$

and substituting in the equation of the Hessian the terms containing u^3, v^3 disappear, and the ratio $u:v$ is determined by a simple equation. It thus appears that we may write

$$u=-3l^2(XX'^2+YY'^2+ZZ'^2)+(1+2l^2)(Y'Z'X+Z'X'Y+X'Y'Z)$$
$$v=\ \ 3l^2(X^2X'+Y^2Y'+Z^2Z')-(1+2l^2)(YZX'+ZXY'+XYZ');$$

hence introducing, as before, the quantities $\xi, \eta, \zeta, \alpha, \beta, \gamma$, we find

$$uX+vX'=3l^2(\gamma\eta-\beta\zeta)+(1+2l^2)(X^2Y'Z'-X'^2YZ);$$

but from the first of the equations (B),

$$X^2Y'Z'-X'^2YZ=\frac{1}{2l}(\gamma\eta-\beta\zeta),$$

and therefore the preceding value of $uX+vX'$ becomes

$$\left(3l^2-\frac{1+2l^2}{2l}\right)(\gamma\eta-\beta\zeta),$$

which is equal to

$$\frac{-1+4l^3}{2l}(\gamma\eta-\beta\zeta).$$

Hence throwing out the constant factor, we find, for the coordinates of the point G, the values

$$\gamma\eta-\beta\zeta,\quad \alpha\zeta-\gamma\xi,\quad \beta\xi-\alpha\eta.$$

11. To find the coordinates of the point O.

Consider O as the point of intersection of the tangents to the Hessian at the points E, F, then the coordinates of O are proportional to the terms of

$$\left\| \begin{matrix} 3l^2X^2-\overline{1+2l^2}YZ, & 3l^2Y^2-\overline{1+2l^2}ZX, & 3l^2Z^2-\overline{1+2l^2}XY \\ 3l^2X'^2-\overline{1+2l^2}Y'Z', & 3l^2Y'^2-\overline{1+2l^2}Z'X', & 3l^2Z'^2-\overline{1+2l^2}X'Y'. \end{matrix} \right\|$$

Hence the x-coordinate is proportional to

$$(3l^2Y^2-\overline{1+2l^2}ZX)(3l^2Z'^2-\overline{1+2l^2}X'Y')-(3l^2Z^2-\overline{1+2l^2}XY)(3l^2Y'^2-\overline{1+2l^2}Z'X'),$$

which is equal to

$$9l^4(Y^2Z'^2-Y'^2Z^2)+3l^2(1+2l^2)YY'(XY'-X'Y)+3l^2(1+2l^2)ZZ'(ZX'-Z'X)$$
$$-(1+2l^2)^2XX'(YZ'-Y'Z);$$

or introducing, as before, the quantities $\xi, \eta, \zeta, \alpha, \beta, \gamma$, to

$$-9l^2\alpha\xi+3l^2(1+2l^2)(\beta\zeta+\gamma\eta)-(1+2l^2)^2\alpha\xi$$
$$=(-1-13l^2-4l^4)\alpha\xi+2l^2(1+2l^2)(\beta\zeta+\gamma\eta).$$

But by the first of the equations (A) $\beta\zeta+\gamma\eta=2l\alpha\xi$, and the preceding value thus becomes $(-1-7l^3+8l^6)\alpha\xi$. Hence throwing out the constant factor the coordinates of the point O are found to be

$$\alpha\xi,\quad \beta\eta,\quad \gamma\zeta.$$

12. The points G, O are conjugate poles of the cubic.

Take a, b, c for the coordinates of G, and a', b', c' for the coordinates of O, we have

$$a,\ b,\ c=\gamma\eta-\beta\zeta,\quad \alpha\zeta-\gamma\xi,\quad \beta\xi-\alpha\eta,$$
$$a',\ b',\ c'=\alpha\xi\quad ,\quad \beta\eta\quad ,\quad \gamma\zeta.$$

These values give $aa'+l(bc'+b'c)$

$$=\alpha\xi(\gamma\eta-\beta\zeta)+l\{\beta\eta(\beta\xi-\alpha\eta)+\gamma\zeta(\alpha\zeta-\gamma\xi)\}$$
$$=\xi\eta(\alpha\gamma+l\beta^2)+\eta^2(-l\alpha\beta)+\zeta^2(l\alpha\gamma)+\xi\zeta(-\alpha\beta-l\gamma^2);$$

or substituting for $\xi\eta$, η^2, ξ^2, $\xi\zeta$ their values in terms of α, β, γ, this is

$$\left(-\frac{2}{l}\gamma^2-\frac{1}{l^2}\alpha\beta\right)(\alpha\gamma+l\beta^2)$$
$$+\left(\frac{\beta^2}{l^2}-4\gamma\alpha\right)(-l\alpha\beta)$$
$$+\left(\frac{\gamma^2}{l^2}-4\alpha\beta\right)(l\alpha\gamma)$$
$$+\left(-\frac{2}{l}\beta^2-\frac{1}{l}\alpha\gamma\right)(-\alpha\beta-l\gamma^2),$$

which is identically equal to zero. Hence, completing the system, we find

$$aa'+l(bc'+b'c)=0,$$
$$bb'+l(ca'+c'a)=0,$$
$$cc'+l(ab'+a'b)=0,$$

equations which show that O (as well as G) is a point of the Hessian, and that the points G, O are corresponding poles of the cubic.

13. The line EF joining a pair of conjugate poles of the cubic is a tangent of the Pippian*.

In fact, the equations (A), by the elimination of α, β, γ, give

$$-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\xi\eta\zeta=0,$$

which proves the theorem.

14. To find the equation of the pair of lines through F, and to show that these lines are tangents of the Pippian.

The equation of the pair of lines considered as the first or conic polar of the conjugate pole E, is

$$X(x^2+2lyz)+Y(y^2+2lzx)+Z(z^2+2lxy)=0.$$

* STEINER'S curve R_{φ}, in the particular case of a cubic basis-curve, is according to definition the envelope of the line EF, that is, the curve R_6 in the particular case in question is the Pippian.

Let one of the lines be

$$\lambda x+\mu y+\nu z=0,$$

then the other is

$$\frac{X}{\lambda}x+\frac{Y}{\mu}y+\frac{Z}{\nu}z=0;$$

and we find

$$\begin{aligned} 2lX\mu\nu-Y\nu^2-Z\mu^2&=0\\ -X\nu^2+2lY\nu\lambda-Z\lambda^2&=0\\ -X\mu^2-Y\lambda^2+2lZ\mu\nu&=0, \end{aligned}$$

any two of which determine the ratios λ, μ, ν.

The elimination of X, Y, Z gives

$$\begin{vmatrix} 2l\mu\nu, & -\nu^2, & -\mu^2\\ -\nu^2, & 2l\nu\lambda, & -\lambda^2\\ -\mu^2, & -\lambda^2, & 2l\lambda\mu \end{vmatrix}=0,$$

which is equivalent to

$$\lambda\mu\nu\{-l(\lambda^3+\mu^3+\nu^3)+(-1+4l^2)\lambda\mu\nu\}=0;$$

or omitting a factor, to

$$-l(\lambda^3+\mu^3+\nu^3)+(-1+4l^2)\lambda\mu\nu=0,$$

which shows that the line in question is a tangent of the Pippian.

15. To find the equation of the pair of lines through O.

The equation of the pair of lines through E is in like manner

$$X'(x^2+2lyz)+Y'(y^2+2lzx)+Z'(z^2+2lxy)=0;$$

and combining this with the foregoing equation,

$$X(x^2+2lyz)+Y(y^2+2lzx)+Z(z^2+2lxy)=0$$

of the pair of lines through F, viz. multiplying the two equations by

$$X^2X'+Y^2Y'+Z^2Z',\quad -(XX'^2+YY'^2+ZZ'^2),$$

and adding, then if as before

$$a:b:c=\gamma\eta-\beta\zeta:\alpha\zeta-\gamma\xi:\beta\xi-\alpha\eta,$$

we find as the equation of a conic passing through the points A, B, C, D, the equation

$$a(x^2+2lyz)+b(y^2+2lzx)+c(z^2+2lxy)=0.$$

But putting, as before,

$$a':b':c'=\alpha\xi:\beta\eta:\gamma\zeta,$$

then a', b', c' are the coordinates of the point O, and the equations

$$\begin{aligned} aa'+l(bc'+b'c)&=0,\\ bb'+l(ca'+c'a)&=0,\\ cc'+l(ab'+a'b)&=0 \end{aligned}$$

show that the conic in question is in fact the pair of lines through the point O.

16. To find the coordinates of the point Γ, which is the harmonic of G with respect to the points E, F.

The coordinates of the point in question are

$$uX-vX',\ uY-vY',\ uZ-vZ',$$

where u, v have the values given in No. 10, viz.

$$u=-3l^2(XX'^2+YY'^2+ZZ'^2)+(1+2l^3)(Y'Z'X+Z'X'Y+X'Y'Z),$$
$$v=\ 3l^2(X^2X'+Y^2Y'+Z^2Z')-(1+2l^3)(YZX'+ZXY'+XYZ');$$

these values give

$$uX-vX'=-3l^2\{2X^2X'^2+(XY'+X'Y)YY'+(XZ'+X'Z)ZZ'\}$$
$$+(1+2l^3)\{(XY'+X'Y)(XZ'+X'Z)+XX'(YZ'+Y'Z\};$$

and therefore

$$uX-vX'=-3l^2\left\{2\alpha^3-\frac{2}{l}\beta\gamma\right\}+(1+2l^3)\left\{\frac{1}{l^2}\beta\gamma-\frac{1}{l}\alpha^3\right\}$$
$$=\ \frac{1}{l^2}(1+8l^3)(-l\alpha^3+\beta\gamma);$$

and consequently, omitting the constant factor, the coordinates of Γ may be taken to be

$$-l\alpha^3+\beta\gamma,\ \ -l\beta^3+\gamma\alpha,\ \ -l\gamma^3+\alpha\beta.$$

17. The line through two consecutive positions of the point Γ is the line EF.

The coordinates of the point Γ are

$$-l\alpha^3+\beta\gamma,\ \ -l\beta^3+\gamma\alpha,\ \ -l\gamma^3+\alpha\beta;$$

and it has been shown that the quantities α, β, γ satisfy the equation

$$-l(\alpha^3+\beta^3+\gamma^3)+(-1+4l^3)\alpha\beta\gamma=0.$$

Hence, considering α, β, γ as variable parameters connected by this equation, the equation of the line through two consecutive positions of the point Γ is

$$\begin{vmatrix} -3l\alpha^3+(-1+4l^3)\beta\gamma, & -3l\beta^3+(-1+4l^3)\gamma\alpha, & -3l\gamma^3+(-1+4l^3)\alpha\beta \\ x, & -2l\alpha\ , & \gamma\ , & \beta \\ y, & \gamma\ , & -2l\beta\ , & \alpha \\ z, & \beta\ , & \alpha\ , & -2l\gamma \end{vmatrix}=0;$$

and representing this equation by

$$Lx+My+Nz=0,$$

we find

$$L=(4l^3\beta\gamma-\alpha^3)(-3l\alpha^3+(-1+4l^3)\beta\gamma)$$
$$+(\alpha\beta+2l\gamma^3)(-3l\beta^3+(-1+4l^3)\gamma\alpha)$$
$$+(\alpha\gamma+2l\beta^3)(-3l\gamma^3+(-1+4l^3)\alpha\beta);$$

or multiplying out and collecting,

$$L=3l\alpha^4+(-1-8l^3)\alpha^3\beta\gamma+(-5l+8l^4)(\alpha\beta^3+\alpha\gamma^3)+(-16l^3+16l^6)\beta^3\gamma^3;$$

but the equation

$$-l(\alpha^3+\beta^3+\gamma^3)+(-1+4l^3)\alpha\beta\gamma=0$$

gives

$$3l\alpha^3=-3l(\alpha\beta^3+\alpha\gamma^3)+(-3+12l^3)\alpha^2\beta\gamma,$$

and we have

$$\begin{aligned}\mathrm{L}&=(-4+4l^3)\alpha^2\beta\gamma+(-8l+8l^4)(\alpha\beta^3+\alpha\gamma^3)+(-16l^6+16l^3)\beta^3\gamma^3\\&=(-4+4l^3)\left(\alpha^2\beta\gamma+2l(\alpha\beta^3+\alpha\gamma^3)+4l^3\beta^3\gamma^3\right)\\&=(-4+4l^3)(\alpha\gamma+2l\beta^3)(\alpha\beta+2l\gamma^3);\end{aligned}$$

or, in virtue of the equations (D),

$$\mathrm{L}=(-4+4l^3)l^2\zeta\xi.l^2\xi\eta=(-4+4l^3)l^4\xi^2\eta\zeta=(-4+4l^3)l^4\xi\eta\zeta.\xi.$$

Hence, omitting the common factor, we find $\mathrm{L}:\mathrm{M}:\mathrm{N}=\xi:\eta:\zeta$, and the equation $\mathrm{L}x+\mathrm{M}y+\mathrm{N}z=0$ becomes

$$\xi x+\eta y+\zeta z=0,$$

which is the equation of the line EF, that is, the line through two consecutive positions of Γ is the line EF; or what is the same thing, the line EF touches the Pippian in the point Γ which is the harmonic of G with respect to the points E, F.

18. The lineo-polar envelope of the line EF, with respect to the cubic, is the pair of lines OE, OF.

The equation of the pair of lines OE, OF, considered as the tangents to the Hessian at the points E, F, is

$$\left.\begin{aligned}&\{(3l^2\mathrm{X}^3-\overline{1+2l^3}\mathrm{YZ})x+(3l^2\mathrm{Y}^3-\overline{1+2l^3}\mathrm{ZX})y+(3l^2\mathrm{Z}^3-\overline{1+2l^3}\mathrm{XY})z\}\\&\times\{(3l^2\mathrm{X}'^3-\overline{1+2l^3}\mathrm{Y}'\mathrm{Z}')x+(3l^2\mathrm{Y}'^3-\overline{1+2l^3}\mathrm{Z}'\mathrm{X}')y+(3l^2\mathrm{Z}'^3-\overline{1+2l^3}\mathrm{X}'\mathrm{Y}')z\}\end{aligned}\right\}=0.$$

And on the left-hand side the coefficient of x^2 is

$$9l^2\mathrm{X}^2\mathrm{X}'^2-3l^2(1+2l^3)(\mathrm{X}^2\mathrm{Y}'\mathrm{Z}'+\mathrm{X}'^2\mathrm{YZ})+(1+2l^3)^2\mathrm{YY}'\mathrm{ZZ}',$$

which is equal to

$$9l^2\alpha^2-3l^2(1+2l^3)(l^2\beta\gamma+\tfrac{1}{l}\alpha^2)+(1+2l^3)^2\beta\gamma,$$

that is,

$$\tfrac{1}{l}(-l+l^4)\{3l\alpha^2+2(1+2l^3)\beta\gamma\};$$

and the coefficient of yz is

$$9l^2(\mathrm{Y}^2\mathrm{Z}'^2+\mathrm{Y}'^2\mathrm{Z}^2)-3l^2(1+2l^3)\left(\mathrm{YY}'(\mathrm{XY}'+\mathrm{X}'\mathrm{Y})+\mathrm{ZZ}'(\mathrm{XZ}'+\mathrm{X}'\mathrm{Z})\right)+(1+2l^3)^2\mathrm{XX}'(\mathrm{YZ}'+\mathrm{Y}'\mathrm{Z}),$$

which is equal to

$$9l^2\left(\tfrac{1}{l^2}\alpha^2-2\beta\gamma\right)-3l^2(1+2l^3)\left(-\tfrac{2}{l}\beta\gamma\right)+(1+2l^3)^2\alpha\left(-\tfrac{1}{l}\alpha\right),$$

that is,

$$\tfrac{1}{l}(-l+l^4)\{(1-4l^3)\alpha^2-6l^2\beta\gamma\}.$$

Hence completing the system and throwing out the constant factor, the equation of the pair of lines is

$$\begin{aligned}&(3l\alpha^2+2(1+2l^3)\beta\gamma,\ 3l\beta^2+2(1+2l^3)\gamma\alpha,\ 3l\gamma^2+2(1+2l^3)\alpha\beta,\\ &\qquad(1-4l^3)\alpha^2-6l^2\beta\gamma,\ (1-4l^3)\beta^2-6l^2\gamma\alpha,\ (1-4l^3)\gamma^2-6l^2\alpha\beta\,\Big)(x, y, z)^2=0.\end{aligned}$$

But the equation of the line EF is $\xi x+\eta y+\zeta z=0$, and the equation of its lineo-polar envelope is

$$\begin{vmatrix} \xi, & \eta, & \zeta \\ \xi, & x, & lz, & ly \\ \eta, & lz, & y, & lx \\ \zeta, & ly, & lx, & z \end{vmatrix}=0;$$

or expanding,

$$(yz-l^2x^2,\ zx-l^2y^2,\ xy-l^2z^2,\ l^2yz-lx^2,\ l^2zx-ly^2,\ l^2xy-lz^2\,\Big)(\xi, \eta, \zeta)^2=0;$$

or arranging in powers of x, y, z,

$$(-l^2\xi^2-2l\eta\zeta,\ -l^2\eta^2-2l\zeta\xi,\ -l^2\zeta^2-2l\xi\eta,\ \tfrac{1}{2}\xi^2+l^2\eta\zeta,\ \tfrac{1}{2}\eta^2+l^2\zeta\xi,\ \tfrac{1}{2}\zeta^2+l^2\xi\eta\,\Big)(x, y, z)^2=0.$$

And if in this equation we replace ξ^2, &c. by their values in terms of α, β, γ, as given by the equations (D), we obtain the equation given as that of the pair of lines OE, OF.

19. It remains to prove the theorem with respect to the connexion of the lines EF, IJ.

The equations (A) show that the two lines

$$\xi x+\eta y+\zeta z=0,$$
$$\alpha x+\beta y+\zeta z=0,$$

(where ξ, η, ζ and α, β, γ have the values before attributed to them) are conjugate polars with respect to the curve of the third class,

$$l(\xi^3+\eta^3+\zeta^3)-3\xi\eta\zeta=0,$$

in which equation ξ, η, ζ denote current line coordinates. The curve in question is of the form $APU+BQU=0$. We have, in fact, identically,

$$3T.PU-4S.QU=(1+8l^3)^2\{l(\xi^3+\eta^3+\zeta^3)-3\xi\eta\zeta\}.$$

It is clear that the curve in question must have the curve $PU=0$ for its Hessian; and in fact, in the formula of my Third Memoir,

$$\begin{aligned}H(6\alpha PU+\beta QU)=&(-2T,\ 48S^2,\ 18TS,\ T^2+16S^2\,\Big)(\alpha, \beta)^2\,PU\\ &+(\ 8S\ ,\ T\ ,-8S^2,\ -TS\ \Big)(\alpha, \beta)^2\,QU.\end{aligned}$$

The coefficient of QU is

$$(8S\alpha+T\beta)(\alpha^2-S\beta^2);$$

and therefore, putting $\alpha=\frac{1}{2}T$, $\beta=-4S$, we find

$$H(3T.PU-4S.QU)=-\tfrac{1}{4}(T^2-64S^3)^2PU.$$

Article No. 20.—*Theorem relating to the curve of the third class, mentioned in the preceding article.*

20. The consideration of the curve $3T.PU-4S.QU=0$, gives rise to another geometrical theorem. Suppose that the line (ξ, η, ζ), that is, the line whose equation is $\xi x+\eta y+\zeta z=0$, is with respect to this curve of the third class one of the four polars of a point (X, Y, Z) of the Hessian, and that it is required to find the envelope of the line $\xi x+\eta y+\zeta z=0$.

We have

$$X : Y : Z = l\xi^2-\eta\zeta : l\eta^2-\zeta\xi : l\zeta^2-\xi\eta,$$

and X, Y, Z are to be eliminated from these equations, and the equation

$$l^2(X^2+Y^2+Z^2)-(1+2l^2)XYZ=0$$

of the Hessian. We have

$$\begin{aligned} X^2+Y^2+Z^2= \;& l^2(\xi^2+\eta^2+\zeta^2)^2 \\ & -3l^2(\xi^2+\eta^2+\zeta^2)\xi\eta\zeta \\ & +9l\xi^2\eta^2\zeta^2 \\ & -(1+2l^2)(\eta^2\zeta^2+\zeta^2\xi^2+\xi^2\eta^2) \\ XYZ= \;& l(\xi^2+\eta^2+\zeta^2)\xi\eta\zeta \\ & +(-1+l^2)\xi^2\eta^2\zeta^2 \\ & -l^2(\eta^2\zeta^2+\zeta^2\xi^2+\xi^2\eta^2), \end{aligned}$$

and thence

$$\begin{aligned} HU= \;& l^2(\xi^2+\eta^2+\zeta^2)^3 \\ & -(l+5l^4)(\xi^2+\eta^2+\zeta^2)\xi\eta\zeta \\ & +(1+10l^2-2l^6)\xi^2\eta^2\zeta^2; \end{aligned}$$

and equating the right-hand side to zero, we have the equation in line coordinates of the curve in question, which is therefore a curve of the sixth class in quadratic syzygy with the Pippian and Quippian.

Article No. 21.—*Geometrical definition of the Quippian.*

21. I have not succeeded in obtaining any good geometrical definition of the Quippian, and the following is only given for want of something better.

The curve

$$T.PU\{P6H(\alpha U+6\beta HU)\}-P6HU\{T(\alpha U+6\beta HU).P(\alpha U+6\beta HU)\}=0,$$

which is derived in what may be taken to be a known manner from the cubic, is in general a curve of the sixth class. But if the syzygetic cubic $\alpha U+6\beta HU=0$ be properly selected, viz. if this curve be such that its Hessian breaks up into three lines, then both the Pippian of the cubic $\alpha U+6\beta HU=0$, and the Pippian of its Hessian will break up into the same three points, which will be a portion of the curve of the sixth class, and discarding these three points the curve will sink down to one on the third class, and will in fact be the Quippian of the cubic.

To show this we may take

$$\alpha U+6\beta HU=x^3+y^3+z^3=0$$

as the equation of the syzygetic cubic satisfying the prescribed condition, for this value in fact gives

$$H(\alpha U+6\beta HU)=-xyz=0,$$

a system of three lines. We find, moreover,

$$P(\alpha U+6\beta HU)=P(x^3+y^3+z^3)=-\xi\eta\zeta$$

and

$$P\{6H(\alpha U+6\beta HU)\}=P(-6xyz)=-4\xi\eta\zeta,$$

the latter equation being obtained by first neglecting all but the highest power of l in the expression of PU, and then writing $l=-1$: we have also $T(\alpha U+6\beta HU)=1$. Substituting the above values, the curve of the sixth class is

$$\xi\eta\zeta\{-4T.PU+P(6HU)\}=0;$$

or throwing out the factor $\xi\eta\zeta$, we have the curve of the third class,

$$-4T.PU+P(6HU)=0.$$

Now the general expression in my Third Memoir, viz.

$$P(\alpha U+6\beta HU)=(\alpha^3+12S\alpha\beta^3+4T\beta^3)PU$$
$$+(\alpha^3\beta-4S\beta^3)QU,$$

putting $\alpha=0$, $\beta=1$, gives

$$P(6HU)=4T.PU-4S.QU,$$

or what is the same thing,

$$-4T.PU+P(6HU)=-4S.QU;$$

and the curve of the third class is therefore the Quippian $QU=0$. It may be remarked, that for a cubic $U=0$ the Hessian of which breaks up into three lines, the above investigation shows that we have $PU=-\xi\eta\zeta$, $P(6HU)=-4\xi\eta\zeta$, and $T=1$, and consequently that $-4T.PU+P(6HU)$ ought to vanish identically; this in fact happens in virtue of the factor S on the right-hand side, the invariant S of a cubic of the form in question being equal to zero; the appearance of the factor S on the right-hand side is thus accounted for *à priori*.

Article No 22.—*Theorem relating to a line which meets three given conics in six points in involution.*

22. The envelope of a line which meets three given conics, the first or conic polars of any three points with respect to the cubic, in six points in involution, is the Pippian.

It is readily seen that if the theorem is true with respect to the three conics,

$$\frac{dU}{dx}=0,\quad \frac{dU}{dy}=0,\quad \frac{dU}{dz}=0,$$

it is true with respect to any three conics whatever of the form

$$\lambda\frac{dU}{dx}+\mu\frac{dU}{dy}+\nu\frac{dU}{dz}=0,$$

that is, with respect to any three conics, each of them the first or conic polar of some point (λ, μ, ν) with respect to the cubic. Considering then these three conics, take as the equation of the line $\xi x+\eta y+\zeta z=0$, and let (X, Y, Z) be the coordinates of a point of intersection with the first conic, we have

$$\xi X+\eta Y+\zeta Z=0,$$
$$X^2+2lYZ=0;$$

and combining with these a linear equation

$$\alpha X+\beta Y+\gamma Z=0,$$

in which (α, β, γ) are arbitrary quantities, we have

$$X : Y : Z=\gamma\eta-\beta\zeta : \alpha\zeta-\gamma\xi : \beta\xi-\alpha\eta;$$

and hence

$$(\gamma\eta-\beta\zeta)^2+2l(\alpha\zeta-\gamma\xi)(\beta\xi-\alpha\eta)=0,$$

an equation in (α, β, γ) which is in fact the equation in line coordinates of the two points of intersection with the first conic. Developing and forming the analogous equations, we find

$$\begin{array}{lllll} (-2l\eta\zeta, & \zeta^2, & \eta^2, & -\eta\zeta-l\xi^2, & l\xi\eta, \quad l\xi\zeta \\ (\zeta^2, & -2l\zeta\xi, & \xi^2, & l\xi\eta, & -\zeta\xi-l\eta^2, \quad l\eta\zeta \\ (\eta^2, & \xi^2, & -2l\xi\eta, & l\zeta\xi, & l\eta\zeta, \quad -\xi\eta-l\zeta^2 \end{array} \Big)(\alpha, \beta, \gamma)^2=0,$$

which are respectively the equations in line coordinates of the three pairs of intersections.

Now combining these equations with the equation $\gamma=0$, we have the equations of the pairs of lines joining the points of intersection with the point $(x=0, y=0)$, and if the six points are in involution, the six lines must also be in involution, or the condition for the involution of the six points is

$$\begin{vmatrix} -2l\eta\zeta, & \zeta^2, & l\xi\zeta, \\ \zeta^2, & -2l\zeta\eta, & l\eta\zeta, \\ \eta^2, & \xi^2, & -\xi\eta-l\zeta^2 \end{vmatrix}=0,$$

that is,

$$4l^2\zeta^2\xi\eta(-\xi\eta-l\zeta^2)+l\eta^2\zeta^4+l\xi^2\zeta^4$$
$$+2l^2\xi^2\eta^2\zeta^2+2l^2\xi^2\eta^2\zeta^2+\zeta^4(-\xi\eta-l\zeta^2)=0;$$

or reducing and throwing out the factor ζ^4, we find

$$-l(\xi^2+\eta^2+\zeta^2)+(-1+4l^2)\xi\eta\zeta=0,$$

which shows that the line in question is a tangent of the Pippian.

It is to be remarked that any three conics whatever may be considered as the first or conic polars of three properly selected points with respect to a properly selected cubic curve. The theorem applies therefore to any three conics whatever, but in this case the cubic curve is not given, and the Pippian therefore stands merely for a curve of the third class, and the theorem is as follows, viz. the envelope of a line which meets any three conics in six points in involution, is a curve of the third class.

Article No. 23.—*Completion of the theory in Liouville, and comparison with analogous theorems of* HESSE.

In order to convert the foregoing theorem into its reciprocal, we must replace the cubic $U=0$ by a curve of the third class, that is we must consider the coordinates which enter into the equation as line coordinates; and it of course follows that the coordinates which enter into the equation $PU=0$ must be considered as point coordinates, that is we must consider the Pippian as a curve of the third order: we have thus the theorem; The locus of a point such that the tangents drawn from it to three given conics (the first or conic poles of any three lines with respect to a curve of the third class) form a pencil in involution, is the Pippian considered as a curve of the third order. This in fact completes the fundamental theorem in my memoirs in *Liouville* above referred to, and establishes the analogy with HESSE's results in relation to the Hessian; to show this I set out the two series of theorems as follows:—

HESSE, in his memoirs "On Curves of the Third Order and Curves of the Third Class," *Crelle*, tt. xxviii. xxxvi. and xxxviii., has shown as follows:—

(α) The locus of a point such that its polars, with respect to the three conics $X=0$, $Y=0$, $Z=0$ (or more generally its polars with respect to all the conics of the series $\lambda X+\mu Y+\nu Z=0$) meet in a point, is a curve of the third order $V=0$.

(β) Conversely, given a curve of the third order $V=0$, there exists a series of conics such that the polars with respect to all the conics of any point whatever of the curve $V=0$, meet in a point.

(γ) The equation of any one of the conics in question is

$$\lambda\frac{dU}{dx}+\mu\frac{dU}{dy}+\nu\frac{dU}{dz}=0,$$

that is, the conic is the first or conic polar of a point (λ, μ, ν) with respect to a certain curve of the third order $U=0$; and this curve is determined by the condition that its Hessian is the given curve $V=0$, that is, we have $V=HU$.

(δ) The equation $V=HU$ is solved by assuming $U=aV+bHV$, for we have then $H(aV+bHV)=AV+BHV$, where A, B are given cubic functions of a, b, and thence $V=HU=AV+BHV$, or $A=1$, $B=0$; the latter equation gives what is alone important, the ratio $a:b$, and it thus appears that there are three distinct series of conics, each of them having the above-mentioned relation to the given curve of the third order $V=0$.

In the memoirs in *Liouville* above referred to, I have in effect shown that—

(α') The locus of a point such that the tangents from it to three conics, represented in line coordinates by the equations $X=0$, $Y=0$, $Z=0$ (or more generally with respect to any three conics of the series $\lambda X+\mu Y+\nu Z=0$) form a pencil in involution, is a curve of the third order $V=0$.

(β') Conversely, given a curve of the third order $V=0$, there exists a series of conics such that the tangents from any point whatever of the curve to any three of the conics, form a pencil in involution.

Now, considering the coordinates which enter into the equation of the Pippian as point coordinates, and consequently the Pippian as a curve of the third order, I am able to add as follows:—

(γ) The equation in line coordinates of any one of the conics in question is

$$\lambda\frac{dU}{d\xi}+\mu\frac{dU}{d\eta}+\nu\frac{dU}{d\zeta}=0,$$

that is, the conic is the first or conic polar of a line (λ, μ, ν) with respect to a certain curve of the third class $U=0$; and this curve is determined by the condition that its Pippian is the given curve of the third order $V=0$, that is, we have $V=PU$.

(δ) The equation $V=PU$ is solved by assuming $U=aPV+bQV$, for we have then $P(aPV+bQV)=AV+BHV$, where A and B are given cubic functions of a, b; and thence $V=PU=AV+BHV$, or $A=1$, $B=0$; the latter equation gives what is alone important, the ratio $a:b$; and it thus appears that there are three distinct curves of the third class $U=0$, and therefore (what indeed is shown in the Memoirs in *Liouville*) three distinct series of conics having the above-mentioned relation to the given curve of the third order $V=0$.

It is hardly necessary to remark that the preceding theorems, although precisely analogous to those of HESSE, are entirely distinct theorems, that is the two series are not connected together by any relation of reciprocity.

Article Nos. 24 to 28.—*Various investigations and theorems.*

24. Reverting to the theorem (No. 18), that the lineo-polar envelope of the line EF is the pair of lines OE, OF; the line EF is any tangent of the Pippian, hence the theorem includes the following one:—

The lineo-polar envelope with respect to the cubic, of any tangent of the Pippian, is a pair of lines.

And conversely,

The Pippian is the envelope of a line such that the lineo-polar envelope of the line with respect to the cubic is a pair of lines.

It is I think worth while to give an independent proof. It has been shown that the equation of the lineo-polar envelope with respect to the cubic, of the line $\xi x+\eta y+\zeta z=0$ (where ξ, η, ζ are arbitrary quantities), is

$$(-P^2\xi^2-2l\eta\zeta,\ -P^2\eta^2-2l\zeta\xi,\ -P^2\zeta^2-2l\xi\eta,\ \tfrac{1}{2}\xi^2+P^2\eta\zeta,\ \tfrac{1}{2}\eta^2+P^2\zeta\xi,\ \tfrac{1}{2}\zeta^2+P^2\xi\eta)(x, y, z)^3=0;$$

and representing this equation by

$$\tfrac{1}{3}(a, b, c, f, g, h)(x, y, z)^3=0,$$

we find

$$bc-f^2=\xi(-\xi^3+8P^2\eta^3+8P^2\zeta^3+12P^2\xi\eta\zeta),$$
$$ca-g^2=\eta(8P^2\xi^3-\eta^3+8P^2\zeta^3+12P^2\xi\eta\zeta),$$
$$ab-h^2=\zeta(8P^2\xi^3+8P^2\eta^3-\zeta^3+12P^2\xi\eta\zeta),$$

$$gh-af=\xi\left(2l^2(\xi^2+\eta^2+\zeta^2)+4l(1+2l^2)\xi\eta\zeta\right)+(1+8l^2)\eta^2\zeta^2,$$
$$hf-bg=\eta\left(2l^2(\xi^2+\eta^2+\zeta^2(+4l(1+2l^2)\xi\eta\zeta\right)+(1+8l^2)\zeta^2\xi^2,$$
$$fg-ch=\zeta\left(2l^2(\xi^2+\eta^2+\zeta^2)+4l(1+2l^2)\xi\eta\zeta\right)+(1+8l^2)\xi^2\eta^2.$$

and after all reductions,

$$abc-af^2-bg^2-ch^2+2fgh$$
$$=[-l(\xi^2+\eta^2+\zeta^2)+(-1+4l^2)\xi\eta\zeta]^2=(\mathrm{PU})^2,$$

or the condition in order that the conic may break up into a pair of lines is $\mathrm{PU}=0$.

25. The following formulæ are given in connexion with the foregoing investigation, but I have not particularly considered their geometrical signification. The lineo-polar envelope of an arbitrary line $\xi x+\eta y+\zeta z=0$, with respect to the cubic,

$$x^3+y^3+z^3+6lxyz=0,$$

has been represented by

$$(a, b, c, f, g, h\,\mathrm{X}x, y, z)^3=0;$$

and if in like manner we represent the lineo-polar envelope of the same line, with respect to a syzygetic cubic,

$$x^3+y^3+z^3+6l'xyz=0,$$

by

$$(a', b', c', f', g', h'\,\mathrm{X}x, y, z)^3=0,$$

then we have

$$a'(bc-f^2)+b'(ca-g^2)+c'(ab-h^2)+2f'(gh-af)+2g'(hf-bg)+2h'(fg-ch)$$
$$=(l^2+2l^2)(\xi^3+\eta^3+\zeta^3)^2$$
$$+(2l'+4l-32l^2l'+8l^4)(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta$$
$$+(24l^2l'^2+48l^4l'^2-72l^2l'+24l^2+3)\xi^2\eta^2\zeta^2,$$

which may be verified by writing $l'=l$, in which case the right-hand side becomes as it should do, $3(\mathrm{PU})^2$. If $l'=-\frac{1+2l^3}{6l^2}$, that is, if the syzygetic cubic be the Hessian, then the formula becomes

$$a'(bc-f^2)+\&\mathrm{c.}=\frac{1}{36l^2}\left\{\begin{array}{l}(1+4l^3+76l^6)(\xi^3+\eta^3+\zeta^3)^2\\+12l^2(-1+26l^3+56l^6)(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta\\+12l(2+57l^3+168l^6+16l^9)\xi^2\eta^2\zeta^2\end{array}\right\}$$

which is equal to

$$\frac{1}{36l^2}\left\{\overline{\mathrm{QU}}^2-24\mathrm{S}.\overline{\mathrm{PU}}^3\right\}.$$

26. The equation

$$(bc'+b'c-2ff'\ldots gh'+g'h-af'-a'f,\ldots\,\mathrm{X}\xi, \eta, \zeta)^2=0$$

is the equation in line coordinates of a conic, the envelope of the line which cuts harmonically the conics

$$(a, b, c, f, g, h\,\mathrm{X}x, y, z)^2=0,$$
$$(a', b', c', f', g', h'\,\mathrm{X}x, y, z)^2=0;$$

and if a, b, &c., a', &c. have the values before given to them, then the coefficients of the equation are

$$bc'+b'c-2ff' = \xi\{-\xi^2+4ll'(l+l')(\eta^2+\zeta^2)+(16ll'-2l^2-2l'^2)\eta\zeta,$$
$$ca'+c'a-2gg' = \eta\{-\eta^2+4ll'(l+l')(\zeta^2+\xi^2)+(16ll'-2l^2-2l'^2)\zeta\xi,$$
$$ab'+a'b-2hh' = \zeta\{-\zeta^2+4ll'(l+l')(\xi^2+\eta^2)+(16ll'-2l^2-2l'^2)\xi\eta,$$

$$gh'+g'h-af'-a'f = \xi\{(l^2+l'^2)(\xi^2+\eta^2+\zeta^2)+(2l+2l'+8l^2l'^2)\xi\eta\zeta\}+\big(1+4ll'(l+l')\big)\eta^2\zeta^2,$$
$$hf'+h'f-bg'-b'g = \eta\{(l^2+l'^2)(\xi^2+\eta^2+\zeta^2)+(2l+2l'+8l^2l'^2)\xi\eta\zeta\}+\big(1+4ll'(l+l')\big)\zeta^2\xi^2,$$
$$fg'+f'g-ch'-c'h = \zeta\{(l^2+l'^2)(\xi^2+\eta^2+\zeta^2)+(2l+2l'+8l^2l'^2)\xi\eta\zeta\}+\big(1+4ll'(l+l')\big)\xi^2\eta^2;$$

and we thence obtain

$$\begin{aligned}(bc'+b'c-2ff', \ldots gh'+g'h-af'-a'f, \ldots)(\xi, \eta, \zeta)^3 \\ = \quad -(\xi^3+\eta^3+\zeta^3)^2 \\ +(l^2+l'^2+16ll')(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta \\ +(6l+6l'+24l^2l'^2)\xi^2\eta^2\zeta^2 \\ +\big(4+16ll'(l+l')\big)(\zeta^2\eta^2+\xi^2\zeta^2+\eta^2\xi^2) \\ =0\end{aligned}$$

as the condition which expresses that a line $\xi x+\eta y+\zeta z=0$ cuts harmonically its lineo-polar envelopes with respect to the cubic and with respect to a syzygetic cubic.

27. To find the locus of a point such that its second or line polar with respect to the cubic may be a tangent of the Pippian. Let the coordinates of the point be (x, y, z); then if $\xi x+\eta y+\zeta z=0$ be the equation of the polar, we have

$$\xi : \eta : \zeta = x^3+2lyz : y^3+2lzx : z^3+2lxy,$$

and the line in question being a tangent to the Pippian,

$$-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^2)\xi\eta\zeta=0.$$

But the preceding values give

$$\begin{aligned}\xi^3+\eta^3+\zeta^3 = (x^3+y^3+z^3)^3+6l\,(x^3+y^3+z^3)xyz+36l^2x^3y^3z^3 \quad +(-2+8l^3)(y^3z^3+z^3x^3+x^3y^3) \\ \xi\eta\zeta \quad = \quad 4l^2(x^3+y^3+z^3)xyz+(1+8l^3)x^3y^3z^3+ \quad 2l \quad (y^3z^3+z^3x^3+x^3y^3);\end{aligned}$$

and we have therefore

$$l(x^3+y^3+z^3)^3+(10l^3-16l^2)(x^3+y^3+z^3)xyz+(1+40l^3-32l^6)x^3y^3z^3=0;$$

or introducing U, HU in place of $x^3+y^3+z^3$, xyz, the equation becomes

$$-\mathrm{S.U}^3+(\mathrm{HU})^3=0,$$

which is the equation of the locus in question.

28. The locus of a point such that its second or line polar with respect to the cubic is a tangent of the Quippian, is found in like manner by substituting the last-mentioned values of ξ, η, ζ in the equation

$$\mathrm{QU}=(1-10l^3)(\xi^3+\eta^3+\zeta^3)-6l^2(5+4l^3)\xi\eta\zeta.$$

We find as the equation of the locus,

$$(1-10l^3)(x^3+y^3+z^3)^3+6l(1-30l^3-16l^6)(x^3+y^3+z^3)xyz$$
$$+6l^3(1-104l^3-32l^6)x^3y^3z^3$$
$$-2(1+8l^3)^3(y^3z^3+z^3x^3+x^3y^3)=0,$$

where the function on the left-hand side is the octicovariant $\Theta_{ii}U$ of my Third Memoir, the covariant having been in fact defined so as to satisfy the condition in question. And I have given in the memoir the following expression for $\Theta_{ii}U$, viz.

$$\Theta_{ii}U=(1-16l^3-6l^6)U^3$$
$$+(6l\quad\quad)U.HU$$
$$+(6l^3\quad\quad)(HU)^3$$
$$-2(1+8l^3)^3(y^3z^3+z^3x^3+x^3y^3).$$

Article Nos. 29 to 31.—*Formulæ for the intersection of a cubic curve and a line.*

29. If the line $\xi x+\eta y+\zeta z=0$ meet the cubic

$$x^3+y^3z^3+6lxyz=0$$

in the points

$$(x_1, y_1, z_1),\ (x_2, y_2, z_2),\ (x_3, y_3, z_3),$$

then we have

$$x_1x_2x_3 : y_1y_2y_3 : z_1z_2z_3=\eta^3-\zeta^3 : \zeta^3-\xi^3 : \xi^3-\eta^3.$$

It will be convenient to represent the equation of the cubic by the abbreviated notation $(1, 1, 1, l)(x, y, z)^3=0$; we have the two equations

$$(1, 1, 1, l)(x, y, z)^3=0,$$
$$\xi x+\eta y+\zeta z\quad\quad=0;$$

and if to these we join a linear equation with arbitrary coefficients,

$$\alpha x+\beta y+\gamma z=0,$$

then the second and third equations give

$$x : y : z=\beta\zeta-\gamma\eta : \gamma\xi-\alpha\zeta : \alpha\eta-\beta\xi;$$

and substituting these values in the first equation, we obtain the resultant of the system. But this resultant will also be obtained by substituting, in the third equation, a system of simultaneous roots of the first and second equations, and equating to zero the product of the functions so obtained*. We must have therefore

$$(1, 1, 1, l)(\beta\zeta-\gamma\eta,\ \gamma\xi-\alpha\zeta,\ \alpha\eta-\beta\xi)^3=(\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)(\alpha x_3+\beta y_3+\gamma z_3);$$

and equating the coefficients of α^3, β^3, γ^3, we obtain the above-mentioned relations.

30. If a tangent to the cubic

$$x^3+y^3+z^3+6lxyz=0$$

* This is in fact the general process of elimination given in SCHLÄFLI'S Memoir, "Ueber die Resultante einer Systemes mehrerer algebraischer Gleichungen," Vienna Trans. 1852.

at a point (x_1, y_1, z_1) of the cubic meet the cubic in the point (x_2, y_2, z_2), then

$$x_2 : y_2 : z_2 = x_1(y_1^3 - z_1^3) : y_1(z_1^3 - x_1^3) : z_1(x_1^3 - y_1^3).$$

For if the equation of the tangent is $\xi x + \eta y + \zeta z = 0$, then

$$x_1^2 x_2 : y_1^2 y_2 : z_1^2 z_2 = \eta^3 - \zeta^3 : \zeta^3 - \xi^3 : \xi^3 - \eta^3,$$

and

$$\xi : \eta : \zeta = x_1^3 + 2ly_1z_1 : y_1^3 + 2lz_1x_1 : z_1^3 + 2lx_1y_1.$$

These values give

$$\begin{aligned}\eta^3 - \zeta^3 &= (y_1^3 - z_1^3)(y_1^3 + z_1^3 + 6lx_1y_1z_1 - 8l^3x_1^3) \\ &= (y_1^3 - z_1^3) \times -(1 + 8l^3)x_1^3,\end{aligned}$$

since (x_1, y_1, z_1) is a point of the cubic, and forming in like manner the values of $\zeta^3 - \xi^3$ and $\xi^3 - \eta^3$, we obtain the theorem.

31. The preceding values of (x_2, y_2, z_2) ought to satisfy

$$(x_1^3 + 2ly_1z_1)x_2 + (y_1^3 + 2lz_1x_1)y_2 + (z_1^3 + 2lx_1y_1)z_2 = 0$$

$$x_2^3 + y_2^3 + z_2^3 + 6lx_2y_2z_2 = 0,$$

in fact the first equation is satisfied identically, and for the second equation we obtain

$$\begin{aligned}x_2^3 + y_2^3 + z_2^3 &= x_1^3(y_1^3 - z_1^3)^3 + y_1^3(z_1^3 - x_1^3)^3 + z_1^3(x_1^3 - y_1^3)^3 \\ &= -x_1^6(y_1^3 - z_1^3) - y_1^6(z_1^3 - x_1^3) - z_1^6(x_1^3 - y_1^3) \\ &= (x_1^3 + y_1^3 + z_1^3)(y_1^3 - z_1^3)(z_1^3 - x_1^3)(x_1^3 - y_1^3) \\ x_2y_2z_2 &= x_1y_1z_1(y_1^3 - z_1^3)(z_1^3 - x_1^3)(x_1^3 - y_1^3),\end{aligned}$$

and consequently

$$x_2^3 + y_2^3 + z_2^3 + 6lx_2y_2z_2 = (x_1^3 + y_1^3 + z_1^3 + 6lx_1y_1z_1)(y_1^3 - z_1^3)(z_1^3 - x_1^3)(x_1^3 - y_1^3) = 0,$$

which verifies the theorem. It is proper to add (the remark was made to me by Professor SYLVESTER) that the foregoing values

$$x_2 : y_2 : z_2 = x_1(y_1^3 - z_1^3) : y_1(z_1^3 - x_1^3) : z_1(x_1^3 - y_1^3)$$

satisfy *identically* the relation

$$\frac{x_2^3 + y_2^3 + z_2^3}{x_2y_2z_2} = \frac{x_1^3 + y_1^3 + z_1^3}{x_1y_1z_1}.$$

Article Nos. 32 to 34.—*Formulæ for the Satellite line and Point.*

32. The line $\xi x + \eta y + \zeta z = 0$ meets the cubic

$$x^3 + y^3 + z^3 + 6lxyz = 0$$

in three points, and the tangents to the cubic at these points meet the cubic in three points lying in a line, which has been called the Satellite line of the given line.

To find the equation of the satellite line; suppose that (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are the coordinates of the point in which the given line meets the cubic; then we have, as before,

$$(1, 1, 1, \Omega)(\beta\zeta - \gamma\eta, \gamma\xi - \alpha\zeta, \alpha\eta - \beta\xi)^3 = (\alpha x_1 + \beta y_1 + \gamma z_1)(\alpha x_2 + \beta y_2 + \gamma z_2)(\alpha x_3 + \beta y_3 + \gamma z_3).$$

The equation of the three tangents is

$$\left.\begin{aligned}\Pi=&[(x_1^2+2ly_1x_1)x+(y_1^2+2lz_1x_1)y+(z_1^2+2lx_1y_1)z]\\ \times&[(x_2^2+2ly_2x_2)x+(y_2^2+2lz_2x_2)y+(z_2^2+2lx_2y_2)z]\\ \times&[(x_3^2+2ly_3x_3)x+(y_3^2+2lz_3x_3)y+(z_3^2+2lx_3y_3)z]\end{aligned}\right\}=0$$

and if we put

$$F=(\xi^2+\eta^2+\zeta^2)^2-24l^2(\xi^2+\eta^2+\zeta^2)\xi\eta\zeta+(-24l-48l^4)\xi^2\eta^2\zeta^2+(-4+32l^6)(\eta^2\zeta^2+\zeta^2\xi^2+\xi^2\eta^2)$$

(F is the reciprocant FU of my Third Memoir), then we have identically

$$F.U-\Pi=(\xi x+\eta y+\zeta z)^3(\xi' x+\eta' y+\zeta' z),$$

and the equation of the satellite line is $\xi' x+\eta' y+\zeta' z=0$. In fact the geometrical theory shows that we must have

$$F.U-N\Pi=(\xi x+\eta y+\zeta z)^3(\xi' x+\eta' y+\zeta' z),$$

and it is then clear that N is a mere number. To determine its value in the most simple manner, write $l=0$, $y=0$, $x=\zeta$, $z=-\xi$, we have then $F.U-N\Pi=0$, where

$$F=\xi^6+\eta^6+\zeta^6-2\eta^3\zeta^3-2\zeta^3\xi^3-2\xi^3\eta^3,\quad U=\zeta^3-\xi^3.$$

The value of Π is $\Pi=F.U$, and we thus obtain $N=1$. For substituting the above values,

$$\begin{aligned}\Pi=&(x_1^3\zeta-z_1^3\xi)(x_2^3\zeta-z_2^3\zeta)(x_3^3\zeta-z_3^3\xi)\\ =&\quad\zeta^3x_1^3x_2^3x_3^3\\ &-\zeta^3\xi^3(x_1^3x_2^3z_3^3+\text{\&c.})\\ &+\zeta\xi^3(x_1^3z_2^3z_3^3+\text{\&c.})\\ &-\xi^3x_1^3z_2^3z_3^3,\end{aligned}$$

and we have

$$\begin{aligned}x_1x_2x_3=&\eta^3-\zeta^3\\ x_1x_2z_3+\text{\&c.}=&\quad3\zeta^3\xi\\ x_1z_2z_3+\text{\&c.}=&-3\zeta\xi^3\\ z_1z_2z_3\quad=&\quad\xi^3-\eta^3,\end{aligned}$$

and thence

$$x_1^3x_2^3z_3^3+\text{\&c.}=9\zeta^4\xi^2+6\zeta\xi^3(\eta^3-\zeta^3)=3\zeta^4\xi^2+6\zeta\xi^3\eta^3$$

$$x_1^3z_2^3z_3^3+\text{\&c.}=9\zeta^3\xi^4-6\zeta^3\xi(\xi^3-\eta^3)=3\zeta^3\xi^4+6\zeta^3\xi\eta^3,$$

and consequently

$$\begin{aligned}\Pi=&\zeta^3(\eta^3-\zeta^3)^2\\ &-\zeta^3\xi.3\zeta\xi^2(\zeta^3+\eta^3)\\ &+\zeta\xi^3.3\zeta^3\xi(\xi^3+\eta^3)\\ &-\xi^3(\xi^3-\eta^3)^2\\ =&(\zeta^3-\xi^3)(\xi^6+\eta^6+\zeta^6-2\zeta^3\eta^3-2\zeta^3\xi^3-2\xi^3\eta^3).\end{aligned}$$

Now considering the equation

$$F.U-\Pi=(\xi x+\eta y+\zeta z)^3(\xi' x+\eta' y+\zeta' z),$$

in order to find ξ', η', ζ' it will be sufficient to find the coefficients of x^3, y^3, z^3 in the

function on the left-hand side of the equation. The coefficient of x^3 in Π is

$$\begin{aligned}&(x_1^3+2ly_1x_1)(x_2^3+2lx_2x_3)(x_3^3+2lx_3y_3)\\ =\;& x_1^3x_2^3x_3^3\\ &+2l(x_1^3x_2^3y_3x_3+\text{ \&c.})\\ &+4l^2(x_1^3y_2x_2y_3x_3+\text{ \&c.})\\ &+8l^3y_1y_2y_3x_1x_2x_3;\end{aligned}$$

and it is easy to see that representing the function

$$(1, 1, 1, I\nabla(\beta\zeta-\gamma\eta,\ \gamma\xi-\alpha\zeta,\ \alpha\eta-\beta\xi)^3$$

by

$$(a, b, c, f, g, h, i, j, k, I\nabla(\alpha, \beta, \gamma)^3,$$

the symmetrical functions can be expressed in terms of the quantities a, b, &c., and that the preceding value of the coefficient of x^3 in Π is

$$\begin{aligned}&a^3\\ &+2l(9hj-6al)\\ &+4l^2(6gk-3fj-3hi+3l^2)\\ &+8l^3bc;\end{aligned}$$

and substituting for a, &c. their values, this becomes

$$\begin{aligned}&(\eta^3-\zeta^3)^3\\ &+2l\{-9(\xi\eta^3+2l\eta\zeta^2)(\zeta^2\xi+2l\zeta\eta^2)\}\\ &+4l^2\{-6(\zeta\xi^2+2l\xi\eta^2)(\xi^2\eta+2l\xi\zeta^2)\\ &\quad+3(\eta\zeta^2+2l\zeta\xi^2)(\zeta^2\xi+2l\zeta\eta^2)\\ &\quad+3(\xi\eta^2+2l\eta\zeta^2)(\eta^2\zeta+2l\eta\xi^2)\}\\ &+8l^3(\xi^3-\eta^3)(\zeta^3-\xi^3),\end{aligned}$$

and reducing, we obtain for the coefficient of x^3 in Π the following expression,—

$$\begin{aligned}&(\eta^3-\zeta^3)^3\\ &-18l\xi^3\eta^3\zeta^3\\ &-24l^2(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta\\ &-24l^2(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3)\\ &+8l^3(\xi^3-\eta^3)(\zeta^3-\xi^3).\end{aligned}$$

Now the coefficient of x^3 in F.U is simply F, which is equal to

$$\begin{aligned}&\xi^6+\eta^6+\zeta^6-2\eta^3\zeta^3-2\zeta^3\xi^3-2\xi^3\eta^3\\ &-24l\xi^3\eta^3\zeta^3\\ &-24l^2(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta\\ &-32l^2(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3)\\ &-48l^4\xi^3\eta^3\zeta^3;\end{aligned}$$

and subtracting, the coefficient of x^3 in F.U$-\Pi$ is

$$\begin{aligned}&\xi^4-2\xi^2\eta^2-2\xi^2\zeta^2\\&-6l\xi^2\eta^2\zeta^2\\&-8l^2(\eta^2\zeta^2+\zeta^2\xi^2+\xi^2\eta^2)\\&-8l^2(\xi^2-\eta^2)(\zeta^2-\xi^2)\\&-48l^2\xi^2\eta^2\zeta^2,\end{aligned}$$

which is equal to

$$(1+8l^2)\xi^2(\xi^2-2\xi\eta^2-2\xi\zeta^2-6l\eta^2\zeta^2).$$

The expression last written down is therefore the value of $\xi^2\xi'^2$, or dividing by ξ^2 we have ξ'^2, and then the values of η', ζ' are of course known, and we obtain the identical equation

$$\text{F.U}-\Pi=$$

$$(\xi x+\eta y+\zeta z)^2\left\{\begin{aligned}&(1+8l^2)(\xi^4-2\xi\eta^2-2\xi\zeta^2-6l\eta^2\zeta^2)x\\&+(1+8l^2)(\eta^4-2\eta\zeta^2-2\eta\xi^2-6l\zeta^2\xi^2)y\\&+(1+8l^2)(\zeta^4-2\zeta\xi^2-2\zeta\eta^2-6l\xi^2\eta^2)z\end{aligned}\right\}$$

and the second factor equated to zero is the equation of the satellite line of $\xi x+\eta y+\zeta z=0$.

33. The point of intersection of the line $\xi x+\eta y+\zeta z=0$ with the satellite line $\xi' x+\eta' y+\zeta' z=0$ is the satellite point of the former line; and the coordinates of the satellite point are at once found to be

$$\begin{aligned}x:y:z=&(\eta^3-\zeta^3)(\eta\zeta+2l\xi^2)\\&:(\zeta^3-\xi^3)(\zeta\xi+2l\eta^2)\\&:(\xi^3-\eta^3)(\xi\eta+2l\zeta^2).\end{aligned}$$

34. If the primary line $\xi x+\eta y+\zeta z=0$ is a tangent to the cubic, then (x_1, y_1, z_1) being the coordinates of the point of contact, we have

$$\xi:\eta:\zeta=x_1^3+2ly_1z_1:y_1^3+2lz_1x_1:z_1^3+2lx_1y_1;$$

these values give as before

$$\eta^3-\zeta^3=-(1+8l^2)x_1^3(y_1^3-z_1^3);$$

and they give also

$$\eta\zeta+2l\xi^2=(1+8l^2)y_1^2z_1^2,$$

and consequently we obtain

$$x:y:z=x_1(y_1^3-z_1^3):y_1(z_1^3-x_1^3):z_1(x_1^3-y_1^3),$$

that is, the satellite point of a tangent of the cubic is the point in which this tangent again meets the cubic.

Article Nos. 35 and 36.—*Theorems relating to the satellite point.*

35. If the line $\xi x+\eta y+\zeta z=0$ be a tangent of the Pippian, then the locus of the satellite point is the Hessian.

Take (x, y, z) as the coordinates of the satellite point, then we have

$$\begin{aligned} x : y : z = &(\eta^3-\zeta^3)(\eta\zeta+2l\xi^2) \\ &:(\zeta^3-\xi^3)(\zeta\xi+2l\eta^2) \\ &:(\xi^3-\eta^3)(\xi\eta+2l\zeta^2); \end{aligned}$$

where the parameters ξ, η, ζ are connected by the equation

$$-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\xi\eta\zeta=0.$$

We have

$$\begin{aligned} y^3+z^3=&(\zeta^3-\xi^3)^3(\zeta\xi+2l\eta^2)^3 \\ &+(\xi^3-\eta^3)^3(\xi\eta+2l\zeta^2)^3, \end{aligned}$$

and it is easy to see that the function on the right-hand side must divide by $\eta^3-\zeta^3$: hence $x^3+y^3+z^3$ will also divide by $\eta^3-\zeta^3$, and consequently by $(\eta^3-\zeta^3)(\zeta^3-\xi^3)(\xi^3-\eta^3)$. We have

$$\begin{aligned} (y^3+z^3)\div(\eta^3-\zeta^3)= \quad & \xi^3\left\{\begin{matrix} -\zeta^3-\zeta^6\eta^3-\zeta^3\eta^6-\eta^9 \\ +3\xi^3(\zeta^6+\zeta^3\eta^3+\eta^6) \\ -3\xi^6(\zeta^3+\eta^3) \\ +\xi^9 \end{matrix}\right\} \\ &+6l\xi^3\eta^3\zeta^3\{-\zeta^3-\zeta^3\eta^3-\eta^6+3\xi^3(\zeta^3+\eta^3)-3\xi^6\} \\ &+12l^3\xi\eta\zeta\{-\eta^3\zeta^3(\eta^3+\zeta^3)+3\xi^3\eta^3\zeta^3-\xi^9\} \\ &+8l^6\quad\{-\eta^6\zeta^6+3\eta^3\zeta^3\xi^6-(\eta^3+\zeta^3)\xi^9\} \end{aligned}$$

and

$$x^3\div(\eta^3-\zeta^3)=(\eta^6-2\eta^3\zeta^3+\zeta^6)(\eta^3\zeta^3+6l\xi^3\eta^3\zeta^3+12l^3\xi^3\eta\zeta+8l^6\xi^9).$$

Adding these values and completing the reduction, we find

$$\begin{aligned} x^3+y^3+z^3\div(\eta^3-\zeta^3)(\zeta^3-\xi^3)(\xi^3-\eta^3)=&-\xi^9-\eta^9-\zeta^9+2\eta^3\zeta^3+2\zeta^3\xi^3+2\xi^3\eta^3 \\ &+18l^2\xi^3\eta^3\zeta^3 \\ &+12l^3(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta \\ &+8l^6(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3); \end{aligned}$$

and we have also

$$\begin{aligned} xyz\div(\eta^3-\zeta^3)(\zeta^3-\xi^3)(\xi^3-\eta^3) \quad = \quad & \xi^3\eta^3\zeta^3 \\ &+2l(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta \\ &+4l^3(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3) \\ &+8l^6\xi^3\eta^3\zeta^3, \end{aligned}$$

and thence

$$\begin{aligned} \{A(x^3+y^3+z^3)+Bxyz\}\div(\eta^3-\zeta^3)(\zeta^3-\xi^3)(\xi^3-\eta^3) \\ = \quad -A(\xi^3+\eta^3+\zeta^3)^3 \\ +(12l^3A+4lB)(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta \\ +\big(18l^2A+(1+8l^3)B\big)\xi^3\eta^3\zeta^3 \\ +\big((4l^3+8l^6)A+4l^3B\big)(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3). \end{aligned}$$

The coefficient of $\eta^2\zeta^2+\zeta^2\xi^2+\xi^2\eta^2$ on the right-hand side will vanish if $(1+2l^3)A+l^3B=0$, or, what is the same thing, if $A=l^3$, $B=-(1+2l^3)$; and substituting these values, we obtain

$$\begin{aligned}\{l^3(x^3+y^3+z^3)-(1+2l^3)\xi\eta\zeta\}\div(\eta^3-\zeta^3)(\zeta^3-\xi^3)(\xi^3-\eta^3)\\ =-l^3(\xi^3+\eta^3+\zeta^3)\\ +(-4l+4l^4)(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta\\ +(-1+8l^3-16l^6)\xi^2\eta^2\zeta^2,\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}l^3(x^3+y^3+z^3)-(1+2l^3)xyz=-(\eta^3-\zeta^3)(\zeta^3-\xi^3)(\xi^3-\eta^3)\\ \times\{-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\xi\eta\zeta\}^3.\end{aligned}$$

Hence the left-hand side vanishes in virtue of the relation between ξ, η, ζ, or we have

$$l^3(x^3+y^3+z^3)-(1+2l^3)xyz=0,$$

which proves the theorem.

36. Suppose that (X, Y, Z) are the coordinates of a point of the Hessian, and let (P, Q, R) be the coordinates of the point in which the tangent to the Hessian at the point (X, Y, Z) again meets the Hessian, or, what is the same thing, the satellite point *in regard to the Hessian* of the tangent at (X, Y, Z). And consider the conic

$$X(x^2+2lyz)+Y(y^2+2lzx)+Z(z^2+2lxy),$$

which is the first or conic polar of the point (X, Y, Z) in respect of the cubic. The polar (in respect to this conic) of the point (P, Q, R) will be

$$\xi x+\eta y+\zeta z=0,$$

where

$$\begin{aligned}\xi&=PX+l(RY+QZ),\\ \eta&=QY+l(PZ+RX),\\ \zeta&=RZ+l(QX+PY);\end{aligned}$$

or putting for (P, Q, R) their values,

$$\begin{aligned}\xi&=(Y^3-Z^3)(X^3-lYZ),\\ \eta&=(Z^3-X^3)(Y^3-lZX),\\ \zeta&=(X^3-Y^3)(Z^3-lXY).\end{aligned}$$

And if from these equations and the equation of the Hessian we eliminate (X, Y, Z), we shall obtain the equation in line coordinates of the curve which is the envelope of the line $\xi x+\eta y+\zeta z=0$. We find, in fact,

$$\begin{aligned}\xi^2+\eta^3+\zeta^3=(Y^3-Z^3)(Z^3-X^3)(X^3-Y^3)\\ \times\begin{cases}l^3(X^3+Y^3+Z^3)^3\\ -3l(X^3+Y^3+Z^3)XYZ\\ +9l^3X^3Y^3Z^3\\ +(1-4l^3)(Y^3Z^3+Z^3X^3+X^3Y^3),\end{cases}\end{aligned}$$

$$\xi\eta\zeta \quad = (Y^3-Z^3)(Z^3-X^3)(X^3-Y^3)$$
$$\times \begin{cases} l^2(X^3+Y^3+Z^3)XYZ \\ +(1-l^2)X^2Y^2Z^2 \\ -l(Y^3Z^3+Z^3X^3+X^3Y^3); \end{cases}$$

and thence recollecting that

$$HU = l^2(X^3+Y^3+Z^3)-(1+2l^2)XYZ,$$

we find

$$-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\xi\eta\zeta = -(Y^3-Z^3)(Z^3-X^3)(X^3-Y^3)(HU)^3,$$

and the equation of the envelope is

$$-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\xi\eta\zeta = 0,$$

which is therefore the Pippian. We have thus the theorem,—

The envelope of the polar of the satellite point *in respect to the Hessian* of the tangent at any point of the Hessian, such polar being in respect of the conic which is the first or conic polar of the point of the Hessian in respect of the cubic, is the Pippian.

Article Nos. 37 to 40.—*Investigations and theorems relating to the first or conic polar of a point of the cubic.*

37. The investigations next following depend on the identical equations

$$\{\alpha(X^3+2lYZ)+\beta(Y^3+2lZX)+\gamma(Z^3+2lXY)\}$$
$$\times\{-XYZ(x^3+y^3+z^3)+(X^3+Y^3+Z^3)xyz\}$$
$$=\{X(x^3+2lyz)+Y(y^3+2lzx)+Z(z^3+2lxy)\}$$
$$\times\{X(Y^3-Z^3)(\gamma y-\beta z)+Y(Z^3-X^3)(\alpha z-\gamma x)+Z(X^3-Y^3)(\beta x-\alpha y)\}$$
$$+\{x(X^3+2lYZ)+y(Y^3+2lZX)+z(Z^3+2lXY)\}$$
$$\times\{-(\alpha YZ+\beta ZX+\gamma XY)(Xx^2+Yy^2+Zz^2)+(\alpha X^3+\beta Y^3+\gamma Z^3)(Xyz+Yzx+Zxy)\},$$

which is easily verified.

I represent the equation in question by

$$K\Upsilon = WL+P\Theta;$$

then considering (x, y, z) as current coordinates, and (X, Y, Z) and (α, β, γ) as the coordinates of two given points Σ and Ω, we shall have $U=0$ the equation of the cubic, $W=0$ the equation of the first or conic polar of Σ with respect to the cubic, $P=0$ the equation of the second or line polar of Σ with respect to the cubic. The equation $\Upsilon=0$ is that of a syzygetic cubic passing through the point Σ: the coordinates of the satellite point in respect to this syzygetic cubic of its tangent at Σ are

$$X(Y^3-Z^3) : Y(Z^3-X^3) : Z(X^3-Y^3);$$

and calling the point in question Σ', then $L=0$ is the equation of a line through the points Σ', Ω. The equation $\Theta=0$ is that of a conic, viz. the first or conic polar of Σ with respect to a certain syzygetic cubic

$$-2(\alpha YZ+\beta ZX+\gamma XY)(x^3+y^3+z^3)+(\alpha X^3+\beta Y^3+\gamma Z^3)xyz = 0,$$

depending on the points Σ, Ω, or, what is the same thing, the conic $\Theta=0$ is a properly selected conic passing through the points of intersection of the first or conic polars of Σ with respect to any two syzygetic cubics; and lastly, K is a constant coefficient. The equation expresses that the points of intersection of

$$(W=0,\ P=0),\ (W=0,\ \Theta=0),\ (L=0,\ P=0),\ (L=0,\ \Theta=0),$$

lie in the syzygetic cubic $\Upsilon=0$.

The left-hand side of the equation may be written

$$-XYZ\{\alpha(X^3+2lYZ)+\beta(Y^3+2lZX)+\gamma(Z^3+2lXY)\}(x^3+y^3+z^3+6lxyz)$$
$$+xyz\{\alpha(X^3+2lYZ)+\beta(Y^3+2lZX)+\gamma(Z^3+2lXY)\}(X^3+Y^3+Z^3+6lXYZ);$$

and it may be remarked also that we have

$$-3XYZ\{\alpha(X^3+2lYZ)+\beta(Y^3+2lZX)+\gamma(Z^3+2lXY)\}$$

equal identically to

$$\{X(Y^3-Z^3)(\gamma Y-\beta Z)+Y(Z^3-X^3)(\alpha Z-\gamma X)+Z(X^3-Y^3)(\beta X-\alpha Y)\}$$
$$-(\alpha YZ+\beta ZX+\gamma XY)(X^3+Y^3+Z^3+6lXYZ).$$

Hence if we assume

$$X^3+Y^3+Z^3+6lXYZ=0,$$

the equation will take the form

$$KU=WL+P\Theta,$$

where the constant coefficient K may be expressed under the two different forms

$$K=-XYZ\{\alpha(X^3+2lYZ)+\beta(Y^3+2lZX)+\gamma(Z^3+2lXY)\}$$
$$=\tfrac{1}{3}\{X(Y^3-Z^3)(\gamma Y-\beta Z)+Y(Z^3-X^3)(\alpha Z-\gamma X)+Z(X^3-Y^3)(\beta X-\alpha Y)\},$$

and W, L, P, Θ have the same values as before. In the present case the point Σ is a point of the cubic: the equation $W=0$ represents the first or conic polar of the point in question, and the equation $P=0$ its second or line polar, which is also the tangent of the cubic. The line $L=0$ is a line joining the point Ω with the satellite point of the tangent at Σ, or dropping altogether the consideration of the point Ω, is an arbitrary line through the satellite point: the first or conic polar of Σ meets the cubic twice in the point Σ, and therefore also meets it in four other points; the conic $\Theta=0$ is a conic passing through these four points, and completely determined when the particular position of the line through the satellite point is given. And, as before remarked, $\Theta=0$ is a conic passing through the points of intersection of the first or conic polars of Σ with respect to any two syzygetic cubics. We have thus the theorem,—

The first or conic polar of a point of the cubic touches the cubic at this point, and besides meets it in four other points; the four points in question are the points in which the first or conic polar of the given point in respect of the cubic is intersected by the first or conic polar of the same point in respect to any syzygetic cubic whatever,

38. The analytical result may be thus stated: putting

$$\varkappa=\alpha YZ+\beta ZX+\gamma XY,\ \lambda=\alpha X^3+\beta Y^3+\gamma Z^3,$$

or, if we please, considering $\varkappa$, λ as arbitrary parameters, then the four points lie in the conic

$$(2\varkappa X, 2\varkappa Y, 2\varkappa Z, -\lambda X, -\lambda Y, -\lambda Z)(x, y, z)^2=0,$$

or, what is the same thing, they are the points of intersection of the two conics

$$Xx^2+Yy^2+Zz^2=0,$$
$$Xyz+Yzx+Zxy=0.$$

39. Considering the four points as the angles of a quadrangle, it may be shown that the three centres of the quadrangle lie on the cubic. To effect this, assume that the conic

$$(2\varkappa X, 2\varkappa Y, 2\varkappa Z, -\lambda X, -\lambda Y, -\lambda Z)(x, y, z)^2=0$$

represents a pair of lines; these lines will intersect in a point, which is one of the three centres in question. And taking x, y, z as the coordinates of this point, we have

$$\begin{aligned} x^2 : y^2 : z^2 : yz : zx : xy = {} & 4\varkappa^2YZ-\lambda^2X^2 \\ & : 4\varkappa^2ZX-\lambda^2Y^2 \\ & : 4\varkappa^2XY-\lambda^2Z^2 \\ & : \lambda^2YZ+2\varkappa\lambda X^2 \\ & : \lambda^2ZX+2\varkappa\lambda Y^2 \\ & : \lambda^2XY+2\varkappa\lambda Z^2; \end{aligned}$$

and we may, if we please, use these equations to find the relation between $\varkappa$, λ. Thus in the identical equation $x^2.y^2-(xy)^2=0$, substituting for x^2, xy, y^2 their values, and throwing out the factor Z, we find $(4\varkappa^2-\lambda^2)XYZ-\varkappa\lambda^2(X^2+Y^2+Z^2)=0$, and thence, in virtue of the equation $X^2+Y^2+Z^2+6lXYZ=0$, we obtain

$$4\varkappa^2+6l\varkappa\lambda^2-\lambda^4=0.$$

But the preceding system gives conversely,

$$\begin{aligned} X^2 : Y^2 : Z^2 : YZ : ZX : XY = {} & 4\varkappa^2yz-\lambda^2x^2 \\ & : 4\varkappa^2zx-\lambda^2y^2 \\ & : 4\varkappa^2xy-\lambda^2z^2 \\ & : \lambda^2yz+2\varkappa\lambda x^2 \\ & : \lambda^2zx+2\varkappa\lambda y^2 \\ & : \lambda^2xy+2\varkappa\lambda z^2. \end{aligned}$$

Hence from the identical relation $X^2.Y^2-(XY)^2=0$, substituting for X^2, XY, Y^2 their values, and throwing out the factor z, we find $(4\varkappa^2-\lambda^2)xyz-\varkappa\lambda^2(x^3+y^3+z^3)=0$, and thence, in virtue of the equation $4\varkappa^2-\lambda^2=-6l\varkappa\lambda^2$, we obtain

$$x^3+y^3+z^3+6lxyz=0,$$

which shows that the point in question lies on the cubic. We have thus the theorem,—

The first or conic polar of a point of the cubic touches the cubic at the point, and

meets it besides in four points, which are the angles of a quadrangle the centres of which lie on the cubic. In other words, the quadrangle is an inscribed quadrangle.

40. To find the equations of the *axes* of the quadrangle, that is of the lines through two centres.

We have

$$(4\kappa^3YZ-\lambda^3X^3)x+(\lambda^3XY+2\kappa\lambda Z^3)y+(\lambda^3ZX+2\kappa\lambda Y^3)z=0$$
$$(\lambda^3XY+2\kappa\lambda Z^3)x+(4\kappa^3ZX-\lambda^3Y^3)y+(\lambda^3YZ+2\kappa\lambda X^3)z=0$$
$$(\lambda^3ZX+2\kappa\lambda Y^3)x+(\lambda^3YZ+2\kappa\lambda X^3)y+(4\kappa^3XY-\lambda^3Z^3)z=0;$$

or arranging these equations in the proper form and eliminating κ^3, $\kappa\lambda$, λ^3, we find

$$\begin{vmatrix} YZx, & Z^3y+Y^3z, & X(-Xx+Yy+Zz) \\ ZXy, & X^3z+Z^3x, & Y(\ Xx-Yy+Zz) \\ XYz, & Y^3x+X^3y, & Z(\ Xx+Yy-Zz) \end{vmatrix}=0;$$

or multiplying out,

$$\begin{aligned} XYZ\{&(Z^3-Y^3)x^3+(X^3-Z^3)y^3+(Y^3-X^3)z^3\} \\ &+x^2yZY^3(-2X^3+Y^3+Z^3)+zx^2YZ^3(2X^3-Y^3-Z^3) \\ &+y^2zXZ^3(-2Y^3+Z^3+X^3)+xy^2ZX^3(2Y^3-Z^3-X^3) \\ &+z^2xYX^3(-2Z^3+X^3+Y^3)+yz^2XY^3(2Z^3-X^3-Y^3)=0. \end{aligned}$$

We may simplify this result by means of the equation $X^3+Y^3+Z^3+6lXYZ=0$, so as to make the left-hand side divide out by XYZ: we thus obtain

$$\begin{aligned} &(Z^3-Y^3)x^3+(X^3-Z^3)y^3+(Y^3-X^3)z^3 \\ &+(-3X^3Y-6lY^2Z)x^2y+(-3Y^3Z-6lZ^2X)y^2z+(-3Z^3X-6lX^2Y)z^2x \\ &+(\ 3XY^3+6lX^2Z)xy^2+(\ 3YZ^3+6lY^2X)yz^2+(\ 3ZX^3+6lZ^2Y)zx^2=0; \end{aligned}$$

or in a different form,

$$\begin{aligned} &(y^3-z^3)X^3+(z^3-x^3)Y^3+(x^3-y^3)Z^3 \\ &+(-3x^2y-6lz^2x)X^2Y+(-3y^2z-6lx^2y)Y^2Z+(-3z^2x-6ly^2z)Z^2X \\ &+(\ 3xy^2+6lyz^2)XY^2+(\ 3yz^2+6lzx^2)YZ^2+(\ 3zx^2+6lxy^2)ZX^2=0, \end{aligned}$$

as the equation of the three axes of the quadrangle.

Article No. 41.—*Recapitulation of geometrical definitions of the Pippian.*

In conclusion, I will recapitulate the different modes of generation or geometrical definitions of the Pippian, obtained in the course of the present memoir. The curve in question is—

1. The envelope of the line joining a pair of conjugate poles of the cubic (see Nos. 2 and 13).

2. The envelope of each line of the pair forming the first or conic polar with respect to the cubic of a conjugate pole of the cubic (see Nos. 2 and 14).

3. The envelope of a line which is the polar of a conjugate pole of the cubic, with respect to the conic which is the first or conic polar of the other conjugate pole in respect to any syzygetic cubic (see Nos. 2 and 9).

4. The locus of the harmonic with respect to a pair of conjugate poles of the cubic of the third point of intersection with the Hessian of the line joining the two conjugate poles (see Nos. 2 and 17).

5. The envelope of a line such that its lineo-polar envelope with respect to the cubic breaks up into a pair of lines (see No. 24).

6. The envelope of a line which meets three conics, the first or conic polars of any three points in respect to the cubic, in six points in involution (see No. 22).

7. The envelope of the second or line polar with respect to the cubic, of a point the locus of which is a certain curve of the sixth order in quadratic syzygy with the cubic and Hessian, viz. the curve $-S.U^2+(HU)^2=0$ (see No. 27).

8. The envelope of a line having for its satellite point a point of the Hessian (see No. 35).

9. The envelope of the polar of the satellite point with respect to the Hessian of the tangent at a point of the Hessian, with respect to the first or conic polar of the point of the Hessian in respect to the cubic (see No. 36).

XXI. *On some of the Products of the Destructive Distillation of Boghead Coal.*—Part I.

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Communicated by Dr. Sharpey, *Sec. R.S.*

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The action of heat on organic substances has been studied under two aspects: in the one the relation of the products to the original matter is seen, and we are enabled to draw theoretical deductions, in most cases of great simplicity; in the other, the relation is not capable of being traced; and it would appear, therefore, at a first glance, that the study of bodies produced in this manner would be less conducive to the advancement of theoretical science. But so far from this holding good, it is not too much to assert that organic chemistry has been more enriched by products of the second kind than the first. The metamorphoses of naphthaline, to which the law of substitution owes so much for its development; the study of aniline, which has so greatly increased our knowledge of the theory of basic combinations; the history of the phenyle series and its numerous homologues,—are, immense as their influence on the progress of chemistry has been, only a few instances of what may be anticipated from the study of products of destructive distillation. It is true that their history has been developed, not because of their origin, but from their intrinsic interest, yet it cannot be denied that no operation has furnished chemistry with products so numerous and diversified in character. It is also unquestionable that no organic type exists unrepresented by bodies of pyrogenous derivation.

Heat is perhaps the only chemical agent to which we can assign no special function; at one time acting as a powerful incentive to oxidation, at another to reduction; it is generally recognized as the most potent of disruptive forces, yet we sometimes find it causing the coalescence and reduplication of atoms; it is evident, therefore, allowing heat to possess these various and apparently antagonistic qualities, that there are few organic bodies capable of withstanding high temperatures whose presence among products of destructive distillation can be looked upon as impossible. The progress of chemical science has moreover shown in repeated instances, that the substances at one time regarded as the rarest and most difficultly obtainable, become in a short period those with which we are most familiar. The present investigation may be considered a case in point, its object being to prove the existence in great quantity in a commercial product, of a class of compounds hitherto only procurable by processes founded on purely theoretical considerations, and requiring considerable care in their prosecution.

The substance, the distillate from which contains the hydrocarbons forming the subject of this communication, is the Torbane-hill mineral or Boghead Coal, worked on a

large scale at Bathgate, near Edinburgh. A very partial examination of it has been published by Dr. GENTHER*, with results which, as we shall afterwards see, are far from correct; for he has assumed that the fractions obtained at particular boiling-points are homogeneous, and consist of different but isomeric hydrocarbons of the formula $C^n H^n$. In addition to paraffine, long known as one of the products, he has detected picoline and phenole; but, nevertheless, his results, on the whole, are rather in favour of the view that the Torbane-hill mineral is not a true coal in the sense generally understood by chemists. I do not enter here upon the much-disputed and litigated question of the nature of the mineral itself, my object being solely to study the chemical relations of the bodies produced by its decomposition under the influence of heat.

The ordinary Boghead naphtha appears in commerce in the form of a nearly colourless fluid of a very characteristic odour, quite different to that obtained from ordinary coal. Its specific gravity is only 0·750 at 15°, and is therefore greatly lower than that from the latter source, for even thoroughly purified benzole has a density of 0·850. Notwithstanding its small density the boiling-point was high, the lowest fraction that could be obtained in the first rectification being between 143° and 148°. That the fluid was a mixture of many bodies of very different boiling-points, was shown by the fact that during the first distillation the mercury in the thermometer steadily rose to the highest range it was safe to allow. This complexity was still better seen by the sudden depression of the boiling-point of the first fraction in the second distillation, for there was then obtained a considerable quantity of fluid between 121° and 126°, the second rectification thus causing a fall of no less than 22°; subsequent to this, the lowering of the initial fraction with the advance in the number of rectifications was less marked, but still decided, and after six distillations I was able to procure a portion boiling as low as 98°·5.

It appearing evident that, previous to attempting any examination, it would be essential to obtain the crude hydrocarbons of nearly constant boiling-point, they were submitted to a tolerably complete fractionation, during which there were made, altogether, no less than one thousand distillations.

An opinion has been hazarded by some persons, that in processes of fractional distillation a continual breaking up is taking place, which they suppose to account for the variations in the point of ebullition: that the supposition is incorrect will be admitted by all chemists who have given close attention to processes of this kind, and the question is placed beyond doubt by the fact that the variation becomes less as the distillations are more numerous, and, at last, the fractions distil almost entirely between the points at which they last came over, and doubtless might, if sufficient time was expended on them, be obtained perfectly steady.

As soon as by the means described fractions were obtained of almost constant boiling-points, I proceeded to ascertain whether the fluids consisted of more than one substance, and, passing over the preliminary trials, it was soon found that more than one series of bodies were present. This was proved by the deportment of the mixed hydrocarbons

* LIEBIG's Annalen, xcvii. p. 277, and Chemical Gazette, 1856, No. 330.

with fuming nitric, or a mixture of nitric and sulphuric acids. If, into a flask of three or four ounces capacity, two drachms of fuming nitric acid be poured, and one drachm of the naphtha be added by small portions, with constant and brisk agitation, allowing the temperature to rise a little, but preventing it going too far by immersing the flask (still kept in motion) in very cold water, a point is reached when the temperature no longer rises, and it is seen that the mixture forms an emulsion of a red-brown colour, a layer of green but brilliant fluid floating on the surface; the whole is then poured into a very narrow conical glass and allowed to rest for a few minutes, by which time the fluid unacted on has risen to the surface. The lower portion having been removed and thrown into water by means of a pipette, furnished with a hollow caoutchouc ball to avoid suction with the lips, the indifferent hydrocarbon is transferred to a flask having a well-fitting stopper and containing a quantity of highly-concentrated nitric acid. When the above operation has been repeated until sufficient of the hydrocarbon has been accumulated in the flask, the latter is violently shaken to thoroughly mix the fluids, and on standing for some time separation into two layers takes place; the lower, consisting of the acid, which has abstracted the chief part of the hydrocarbons removeable by it, is withdrawn by means of a pipette, and the operation is several times repeated. It is absolutely essential to correct results that this treatment be sufficiently performed. The acid being removed for the last time, the fluid is washed with a solution of caustic potash, which, by absorbing the nitrous fumes, removes the green colour. The indifferent hydrocarbon is then digested with sticks of potash to remove the water, and when apparently dry, is distilled several times over sodium.

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The fluids obtained as above are perfectly colourless, of a pleasant odour, resembling may-blossoms, very volatile, even at low temperatures, and having a density of about 0·725. If pieces of sodium be rapidly cut from a mass, so as to have only a very thin layer of soda, and are then thrown into the perfectly dry hydrocarbon, the coating of oxide is dissolved, the metal appearing of the lustre of silver, and may probably be thus kept for any length of time.

My first experiments were made upon the fraction boiling between 115° and 121°, merely because that happened to be the largest, and it soon became evident that the hydrocarbon obtained by the process described, possessed the composition and the same degree of condensation in the state of vapour as butyle (valyle of Kolbe). But as it is evident that the formula $C^{6}H^{13}$ corresponds also to the hydruret of the radical of the caprylic alcohol, it was necessary to ascertain the correct boiling-points of the bodies obtained, because if coinciding with those of the radicals, it would be strong evidence of their identity with them. But the composition of the radicals differing but little, it is evident that analysis alone, however carefully made, cannot decide the question; fortunately, however, the addition or subtraction of $C^{2}H^{2}$ has a very considerable influence on their vapour-density, as will at once be seen by reference to the annexed Table, where the increase is compared with the rise of carbon and diminution of hydrogen as the boiling-point becomes higher.

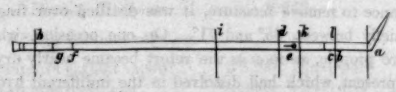
		Boiling-point.	Vapour-density.	Per-centage of carbon.	Per-centage of hydrogen.
Propyle	$C^{12}H^{14}$	65°	2·98	83·72	16·28
Butyle	$C^{16}H^{18}$	106°	3·94	84·21	15·79
Amyle	$C^{20}H^{22}$	158°	4·91	84·51	15·49
Caproyle	$C^{24}H^{26}$	202°	5·88	84·70	15·30

The boiling-points of the indifferent hydrocarbons being such important evidence, it has been considered proper to give positive proof that the fractions analysed were those whose constitution exactly tallied with the bodies with which they were sought to be identified; in each case therefore the vapour-densities of several fractions near the known boiling-point of the radicals were determined; and it is submitted, that it is no slight proof of the correctness of the view advanced relative to the nature of the bodies isolated, that in almost every instance the fraction which gave the correct vapour-density was that which corresponded in boiling-point with the radical. In order to save space, the numerous determinations made are condensed into Tables. In one or two of these merely trial experiments, it will be observed that the regularity of the series is slightly interrupted; where this happens, it will be found to arise from the fact of the bodies not having been obtained from the fraction corresponding to their own boiling-point; thus in the Table containing the amyle experiments, the fourth and sixth determinations are not in perfect harmony with the rest; in the third, the fraction used distilled between 160° and 165°, but it was obtained from the portion of the crude hydrocarbons boiling between 150° and 160°; this makes the density come out somewhat lower than it would have been had the fraction been extracted from the hydrocarbons boiling between 160° and 165°. In general, however, it will be found that the increase of density with the rise of boiling-point is quite as steady as could have been anticipated.

It has been stated that the composition of the fractions varied so little as the boiling-points became higher, that I did not imagine myself justified in considering the nature of any fraction established by the mere results of analysis; but although vapour-density has been chiefly relied on as the means of research, it is believed, nevertheless, from the extreme care with which the combustions were made, that they represent very nearly the true per-centages of carbon and hydrogen.

Chemists are aware that the accurate analysis of volatile fluids, having a high per-centage of carbon, is a somewhat fatiguing process, from the constant attention indispensable, and the slowness with which it is necessary to advance the fire along the tube. By the following method of burning, which is described in the hope that it may be found useful to others, less time and far less attention are required for the determination of the carbon and hydrogen in a body like propyle or butyle, than for the combustion of an ordinary solid. For this purpose the fluid is contained in one bulb, and is driven out at the commencement of the analysis into a column of thirteen or fourteen centimetres of cold oxide of copper: this latter portion is never directly heated until the end of the analysis, the fluid being volatilized by the heat conducted by the oxide. Oxygen

is, of course, passed over the reduced metal to remove any carbon which may have combined with it. The tube is arranged as in the annexed sketch.



At *a* is placed a small plug of recently ignited asbestos, about two centimetres long; then a mixture of fused and powdered chlorate of potash and oxide of copper as far as *b* (about seven centimetres); then at *c* two centimetres of asbestos, followed to *k* by the same length of coarse-grained oxide of copper. The substance is placed at *e*, the tube to *f* being filled completely by granular oxide of copper of about the coarseness of ordinary gunpowder, and, finally, at *g* is placed another plug of asbestos. The combustion is commenced by heating the end *a* to redness, and between the screens *h* and *i*; when fully red, the space between the screens *k* and *l* is filled with live coals, the whole of the fluid being immediately driven amongst the oxide of copper occupying the space from *d* to *i*. In a short time this oxide becomes sufficiently hot to cause a perfectly steady distillation of the fluid over the red-hot column of oxide from *i* to *h*, where it is burnt. No further attention is requisite until carbonic acid ceases to enter the bulbs, when the screen *i* is removed back; but it is always found, if the fluid is moderately volatile, that very little remains. When the tube is red-hot from *h* to *l*, the mixture of oxide and chlorate is slowly heated to redness. Combustions made in this manner seldom fail to yield good results. One great advantage of the method is, that if the space from *i* to *d* be of the proper length, there is no fear of either a sudden rush of gas or of the operation being tediously slow. All the fluid analyses given in this paper were made in this manner.

On the Hydrocarbons unacted on by Sulphuric and Nitric Acids.

The lowest fraction obtained during the rectifications was between 65° and 71°; the quantity was far from large; and although working on a tolerably extensive scale, great care was requisite to prevent loss by evaporation during the distillation. The density at 15°·5 of the crude hydrocarbons was only 0·735, although containing a considerable percentage of benzole. On treatment with highly concentrated fuming nitric acid, in the manner previously described, and placing the acid and nitrocompound in a conical glass, for the purpose of facilitating the separation of the hydrocarbon unacted on by acids, I was surprised to find that no separation whatever took place; and on throwing the mixture into water, no nitrocompound fell to the bottom, the whole rising to the surface in the form of a pale green layer. The latter was therefore separated, placed in a retort, and the heat of a water-bath applied; a considerable quantity of a colourless fluid distilled over, which was treated several times with fuming nitric acid to remove any of the other hydrocarbons present, the last traces being retained with such tenacity that the process had to be repeated many times before the separated acid no longer

produced milkiness when thrown into water, indicating the presence of the nitrocompound. After washing several times with solution of potash, and digestion with sticks of the latter substance to remove moisture, it was distilled over fragments of sodium, and came over chiefly between 65° and 71°. On one occasion, when distilling over sodium some impure propyle, as soon as the retort became nearly dry, a small quantity of nitrocompound present, which had dissolved in the indifferent hydrocarbon, reacted on the metal with great violence, the latter becoming ignited, torrents of smoke being evolved, and large quantities of carbon at the same time depositing on the interior of the retort and receiver: when the radical is in a state of purity, sodium remains perfectly brilliant under it during the distillation.

In order to ascertain the boiling-point of propyle, I commenced by determining the vapour-density of the fractions, beginning with the portion distilling, in the sixteenth rectification, between 72° and 79°: the experiments were made by the method of DUMAS, the second being taken at a slightly higher temperature than the first.

	I.	II.
Temperature of air	20°	20°
Temperature of vapour. . .	104°	112°
Pressure	769 mm.	769 mm.
Excess of weight of balloon .	·4673 grm.	·4817 grm.
Capacity of balloon	294 cub. cent.	307 cub. cent.
Residual air	11 cub. cent.	5 cub. cent.

Experiment.		Theory.
I.	II.	$C^{12}H^{14}$=4 volumes.
3·10	3·08	2·97

It appearing, therefore, that the boiling-point was considerably below 79°, the fraction distilling between 65° and 70° was taken for further examination. On combustion, it was found that—

I. ·1678 gramme yielded ·5148 carbonic acid and ·2521 water.
II. ·1673 gramme yielded ·5132 carbonic acid and ·2450 water.

	Experiment.		Calculation.		
	I.	II.			
Carbon . . .	83·7	83·7	C^{12}	72	83·7
Hydrogen . .	16·7	16·3	H^{14}	14	16·3
				86	100·0

The following are the details of a determination of the vapour-density of the same portion of fluid by GAY-LUSSAC's method:—

Weight of fluid	·1098 grm.
Temperature of vapour . .	100°
Observed volume of vapour.	43·5 cub. cent.

Atmospheric pressure . .	772·1 mm.
Difference of mercury level .	85·4 mm.
Density of vapour	2·956

The formula $C^{12} H^{14}$=4 volumes requires—

12 volumes carbon vapour . .	0·8290 . 12=9·948
28 volumes hydrogen . . .	0·0692 . 28=1·938
	$\frac{11·886}{4}$=2·972.

Experiment.	Theory.
2·956	2·972

The substance unacted on by acids boiling at 65°, as extracted from the Torbane-hill mineral distillate, has therefore not only the composition, but also the vapour-density of the radical of the propylic series. I shall show, further on, that its boiling-point also coincides with what might be expected. It is evident that propyle might be procured by electrolysis of butyric acid, and I propose eventually to subject the hydrocarbons from both sources to a careful comparison. It may be mentioned that propyle has not previously been obtained. Its density at 18° was found to be 0·6745.

Nitrocompound from fraction boiling between 66° *and* 71°.

This substance, which, it has been said, floated on water while holding the radical in solution, was much heavier than that fluid after the hydrocarbon had been distilled off. Treated with alcoholic potash great heat was produced, the solution becoming blood-red, and a peculiar odour being evolved. The mixture was placed in an iron alembic, and the alcohol distilled off on the water-bath; the latter was then removed, and the distillation continued over the naked fire. An abundance of milky fluid containing an alkaloid came over, accompanied by a small quantity of non-basic oil. At last an explosive evolution of vapour ensued; the chief portion of the contents of the alembic being projected into the neck, and there becoming solidified, stopped the distillation. The non-basic oil was removed by addition of hydrochloric acid to dissolve the base, and then passing through a wet filter. The undissolved portion solidified on the filter to a mass of golden-brown crystals having the characters of azobenzide. A large quantity of aniline was found in the basic portion of the distillate. The nitrocompound contained, therefore, a considerable amount of nitrobenzole. The study of the nitrocompounds and their products is reserved for a future communication.

Butyle.

The following experiments were the first made with the radicals from Boghead naphtha, but are placed here in their proper sequence with regard to the other homologues.

It has been found that the separation of the hydrocarbons can be effected by other

means than those adopted for the preparation of the radicals; the details of the method belong to a series of experiments to be given in Part II. It is to be understood, therefore, that the nitric acid process was adhered to for the isolation of the bodies described in this paper.

After preparation of the butyle, I proceeded to determine the vapour-density of the fractions, the results being embodied in the annexed Table.

Vapour-densities of fractions from 104° *to* 129°, *after treatment with nitric acid, &c.*

Boiling-point.	Excess of weight of balloon.	Temperature of vapour.	Temperature of air.	Pressure.	Capacity of balloon.	Residual air.	Density.
°	grm.	°	°	mm.	cub. cent.	cub. cent.	
104 to 107	·5332	154	19	752	298·5	·5	3·63
104 to 110	·4365	171	16	752	268·0	5·0	3·66
107 to 110	·5030	163	16	761	281·5	1·5	3·73
110 to 116	·4950	173	16	761	277·5	2·0	3·82
121 to 127	·5489	179	18	760	274·5	0·0	4·11
126 to 129	·5545	160	14	750	268·5	2·3	4·09

It was evident from the foregoing determinations, that the boiling-point of the hydrocarbon having the formula $C^6 H^{16}$ was between 116° and 121°, but this being higher than was anticipated, it appeared probable that the fractions used had not been treated sufficiently with nitric acid: a portion of somewhat lower boiling-point than those in the Table was therefore repeatedly acted on until the impurity was entirely removed, and the separated acid, on being diluted with water, yielded not the slightest milkiness. The annexed were the results obtained with the hydrocarbon so purified:—

	Portion distilling, 100° to 101°.	Portion distilling, 101° to 103°.
Temperature of air	17°	17°
Temperature of vapour	145°	148°
Pressure	776·2 mm.	776·2 mm.
Excess of weight of balloon	·5545 grm.	·5305 grm.
Capacity of balloon	308 cub. cent.	295 cub. cent.
Residual air	4·5 cub. cent.	5·0 cub. cent.
Density of vapour	3·63	3·66

The repeated treatment with nitric acid, although it had the effect of raising the density, did it to so slight a degree, that it was clear the boiling-point of butyle from the Torbane-hill mineral was much higher than that found by other observers, and lay between 116° and 121°. Analysis gave the numbers annexed:—

I. ·3325 butyle gave 1·0246 carbonic acid and ·4704 water.
II. ·1783 butyle gave ·5505 carbonic acid.
III. ·1841 butyle gave ·5673 carbonic acid and ·2604 water.
IV. ·1687 butyle gave ·5189 carbonic acid and ·2421 water.

	Experiment.				Mean.	Calculation.		
	I.	II.	III.	IV.				
Carbon . . .	84·1	84·2	84·0	83·9	84·1	C^{16}	96	84·2
Hydrogen . .	15·7	——	15·7	15·9	15·8	H^{18}	18	15·8
							114	100·0

The fourth analysis was made upon a different preparation to the others.

Kolbe's radical obtained by the electrolysis of butyric acid gave him the following numbers, which are sensibly the same as the above:—

	Kolbe.	
Carbon	84·1	84·0
Hydrogen . . .	15·9	15·8

Annexed is a determination of the vapour-density of the fraction distilling between 116° and 121°.

Excess of weight of balloon .	·5270 grm.
Temperature of vapour . .	167°
Temperature of air	16°
Pressure	753 mm.
Capacity of balloon . . .	280 cub. cent.
Residual air	1·0 cub. cent.

The formula $C^{16}H^{18}$ requires—

16 volumes carbon vapour. .	0·8290 . 16 = 13·264
36 volumes hydrogen . . .	0·0692 . 36 = 2·491
	$\frac{15·755}{4} = 3·9387$

Experiment.	Kolbe.	Wurtz.	Theory = 4 vols.
3·883	4·053	4·070	3·9387

Although the above numbers are exceedingly near those required by theory, it follows that the boiling-point of the hydrocarbon having the same formula as butyle from Boghead coal, is decidedly higher than that found by Kolbe and Wurtz. The boiling-point of Kolbe's hydrocarbon, obtained by electrolysis of valerianic acid, was 108°, and that of butyle, obtained by Wurtz by cohobating iodide of butyle over sodium, was 106°, whereas my hydrocarbon distilled in the sixteenth rectification between 116° and 121°. It will be seen that in the first and third analysis there is a tendency towards a deficiency in the hydrogen; on searching into the cause of this, it was found that a trace of one of the homologues of olefiant gas, to be described in the second part of this paper, remained in the fluid. This arose from insufficient treatment with nitric acid. The fourth analysis was made on a perfectly pure product. The results obtained further on with the other radicals are perfectly in accordance with those of other chemists. The density of the butyle, as analysed, was found to be 0·6945 at 18°, agreeing perfectly

with KOLBE's determination made at the same temperature, which gave 0·6940. It undergoes considerable expansion with even a small rise of temperature, for WURTZ found the density at 0° to be 0·7057.

Amyle.

The experiments previously made having indicated somewhat too high a boiling-point for the butyle, it was considered necessary to examine the fractions having boiling-points at all approaching to that of amyle with great care, in order to ascertain whether the same difference would be found. The following Table contains the results:—

Vapour-densities of fractions from 154° *to* 169°, *after treatment with nitric acid, &c.*

Boiling-point.	Excess of weight of balloon.	Temperature of vapour.	Temperature of air.	Pressure.	Capacity of balloon.	Residual air.	Density.
°	grm.	°	°	mm.	cub. cent.	cub. cent.	
154 to 160	·6187	210	19	756	274	4·0	4·82
160 to 162	·7581	208	21	755·5	305·5	0	5·02
160 to 165	·7855	193	19	752	305·5	1·5	5·02*
163 to 164	·7518	210	20	755·5	298	0	5·11
166 to 169	·6980	207	18	757	275	0	5·11†

It was quite obvious, from the numbers condensed into the above Table, that the amyle from the Torbane-hill mineral boiled between 154° and 160°; but the first experiment which was made upon the entire fluid boiling between these points proving too low, a fraction was selected distilling in the sixteenth rectification between 157° and 160°.

·1949 gramme gave ·6032 carbonic acid and ·2757 water.

	Experiment.	Calculation.		
Carbon	84·4	C^{10}	120	84·5
Hydrogen . . .	15·7	H^{22}	22	15·5
			142	100·0

The following are the results obtained by FRANKLAND and WURTZ:—

	Frankland.			Wurtz.
Carbon . . .	84·2	84·6	——	84·2
Hydrogen . .	15·4	15·2	15·4	15·7

To confirm the result obtained in the analysis of the amyle from the above fraction, the density of its vapour was ascertained.

Excess of weight of balloon . . .	·6746 grm.
Temperature of vapour	212°

* From fraction of crude naphtha distilling between 154° and 160°.

† This result was obtained during the preliminary experiments; the fluid had not been absolutely freed from bodies of lower boiling-point.

Temperature of air	21°
Pressure	767·3 mm.
Capacity of balloon	283·5 cub. cent.
Residual air	·5 cub. cent.

The formula $C^{20} H^{22}$ requires—

20 volumes carbon vapour	0·8290.20	=16·580
44 volumes hydrogen	0·0692.44	= 3·045
		19·625
		$\frac{19·625}{4}$ = 4·906

Experiment.	Frankland.	Wurtz.	Theory.
4·930	4·899	4·956	4·906

The boiling-point of amyle, according to the most recent experiments on the subject*, is 158°, which agrees quite as nearly as could be expected with the above result. The density of the fluid was found to be (at 18°) 0·7365. Frankland found it 0·7704 at 11°, and Wurtz 0·7413 at 0°.

When Boghead amyle is cohobated with monohydrated nitric acid for some days, a large quantity of a nitro-acid is formed, which yields a silver salt of very sparing solubility. This acid will be described in a future communication, it being my intention to compare the products of the action of oxidizing agents on the radical from the Torbanehill distillate, with those from amyle prepared by the action of sodium on the iodide.

Caproyle.

This radical was first isolated by Messrs. Brazier and Gossleth† in the course of their researches on caproic and œnanthylic acids; they obtained it by the electrolysis of œnanthylate of potash. Remembering that, as obtained by them, it had a boiling-point of 202°, I commenced in searching for it by taking the vapour-densities of the fractions a little above and below that point, and obtained the numbers annexed:—

	Boiling-point, 196° to 199°.	Boiling-point, 202° to 208°.
Excess of weight of balloon	·8855 grm.	·6680
Temperature of vapour	230°	233°
Temperature of air	15°	14°
Pressure	739·4 mm.	760·2 mm.
Capacity of balloon	322·5 cub. cent.	225 cub. cent.
Residual air	4·0 cub. cent.	·5 cub. cent.
Density	5·74	6·03

The hydrocarbon from the Boghead naphtha, of which I was in search, boiled therefore

* Wurtz, Ann. de Chimie et de Phys. 2me série, tome xliv. p. 282.

† Quarterly Journal of the Chemical Society, vol. iii. p. 226.

between 199° and 202°. The following is a combustion of the fluid distilling at that temperature:—

·1578 gramme caproyle gave ·4897 carbonic acid and ·2238 water.

	Experiment.		Calculation.	
Carbon	84·6	C^{24}	144	84·7
Hydrogen	15·8	H^{26}	26	15·3
			170	100·0

The results of the analysis of this radical by other observers are as follows:—

Brazier and Gossleth.		Wurtz.
84·49	84·54	84·25
15·60	15·44	15·49

A determination of the vapour-density proved that the fraction boiling at 202° had not only the per-centage composition, but exactly the same condensation as the radical caproyle, thus:—

Excess of weight of balloon	·8021 grm.
Temperature of vapour	231°
Temperature of air	15°
Pressure	739·4 mm.
Capacity of balloon	283·0 cub. cent.
Residual air	1·0 cub. cent.
Density	5·83

The formula $C^{24}H^{26}$ requires—

24 volumes carbon vapour	0·8290.24=19·896
52 volumes hydrogen	0·0692.52= 3·598
	$\frac{23·494}{4}$=5·8735

Experiment.	Wurtz.	Theory.
5·83	5·983	5·8735

It was found that as the radicals of higher boiling-point were reached, greater difficulty existed in getting rid of the last traces of nitrocompound: although repeatedly washed with monohydrated nitric acid to remove any that might remain in solution, and then with strong potash, the radical so obtained did not, at first, distil perfectly colourless, and, moreover, retained a slight odour of the nitrocompound. The fluid in this condition left a yellow residue on distillation. A few rectifications over sodium entirely removed the impurity, and the radical as analysed above was perfectly colourless and almost inodorous. Its density at 18° was 0·7568. Wurtz found it at 0° 0·7574. Messrs. Brazier and Gossleth do not state the density of the caproyle analysed by them.

It is submitted that the foregoing experiments prove that the distillate from the Torbane-hill mineral contains, in addition to several other substances, a series of hydrocarbons having the per-centage composition, density in the fluid and gaseous states, and also the boiling-point of the alcohol radicals. It is perhaps to be regretted that in investigating these bodies, we are unable to avail ourselves of active affinities of a kind tending to yield easily procured and definite compounds, the study of which would remove all doubt as to identity. It is also peculiarly unfortunate that the boiling-points of simple and compound radicals, as at present determined, show no fixed law; in fact, if we examine the only data in our possession on the subject, namely, the numbers given by WURTZ, we find no less than nine different values for the increment of $C^2 H^2$, as may be seen from the following Tables:—

I.	Boiling-point.
Ethylamyle $C^{14} H^{13}$	88°
Ethylbutyle $C^{12} H^{14}$	62°
Difference for $C^2 H^2$	26°

II.	Boiling-point.
Methylcaproyle $C^{14} H^{13}$	82°
Ethylbutyle $C^{12} H^{14}$	62°
Difference for $C^2 H^2$	20°

III.	Boiling-point.
Butyle $C^{16} H^{18}$	106°
Ethylamyle $C^{14} H^{13}$	88°
Difference for $C^2 H^2$	18°

IV.	Boiling-point.
Butyle $C^{16} H^{18}$	106°
Ethylbutyle $C^{12} H^{14}$	62°
2)	44°
Difference for $C^2 H^2$	22°

V.	Boiling-point.
Butyle $C^{16} H^{18}$	106°
Methylcaproyle $C^{14} H^{13}$	82°
Difference for $C^2 H^2$	24°

VI.	
Butylamyle $C^{18} H^{20}$	132°
Butyle $C^{16} H^{18}$	106°
Difference for $C^2 H^2$	26°

VII.	
Butylamyle $C^{18} H^{20}$	132°
Ethylamyle $C^{14} H^{13}$	88°
2)	44°
Difference for $C^2 H^2$	22°

VIII.	
Butylamyle $C^{18} H^{20}$	132°
Methylcaproyle $C^{14} H^{13}$	82°
2)	50°
Difference for $C^2 H^2$	25°

IX.	
Amyle $C^{20} H^{22}$	158°
Butylamyle $C^{18} H^{20}$	132°
Difference for $C^2 H^2$	26°

X.	
Butylcaproyle $C^{20} H^{22}$	155°
Butylamyle $C^{18} H^{20}$	132°
Difference for $C^2 H^2$	23°

XI.	
Butylcaproyle $C^{20} H^{22}$	155°
Butyle $C^{16} H^{18}$	106°
2)	49°
Difference for $C^2 H^2$	24°·5

XII.	
Amyle $C^{20} H^{22}$	158°
Butyle $C^{16} H^{18}$	106°
2)	52°
Difference for $C^2 H^2$	26°

XIII.	
Caproyle $C^{24} H^{26}$	202°
Butylcaproyle $C^{20} H^{22}$	155°
2)	47°
Difference for $C^2 H^2$	23°·5

XIV.	
Caproyle $C^{24} H^{26}$	202°
Amyle $C^{20} H^{22}$	158°
2)	44°
Difference for $C^2 H^2$	22°

The following nine values are therefore obtained as the differences for $C^2 H^2$; the Roman numerals indicate the number of the Table:—

III.	II.	IV., VII. and XIV.	X.	XIII.	V.	XI.	VIII.	I., VI., IX. and XII.
18°	20°	22°	23°	23°·5	24°	24°·5	25°	26°

Several curious facts become apparent from inspection of these numbers, not the least

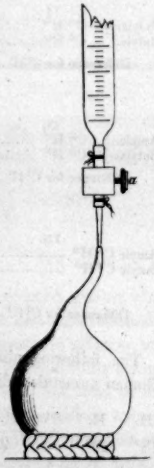
of which is, that, contrary to what has been found to hold with organic bases, isomeric radicals have the same, or nearly the same boiling-points, thus:—

Ethylbutyle=$C^{12}H^{14}$.	Propyle=$C^{12}H^{14}$.
Boiling-point.	Boiling-point.
62°	65°
Ethylamyle=$C^{14}H^{16}$.	Methylcaproyle=$C^{14}H^{16}$.
Boiling-point.	Boiling-point.
88°	82°
Amyle=$C^{20}H^{22}$.	Butylcaproyle=$C^{20}H^{22}$.
Boiling-point.	Boiling-point.
158°	155°.

If we consider the great difficulty of obtaining the radicals absolutely pure, we cannot be surprised at the fact that these numbers do not exactly accord.

It is not unworthy of note, that the great difference in the boiling-points of propyle and butyle from Boghead naphtha, which might have been urged as evidence against the view I have taken of their constitution, is, in fact, strong proof of the correctness of it, for the amount of that difference is 53° or 26° for C^2H^2, which happens to be the number occurring most frequently with the simple and compound radicals, and, moreover, it causes the Boghead propyle to have almost exactly the boiling-point of its isomer, ethylbutyle.

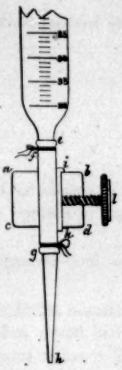
With regard to the fact of isomeric bases differing in boiling-point while isomeric radicals appear not to do so, it remains to be seen how much in the former case the variations are caused by difference of type.



In making the vapour-density determinations, I have endeavoured to find a method of shortening the time required for the performance of some of the operations. One of the points to which I turned my attention was, to avoid the necessity of filling the balloon a second time with mercury or water in order to determine the residual air. For this purpose, as soon as the point of the balloon has been broken off under mercury and the latter has entered the globe, it is to be placed on a straw ring beneath a burette filled with the metal, as in the engraving. The tube is divided into cubic centimetres and fractions. Having so adjusted the position of the balloon that the fine tube from the burette just enters the neck, the screw *a* is turned sufficiently to allow a slow stream of metal to flow until the condensed fluid exactly arrives at the opening, the altered level in the burette is then read off. It is evident that the difference between the first and second readings indicates a quantity of mercury exactly equal in volume to the residual air. The condensed fluid being removed with the pipette, the tap is again

opened, and mercury permitted to flow in until the globe is quite filled. The rest of the process is continued as usual. Where several densities are to be determined in succession, this process saves more time than would readily be believed.

The compression tap used in this experiment is capable of withstanding a very considerable pressure of mercury, and is so manageable that the smallest portion of metal may be added at will; it is also particularly well adapted for various volumetric and gaseous operations. The following is the method of construction. A block of wood, *a*, *b*, *c*, *d*, has a square hole mortised in it, through which passes the vulcanized tube, *e*, *f*, *g*, *k*, attached at *e*, *f* to the lower orifice of the burette. A piece of hard wood, *i*, *k*, also passes through the mortise, and is retained in its place by the flange *i*. It is evident that on turning the head of the screw, *l*, the piece *i* will be pressed against the vulcanized tube and collapse it.



In order to facilitate comparison of the indifferent hydrocarbons from Boghead naphtha with the radicals, I annex the following Table, containing their boiling-points and densities in the fluid and gaseous state, according to various observers.

Comparative Table of the Physical Properties of the Radicals.

Radicals.		Boiling-Points.					Densities.				Vapour-densities.				
		Frankland.	Kolbe.	Wurtz.	Brazier and Gossleth.	C. G. Williams.	Frankland, at 11°.	Kolbe, at 18°.	Wurtz, at 0°.	C. G. Williams, at 18°.	Frankland.	Kolbe.	Wurtz.	C. G. Williams.	Theory.
						Mean.									
Propyle	$C^{12}H^{14}$	...	...	...	...	68°	...	...	...	0·6745	...	...	...	2·96	2·97
Butyle	$C^{16}H^{18}$	...	108°	106°	...	119°	...	0·6940	0·7057	0·6945	...	4·053	4·07	3·88	3·94
Amyle	$C^{20}H^{22}$	155°	...	158°	...	159°	0·7704	...	0·7413	0·7365	4·899	...	4·956	4·93	4·91
Caproyle	$C^{24}H^{26}$	...	...	202°	202°	202°	...	...	0·7574	0·7568	...	...	5·983	5·83	5·87

XXII. *On an Element of Strength in Beams subjected to Transverse Strain, named by the author "The Resistance of Flexure."*—Second Paper.

By William Henry Barlow, *Esq., F.R.S. &c.*

Received March 12,—Read March 26, 1857.

In my former paper on this subject, I pointed out the existence of an element of strength in beams when subjected to transverse strain, which had been omitted in the generally admitted theory.

The forms of beam employed in the experiments described in that paper were only of two kinds, namely, solid rectangular bars, and open beams or girders.

In the experiments given in the present paper I have employed other forms, namely, square bars broken on their sides, square bars broken on their angles, round bars, beams of the I section broken with the flanges horizontal, and similar beams broken with the flanges vertical ⊢; the object of these experiments being, to elucidate the general bearing of the subject more clearly, and to determine with greater precision than was attempted in my former paper, the laws which govern this resistance.

The following are the results of the experiments in which, for the purpose of more easy reference, I have numbered the several forms of section in continuation of those described in my former paper; and have also included the results of those experiments.

Summary of Experiments on Transverse Strength. Solid and open beams.
Length of bearing 60 inches.

Number and form of section.	Total depth of beam.	Depth of metal.	Distance between the bars.	Breadth of bar.	Total sectional area.	Breaking weight.
	in.	in.	in.	in.	sq. in.	lbs.
No. 1	2·015	2·015	Nil.	·975	1·965	1664
	2·020	2·020	Nil.	·980	1·980	1888
	2·073	2·073	Nil.	1·030	2·135	2084
	2·040	2·040	Nil.	·990	2·020	1916
Mean.....	2·012	2·012	Nil.	·994	2·025	1888
No. 2	2·54	1·98	·56		2·01	2188
	2·53	1·98	·55		2·00	2748
	2·49	1·98	·51		1·96	2412
	2·50	1·95	·55		1·95	2524
Mean.....	2·51	1·97	·54	1·005	1·98	2468

Summary of Experiments on Transverse Strength (continued).

Number and form of section.	Total depth of beam.	Depth of metal.	Distance between the bars.	Breadth of bar.	Total sectional area.	Breaking weight.
	in.	in.	in.	in.	sq. in.	lbs.
No. 3	3·02	2·04	·98		2·02	3028
	3·00	2·00	1·00		2·00	3224
	3·00	1·99	1·01		1·98	3112
	3·00	1·99	1·01		1·98	2972
Mean.....	3·01	2·01	1·00	·995	2·00	3084
No. 4	3·99	1·99	2·00		2·00	4204
	4·00	1·97	2·03		1·96	4260
	3·99	1·94	2·05		1·96	4204
	4·01	1·97	2·04		1·99	4745
Mean.....	4·00	1·97	2·03	1·005	1·98	4353
No. 5	4·02	2·98	1·04		2·287	5050
	4·05	3·01	1·04		2·290	5125
	4·05	3·01	1·04		2·290	4985
	4·04	3·04	1·00		2·420	5405
Mean.....	4·04	3·01	1·03	·771	2·322	5·141
No. 6.....	4·02	1·50	2·52		2·26	5212
	4·05	1·50	2·55		2·27	5125
	4·03	1·47	2·56		2·19	4845
	4·06	1·45	2·61		2·20	5405
Mean.....	4·04	1·48	2·56	1·507	2·23	5147
No. 7	4·05	1·55	2·50		2·38	5685
	4·10	1·59	2·51		2·45	6525
	4·08	1·57	2·51		2·38	5965
	4·05	1·53	2·52		2·32	5825
Mean.....	4·07	1·56	2·51	1·525	2·38	6000

Summary of Experiments on Transverse Strength. Square and round bars of one inch sectional area.

Length of bearing 60 inches.

Square bars broken on their sides.

Number and form of section.	Depth.	Breadth.	Sectional area.	Breaking weight.
	in.	in.	sq. in.	lbs.
No. 8	1·010	1·020	1·030	505
	1·010	1·025	1·035	505
	1·010	1·020	1·030	561
	1·020	1·025	1·045	533
	1·000	1·020	1·020	533
Mean	1·010	1·020	1·032	527

Cylindrical bars.

Number and form of section.	Mean diameter.		Sectional area.	Breaking weight.
	in.		sq. in.	lbs.
No. 9	1·145		1·030	519
	1·113		·972	505
	1·115		·976	449
	1·118		·981	449
	1·120		·985	449
Mean	1·122		·989	474

Square bars broken on their angles.

Number and form of section.	Depth.	Side of square.	Sectional area.	Breaking weight.
	in.		sq. in.	lbs.
No. 10	1·442	1·020	1·040	449
	1·467	1·037	1·076	421
	1·450	1·025	1·050	449
	1·428	1·010	1·020	449
	1·428	1·010	1·020	477
Mean	1·443	1·020	1·041	449

Summary of Experiments on Transverse Strength. Square and round bars of about four inches sectional area.

Length of bearing 60 inches.

Square bars broken on their sides.

Number and form of section.	Depth.	Breadth.	Sectional area.	Breaking weight.
	in.	in.	sq. in.	lbs.
No. 11.	1·985	2·020	4·010	3303
	1·990	2·015	4·010	3303
	2·010	2·010	4·040	3443
	2·000	1·990	3·980	3863
Mean	1·996	2·009	4·010	3478

Cylindrical bars.

Number and form of section.	Mean diameter.	Breadth.	Sectional area.	Breaking weight.
			sq. in.	lbs.
No. 12.	2·52		4·987	4283
	2·52		4·987	4283
	2·52		4·987	4003
	2·51		4·948	4003
Mean	2·52		4·977	4143
No. 13.	2·20		3·801	3068
	2·20		3·801	2988
	2·19		3·767	3388
	2·20		3·801	3228
	2·19		3·767	2988
Mean	2·20		3·787	3132

Square bars broken on their angles.

Number and form of section.	Depth.	Side of square.	Sectional area.	Breaking weight.
	in.		sq. in.	lbs.
No. 14.	2·835	2·005	4·020	3128
	2·842	2·010	4·040	3268
	2·842	2·010	4·040	2848
	2·820	1·994	3·976	2708
Mean	2·835	2·005	4·020	2988

Summary of Experiments on Transverse Strength. Compound Sections.

Length of bearing 48 inches.

Number and form of section.	Total depth.	Depth of metal in flanges.	Distance between flanges.	Breadth of flanges.	Breadth of middle rib.	Total breadth.	Sectional area.	Breaking weight.
	in.	in.	in.	in.	in.	in.	sq. in.	lbs.
No. 15.	1·97	·99	·98	1·44	·55	1·99	2·51	3310
	2·00	1·00	1·00	1·50	·47	1·97	2·47	3560
	2·01	1·01	1·00	1·54	·48	2·02	2·52	3735
	2·08	1·11	·97	1·54	·53	2·07	2·81	3910
	2·07	1·06	1·01	1·50	·52	2·02	2·67	4528
	2·07	1·02	1·05	1·57	·47	2·04	2·57	4563
	2·06	1·04	1·02	1·56	·53	2·09	2·71	4423
Mean	2·04	1·03	1·00	1·53	·50	2·03	2·60	4004

Number and form of section.	Total depth.	Depth of centre rib.	Breadth of flanges.	Breadth of centre rib.	Total breadth.	Sectional area.	Breaking weight.
	in.	in.	in.	in.	in.	sq. in.	lbs.
No. 16.	1·97	·50	·98	1·00	1·98	2·43	2368
	1·96	·48	1·00	·96	1·96	2·42	2288
	2·05	·55	1·10	·92	2·02	2·76	3128
	2·04	·51	1·02	1·00	2·02	2·59	2568
	2·06	·50	1·06	·98	2·04	2·67	2420
	2·05	·50	1·02	1·02	2·04	2·60	2648
	2·05	·52	1·04	1·00	2·04	2·63	2568
Mean	2·02	·51	1·03	·98	2·02	2·59	2569

The neutral axis having been shown in my former paper to be in the centre of gravity of the section, we are enabled to test the accuracy of the existing theory, by comparing the resistance at the outer fibres or particles of each of the forms of beam, calculated upon that theory, with the actual tensile strength of the metal obtained by direct experiment.

In any bar or beam, supported at the ends and loaded in the centre,—

Let f represent the ultimate tension*,

l the length,

W the weight applied in the centre,

d the depth,

and x any variable distance from the neutral axis.

Then $\frac{fx}{d}$ will be the tension at the distance x, and according to the principle of LEIBNITZ, the sum of all these resistances at the moment of rupture will be

$$\int \frac{fx^2}{d}\,dx;$$

* In those materials in which the resistance to compression is less than that of tension, f must be taken to represent the ultimate resistance to compression.

and including the equal resistance to compression

$$2\int\frac{fx^2}{d}dx,$$

which taken between the limits $x=0$ and $x=d$, becomes

$$\frac{2fd^2}{3}=\tfrac{1}{4}l\mathrm{W}.$$

In the case of rectangular bars, if the breadth $=b$, this expression becomes

$$\tfrac{2}{3}fbd^2=\tfrac{1}{4}l\mathrm{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

In girders or bars of other forms, if $y=$ the double ordinate corresponding with the distance x, the general expression will be

$$\int\frac{fx}{d}yxdx;$$

and when the form of section is symmetrical above and below the neutral axis,

$$2\int\frac{fx}{d}yxdx=\tfrac{1}{4}l\mathrm{W}.$$

From this general expression we obtain the following for the several forms experimented upon:—

In the square bar broken on its angle, if

$d=$ half the depth,

$y=2(d-x)$,

the complete integral of this between the limits $x=0$ and $x=d$ will be

$$\frac{fd^3}{3}=\tfrac{1}{4}l\mathrm{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

In like manner, in round bars, if

$d=$ half-depth or radius,

$y=2\sqrt{(d^2-x^2)}$,

$\pi=3{\cdot}1416\ldots$,

then the complete integral is

$$\frac{\pi fd^3}{4}\tfrac{1}{4}l\mathrm{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

In the open beam, the expression for the resistance is the same as in the rectangular bar, except that here denoting the half-depth by D, and the half-distance between the bars by d, the expression

$$2bf\int\frac{x}{d}xdx$$

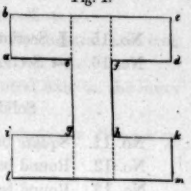
must be taken between the limits $x=d$ and $x=\mathrm{D}$, which gives

$$\frac{2bf}{3}\left(\frac{\mathrm{D}^3-d^3}{\mathrm{D}}\right)=\tfrac{1}{4}l\mathrm{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

In the case of the section No. 15, broken with the flanges horizontal (see fig. 1),

D= depth.
b = breadth of the centre rib.
b' = breadth of the flanges $ae+fd$.
d = half-distance of the flanges.

Fig. 1.



The expression for the centre rib is

$$\tfrac{2}{3}fbD^3,$$

and for the flanges

$$\frac{2b'f}{3}\left(\frac{D^3-d^3}{D}\right);$$

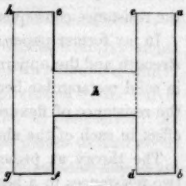
and consequently the resistance to the whole section will be

$$\frac{2f}{3}\left(bD^2+\frac{b'(D^3-d^3)}{D}\right)=\tfrac{1}{4}lW. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

In like manner, for section No. 16, broken with the flanges vertical (see fig. 2),

d = half-depth of the flange $abcd$.
b = width of the two flanges $=he+ca$.
d' = depth of the centre rib.
b' = breadth of the centre rib between the flanges.

Fig. 2.



Then $\frac{2fbd^2}{3}$ = resistance of the flanges,

and $\frac{2fb'd'^3}{3d}$ = resistance of centre rib;

and consequently the total resistance will be

$$\frac{2f}{3}\left(bd^2+\frac{b'd'^3}{d}\right)=\tfrac{1}{4}lW. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

With these formulæ we are enabled to calculate the resistance of the outer fibre under this generally accepted theory, in each of the sections.

The following Table shows the results:—

	Form of section.	Length of bearing. in.	Breaking weight. lbs.	Value of f, or the calculated resistance at the outer fibre.
No. 6.	Open girder	60	5147	25,271
No. 7.	Open girder	60	6000	27,908
No. 4.	Open girder	60	4339	28,032
No. 3.	Open girder	60	3119	31,977
No. 2.	Open girder	60	2468	35,386
No. 5.	Open girder	60	5141	37,408
No. 1.	Solid rectangular 2 × 1 inches .	60	1888	41,709
No. 8.	Square 1 × 1 inch	60	527	45,630
No. 9.	Round bar 1 inch area. . . .	60	474	51,396
No. 10.	Square bar broken diagonally .	60	449	53,966

Compound Sections.

	Form of section.	Length of bearing. in.	Breaking weight. lbs.	Value of f, or the calculated resistance at the outer fibre.
No. 15.	I Section, flanges horizontal .	48	4008	37,508
No. 16.	⊢ Section, flanges vertical . .	48	2569	43,358

Solid bars of 4 inches sectional area and upwards.

No. 11.	Square bar broken on its side .	60	3478	39,094
No. 12.	Round bar $2\frac{1}{2}$ inches diameter .	60	4143	39,560
No. 13.	Round bar $2\frac{1}{4}$ inches diameter .	60	3132	44,957
No. 14.	Square bar broken on its angle .	60	2988	47,746

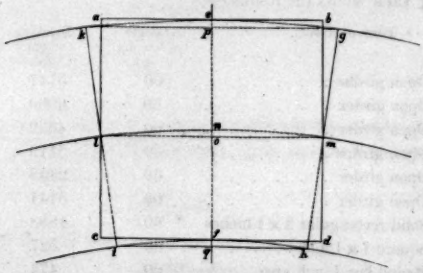
It will be seen from these results, that the apparent resistance at the outer fibre, computed on the principles of this theory, varies from 25,271 lbs. to 53,966 lbs.; while the tensile strength of the metal, as obtained by experiments on direct tension, averages only 18,750 lbs. This discrepancy and variation will be found to arise from the omission of the resistance consequent on the molecular disturbance accompanying curvature.

In my former paper a formula was given by which the difference between the tensile strength and the apparent resistance at the outer fibre could be computed, approximately, in solid rectangular beams and open girders. I now propose to trace the operation of the resistance of flexure, considered as a separate element of strength, and to show its effect in each of the above forms of section.

The theory at present acted upon, proceeds on the assumption that there are only two resistances in a beam, namely, tension and compression; but this supposition fails to account, not only for the strength, but also for the visible changes of figure which arise under transverse strain.

If *abdc* (fig. 3) represent the centre portion of a solid rectangular beam before any

Fig. 3.



strain is applied, *kghi* is the figure which this portion will assume when subjected to

transverse strain, the beam being supposed to be supported at the centre *f*, and loaded at its extremities.

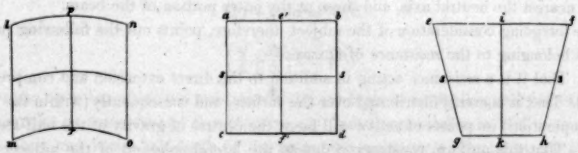
In this change of figure it will be observed that there are three effects:—

First, an extension of the fibres or particles, commencing at the neutral axis *lnm*, and increasing to the upper portion of the beam.

Secondly, a compression of the fibres or particles from the neutral axis to the lower portions of the beam; and

Thirdly, the planes or surfaces *alc* and *bmd* are forced downwards to the distance *ep*, *no* and *fq*.

There are, in fact, two distinct changes of figure:—



There is the change produced by the tension and compression, which, if acting alone, would result in the figure *efhg*; and there is the change produced by curvature, which, if acting alone, would result in the figure *lpnorm*. The effect produced by the curvature is, to cause the sides or planes *bd* and *ac* to descend parallel to themselves; the effect produced by the tension and compression is, to cause these planes to turn about the neutral axis. The combination of these effects is necessary to produce the figure which a beam assumes when placed under transverse strain; and the changes of figure point out distinctly the nature of the resistances. For as it was shown by the measurements taken in the experiments on the neutral axis, that the lines or planes corresponding to *ac* and *bd* remained straight, whatever was the amount of their angular motion, it follows that the tensions and compressions will increase in an arithmetical ratio from the neutral axis to the outer portions of the beam. But the effect of flexure causes the planes corresponding to *ac* and *bd* to descend an equal extent throughout their surfaces; the resistance to this change of figure will therefore be a force distributed evenly over the whole surface.

If *abcd* were a series of horizontal laminæ, these two changes of figure might be obtained separately; *efhg* being the result of the strains applied in the direction of the length, and *lnom* that of a strain applied at right angles to the length.

But if the laminæ are all united together, the elastic reaction of the mass causes certain fixed relations to be established between the curvature and the angles formed by the planes which were at right angles to the length, prior to the strain being applied.

Of these relations, it is sufficient for the present purpose to point out that which subsists between the degree of extension and compression, and the amount of curvature.

Referring again to fig. 3, if b represents any point in the upper surface of a solid beam, before strain is applied, and g the same point when loaded, br* will vary directly as rg. But rg represents the difference between the extension of the fibre, at or nearest the neutral axis, and that at the outer portion of the beam; therefore the resistance to flexure will vary directly as this difference.

In the case of the open beam, the resistance to flexure being only due to that of the bar deflected, whereas the ultimate deflection of the beam is equal to that of a solid beam of the same total depth, the resistance of flexure in the open beam will be to that of the solid beam, at the moment of rupture, as the depth of the bar to the half-depth of the beam; and this is also proportional to the difference between the extension of the fibres nearest the neutral axis, and those at the outer portion of the beam.

The foregoing consideration of the subject, therefore, points out the following properties as belonging to the resistance of flexure:—

1st. That it is a resistance acting in addition to the direct extension and compression.

2nd. That it is evenly distributed over the surface, and consequently (within the limits of its operation) its points of action will be at the centres of gravity of the half-section.

3rd. That this uniform resistance is due to the lateral cohesion of the adjacent surfaces of the fibres or particles, and to the elastic reaction which thus ensues between the portions of a beam unequally strained.

4th. That it is proportional to, and varies with, the inequality of strain between the fibres or particles nearest the neutral axis and those most remote.

We are enabled, under the above-mentioned conditions, to arrive at the relation between the straining and resisting forces in any of the forms of section experimented upon, as resulting from the combined effect of the resistances of tension, compression and flexure.

Using the same letters as before to represent the tension, weight, length, depth, &c., let $\varphi =$ the resistance of flexure acting as a force evenly spread over the surface of the section.

Then, instead of the expression $\frac{fx}{d}$, as representing the resistance at the distance x, we shall have, according to the preceding view, the expression

$$\frac{fx}{d}+\varphi,$$

and these forces acting as before, the moment will be

$$\left(\frac{fx}{d}+\varphi\right)x.$$

The sum of these moments, including those above and below the neutral axis, will be

$$2\int\left(\frac{fx}{d}+\varphi\right)x\,dx,$$

which, taken between the limits $x=0$ and $x=d$, becomes

$$2\left(\tfrac{1}{3}f+\tfrac{1}{2}\varphi\right)d^2=\tfrac{1}{4}lW.$$

* r, which is not represented in the figure, is the intersection of the lines bm, pg.

Taking $y =$ the double ordinate corresponding to the distance x, the general expression, when the sections are symmetrical above and below the neutral axis, will be

$$2\int\left(\frac{fx}{d}+\varphi\right)yx\,dx=\tfrac{1}{4}l\mathrm{W}.$$

From this general expression the following are obtained for the several forms experimented upon:—

First, in the case of the square or rectangular bar,

$$2(\tfrac{1}{3}f+\tfrac{1}{2}\varphi)bd^2=\tfrac{1}{4}l\mathrm{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

For the square when broken angleways,

$$(\tfrac{1}{3}f+\tfrac{1}{2}\varphi)d^3=\tfrac{1}{4}l\mathrm{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

For the round bars,

$$\left(\frac{\pi f}{4}+\frac{4\varphi}{3}\right)d^3=\tfrac{1}{4}l\mathrm{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

For the open bar, since the resistance to flexure depends on the inequality of extension between the part nearest and that most remote from the neutral axis, if $d' =$ the depth of the bar, and D the half-depth of the beam, the resistance to flexure at the moment of rupture will be $\varphi\frac{d'}{\mathrm{D}}$, or multiplied by $d'b$,

$$=\frac{d'^2}{\mathrm{D}}b\varphi;$$

and this resistance acting at the distance $\mathrm{D}-\frac{d'}{2}$, we have, for the whole resistance,

$$2b\left\{\frac{(\mathrm{D}^3-d'^3)f}{3\mathrm{D}}+\frac{d'^2}{\mathrm{D}}\left(\mathrm{D}-\frac{d'}{2}\right)\varphi\right\}=\tfrac{1}{4}l\mathrm{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

In the case of section No. 15 (fig. 1), broken with the flanges horizontal, the expression for the centre part will be

$$2(\tfrac{1}{3}f+\tfrac{1}{2}\varphi)b\mathrm{D}^2;$$

and for the flanges,

$$2b'\left\{\frac{(\mathrm{D}^3-d'^3)f}{3\mathrm{D}}+\frac{d'^2}{\mathrm{D}}\left(\mathrm{D}-\frac{d'}{2}\right)\varphi\right\};$$

and consequently for the whole section,

$$2(\tfrac{1}{3}f+\tfrac{1}{2}\varphi)b\mathrm{D}^2+2b\left\{\frac{(\mathrm{D}^3-d'^3)f}{3\mathrm{D}}+\frac{d'^2}{\mathrm{D}}\left(\mathrm{D}-\frac{d'}{2}\right)\varphi\right\}=\tfrac{1}{4}l\mathrm{W}. \quad . \quad . \quad . \quad (11.)$$

And lastly, for section No. 16 (fig. 2), broken with the flanges vertical, the expression for the flanges will be

$$2(\tfrac{1}{3}f+\tfrac{1}{2}\varphi)b'd^2;$$

and for the centre part,

$$2(\tfrac{1}{3}f+\tfrac{1}{2}\varphi)\frac{b'd'^3}{d};$$

and therefore, for the whole section,

$$2(\tfrac{1}{3}f+\tfrac{1}{2}\varphi)\left(b'd^2+\frac{b'd'^3}{d}\right)=\tfrac{1}{4}l\mathrm{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

These formulæ, applied to the several forms of beams experimented upon, give the following equations:—

No. 1. $\cdot 67062f + 1\cdot 0059\ \varphi = 28320$
No. 2. $1\cdot 0425\ f + 1\cdot 1813\ \varphi = 37020$
No. 3. $1\cdot 4473\ f + 1\cdot 3388\ \varphi = 46260$
No. 4. $2\cdot 3297\ f + 1\cdot 4698\ \varphi = 65295$
No. 5. $2\cdot 0625\ f + 2\cdot 2043\ \varphi = 77115$
No. 6. $3\cdot 0564\ f + 1\cdot 3512\ \varphi = 77205$
No. 7. $3\cdot 2227\ f + 1\cdot 5059\ \varphi = 90000$
No. 8. $\cdot 1734\ f + \cdot 2601\ \varphi = 7905$
No. 9. $\cdot 13867f + \cdot 23541\varphi = 7110$
No. 10. $\cdot 12519f + \cdot 25039\varphi = 6735$
No. 11. $1\cdot 3336\ f + 2\cdot 0009\ \varphi = 52170$
No. 12. $1\cdot 5708\ f + 2\cdot 6666\ \varphi = 62145$
No. 13. $1\cdot 0454\ f + 1\cdot 7746\ \varphi = 46980$
No. 14. $\cdot 9484\ f + 1\cdot 8968\ \varphi = 44820$
No. 15. $1\cdot 281\ f + 1\cdot 126\ \varphi = 48048$
No. 16. $\cdot 711\ f + 1\cdot 066\ \varphi = 30828$

If the metal were of precisely uniform strength, f and φ would be precisely constant quantities, and their value might be obtained from any two of these equations; but as considerable variation occurs in the strength, even in castings of the same dimensions, and as a reduction of strength, per unit of section, is known to arise when the thickness of the metal is increased, the values of f and φ will necessarily vary, and can only be ascertained in each experiment by first establishing the ratio they bear to each other.

For this purpose the first ten experiments may be used, all of which were made of metal of from three-quarters to one inch in thickness, the mean tensile strength of which was ascertained by direct experiment to be 18750 lb. per inch.

Using this value of f in each case, we have

No. 1. $\varphi = \dfrac{28320 - \cdot 67062 \times 18750}{1\cdot 0059} = 15654$

No. 2. $\varphi = \dfrac{37020 - 1\cdot 0425 \times 18750}{1\cdot 1813} = 14748$

No. 3. $\varphi = \dfrac{46260 - 1\cdot 4473 \times 18750}{1\cdot 3388} = 14284$

No. 4. $\varphi = \dfrac{65295 - 2\cdot 3297 \times 18750}{1\cdot 4698} = 14667$

No. 5. $\varphi = \dfrac{77115 - 2\cdot 0625 \times 18750}{2\cdot 2043} = 17442$

No. 6. $\phi = \frac{77905 - 3{\cdot}0564 \times 18750}{1{\cdot}3512} = 14725$

No. 7. $\phi = \frac{90000 - 3{\cdot}2227 \times 18750}{1{\cdot}5059} = 19640$

No. 8. $\phi = \frac{7905 - 1{\cdot}734 \times 18750}{{\cdot}2601} = 17892$

No. 9. $\phi = \frac{7110 - {\cdot}13867 \times 18750}{{\cdot}23541} = 19158$

No. 10. $\phi = \frac{6735 - {\cdot}12519 \times 18750}{{\cdot}25039} = 17523.$

Mean value of $\phi = 16573$.

Ratio of f to ϕ, as 1 to ·847.

If we use the following experiments of Mr. HODGKINSON's on the breaking weight of inch bars, of which the tensile strength was ascertained by direct experiment, the following results are obtained:—

Description of iron.	Transverse strength of the bar, 54 inches bearing.	Tensile strength per inch of the metal.	Computed value of ϕ*.	Ratio of f to ϕ.
	lbs.	lbs.	lbs.	
Carron iron No. 2, cold blast	476	16,683	14,582	1 to ·874
Carron iron No. 2, hot blast	463	13,505	15,999	1 to 1·185
Carron iron No. 3, cold blast	446	14,200	14,617	1 to 1·029
Carron iron No. 3, hot blast	527	17,755	14,621	1 to ·824
Devon iron No. 3, hot blast	537	21,907	14,393	1 to ·657
Buffery iron No. 1, cold blast	463	17,466	13,358	1 to ·765
Buffery iron No. 1, hot blast	436	13,434	14,588	1 to 1·086
Coed-Talon iron No. 2, cold blast	413	18,855	9,732	1 to ·516
Coed-Talon iron No. 2, hot blast	416	16,676	11,347	1 to ·682
Low Moor iron No. 3, cold blast	467	14,535	15,528	1 to 1·066
Mean	464	16,502	14,076	1 to ·853

These results indicate that the ratio between the resistance of tension and the resistance of flexure varies in different qualities of metal, and this supposition appears confirmed by other experiments on rectangular bars, given in the 'Report of the Commissioners on the Application of Iron to Railway Structures.' The mean result, however, accords nearly with that of my own experiments, and shows that the resistance of flexure, computed as a force evenly distributed over the section, is almost nine-tenths of the tensile resistance.

Employing this ratio of the values of f and ϕ, and applying it to the equations result-

* The sign ϕ was employed in my former paper to indicate the difference between the tensile force and the apparent resistance at the outer fibre. It is here used as the measure of the resistance considered as acting evenly over the surface; hence the value of ϕ, as here employed, will be two-thirds of the difference between the tensile resistance and the apparent resistance at the outer fibre in the rectangular bar.

ing from the experiments on the tensile strength of the metal, as derived from each form of section, the deduced values of f will be as follows:—

	Form of Girder.	
No. 1.	Solid rectangle	f=17,971
No. 2.	Open girder	f=17,582
No. 3.	Open girder	f=17,442
No. 4.	Open girder	f=17,882
No. 5.	Open girder	f=19,058
No. 6.	Open girder	f=18,070
No. 7.	Open girder	f=19,659
No. 8.	Square bar, 1 inch, section broken on its side	f=19,399
No. 9.	Round bar, 1 inch, section	f=20,236
No. 10.	Square bar, 1 inch, section broken on its angle	f=19,213
No. 11.	Square bar, 4 inches, section broken on its side	f=16,644
No. 12.	Round bar, 2½ inches diameter	f=15,902
No. 13.	Round bar, 2¼ inches diameter	f=17,778
No. 14.	Square bar, 4 inches, sectional area broken on its angle	f=16,878
No. 15.	Compound section, flanges horizontal	f=20,942
No. 16.	Compound section, flanges vertical	f=18,460

The results thus obtained, though not perfectly regular, are within the limits of the variation exhibited by the metal, as shown by the experiments on direct tension given in the former paper.

If the results be classified,—

The mean tensile strength, as obtained from the open girders, Experiments Nos. 2, 3, 4, 5, 6 and 7, is	18282
From the solid bar, No. 1	17971
From the inch bars, square, round, and square bars broken diagonally, Nos. 8, 9 and 10	19616
From the bars of 4 inches sectional area, square, round, and square bars, broken diagonally, Nos. 11, 12, 13, and 14	16800
From the compound sections, in which the metal was half an inch thick	19701

The variation in strength, as exhibited between the small and the large bars, is in accordance with the experiments made by Lieut.-Colonel James, and recorded in the 'Report of the Commissioners upon the Application of Iron to Railway Structures.'

The results obtained in all these varieties of form of section being so far satisfactory, it appeared desirable to test the application of the formulæ to other known experiments, of which the following may be given as examples:—

First, an experiment made by Mr. HODGKINSON, and given in TREDGOLD's 'Treatise on Cast Iron,' 4th edit. The form of the beam is as shown in the figure. In this case,—

$D = 2{\cdot}5625.$
$b = {\cdot}29.$
$b' = 1{\cdot}47$, mean.
$d = 2{\cdot}1573.$
$d' = 1{\cdot}405$, mean.
$l = 54.$
$W = 6678.$

And employing the formula used for the section No. 16, we have

$$2{\cdot}8631f + 2{\cdot}3476\phi = 90153;$$

and if ϕ be taken at nine-tenths of f,

$$f = 15086.$$

The tensile strength thus computed, accords very closely with the quality of metal employed by Mr. HODGKINSON in that and other experiments made by him at that time on various forms of girders.

In the Reports on the 'Strength and other properties of Metals for Cannon,' made by the Officers of the Ordnance Department of the United States Government, some experiments are given upon the transverse strength of square and round bars of cast iron. These experiments were made with very great care by Major WADE, for the purpose of testing various qualities of metals and modes of treatment, by frequent re-casting, and by keeping the metal for various periods of time under fusion. From each experiment, a constant is derived for the purpose of comparing the relative strengths of metal; and in endeavouring to obtain the constant for round iron, Major WADE has employed the usually accepted theory of the transverse strain. He appears, however, to have found that the formula is defective, for he observes at page 21 of the Report,—"A trial was made with cylindrical bars in lieu of square bars. These generally broke at a point distant from that pressed, and the results were so anomalous that the use of them was soon abandoned. The formula by which the strength of the round bars is computed, appears to be not quite correct; for the unit of strength in the round bars is uniformly much higher than in the square bars cast from the same kind of iron."

The following are the experiments on the round bars, with those on the square bars from the same metal; and it will be seen, that if the tensile strength of the metal be computed by the formula here given, including the resistance to flexure, the discrepancy pointed out by Major WADE disappears; and the tensile resistance, whether obtained for the round or the square bars, agrees very nearly with that derived from the experiments on direct tension under like circumstances.

Experiments on the Transverse Strength of Cast Iron, made by the Officers of the Ordnance Department of the United States Government.

Square bars. Length of bearing 20 inches.						
Description of iron.	Number of experiment.	Hours in fusion.	Breadth.	Depth.	Breaking weight.	Tensile resistance calculated from the formula, including resistance of flexure.
			in.	in.	lbs.	lbs.
Franklin iron.	9	1½	2·025	2·058	12,712	18,920
Second fusion...	10	2	2·000	2·054	12,712	19,233
	11	2½	1·994	2·008	13,950	22,149
	12	2½	1·989	2·013	11,700	18,531
Third fusion ...	79	2½	1·975	1·999	14,569	23,566
	80	2½	1·977	2·008	13,387	21,440
	21	0	2·025	1·980	12,987	20,882
	22	0	2·020	1·990	13,365	21,330
	23	1	2·030	1·990	15,363	24,396
Third fusion ...	24	1	2·030	1·990	14,616	23,211
	25	2	2·020	2·050	13,788	20,735
	26	2	2·050	2·070	14,850	21,582
	27	3	2·025	2·060	16,056	23,852
	28	3	2·035	2·020	16,722	25,708
	29	½	1·978	2·003	12,994	20,904
Third fusion ...	30	1½	1·930	2·003	15,300	25,226
	31	3	1·977	2·028	15,862	24,904
	32	3½	2·010	2·008	16,172	25,473

Round bars. Length of bearing 20 inches.					
Description of iron.	Number of experiment.	Hours in fusion.	Diameter.	Breaking weight.	Tensile resistance computed from the formula, including resistance of flexure.
			in.	lbs.	lbs.
Franklin iron...	37	1	1·975	7,920	20,711
Second fusion...	38	2	1·950	9,270	25,188
	39	3	1·953	9,481	25,644
	40	4	1·975	7,920	20,711
	81	½	2·415	16,425	23,493
Third fusion ...	82	1½	2·420	18,141	25,788
	83	2½	2·420	20,419	29,093
	84	2½	2·420	19,997	28,425
	85	2½	2·420	18,225	25,907
	33	½	1·960	10,437	27,927
Third fusion ...	34	1½	1·970	8,665	22,833
	35	3	2·000	11,112	27,984
	36	3½	1·960	10,606	28,378

The tensile strength of the same metal, as ascertained by direct experiment, is thus stated at page 44 of the Report:—

No. of fusion.	Franklin iron.		
	6-pounder gun, 3rd fusion.	Gun No. 61, 2nd and 3rd fusions.	Mean.
	lbs.	lbs.	lbs.
1st	25,969	15,861	20,915
2nd	29,143	20,420	24,781
3rd	27,755	24,383	26,569
4th	30,039	25,773	27,906
7th	.....	29,690	

Although not bearing directly on the subject of this paper, I cannot refrain from calling attention to the extraordinary development of strength in cast iron, obtained in the experiments made by the United States Government. It will be seen, on referring to the Reports from which the above Tables are taken, that by frequent recasting and keeping the metal under fusion during periods of from three to four hours, an increase of 60 per cent. is obtained; and that the strength of the American iron so treated is more than double that of English under the usual mode of manufacture.

The general accordance presented between the value of the tensile resistance, obtained by direct experiment, and that computed by means of the foregoing formulæ in so many varieties of form of section, is such as to confirm the view here taken of the laws which govern the action of the resistance of flexure.

It remains only to refer to two points connected with it, first, as to the ratio it bears to the tensile resistance. If the metal were homogeneous and the elasticity perfect, it is probable that the resistance of flexure would be precisely equal to the tensile resistance, instead of bearing the ratio of nine-tenths, as found by experiment. It is evident, however, that it varies in different qualities of metal, and that the tensile resistance does not bear a constant ratio to the transverse strength.

The following Table, taken from Major WADE's valuable Reports, shows, that with the same metal and different modes of casting, an increase of transverse strength is obtained, while a decrease takes place in the tensile resistance.

Guns.	Transverse strength.		Tensile strength.		Specific gravity.	
	Bar cut from gun.	Bar cast separate.	Bar cut from gun.	Bar cast separate.	Bar cut from gun.	Bar cast separate.
6-pounder gun, No. 6 ...	8415	9,880	30,234	29,143	7·196	7·263
6-pounder gun, No. 8 ...	9233	9,977	31,087	30,039	7·278	7·248
8-inch gun, No. 64	8575	10,176	26,367	24,583	7·276	7·331
Mean	8741	10,011	29,229	27,922	7·250	7·281
Proportional	1·000	1·145	1·000	·955	1·000	1·004

From the above, it appears that with a decrease of about one-twentieth in the tensile strength, there is an increase of nearly three-twentieths in the transverse strength.

It is easy to conceive also, that though the resistance of flexure might be supposed to maintain nearly the same proportion to the tensile resistance in bodies similarly consti-

tuted, as for example crystalline substances, yet great variation may be expected to occur between crystalline and malleable and fibrous substances.

The only other point to be referred to is, as to the limit of action of the resistance of flexure. It appears evident that in all the simple solid sections, the points of action of the resistance of flexure are the centres of gravity of the half-section; while in the compound sections it is necessary to compute the centre rib and flanges as for two separate beams in which the resistance of flexure is different, and has its point of action at the centre of gravity of the separate portions.

It would appear that the elastic reaction developes this resistance to the full extent, when the section is such that a straight line may be drawn from every point at the outer portion to every point at the neutral axis within the section; but that if the form of section is such that straight lines drawn from the outer fibres, or particles, to the neutral axis fall without the section (as in the case in the compound sections, Nos. 15 and 16), then it must be treated as two separate beams, each having that amount of resistance of flexure due to the depth of the metal contained in it.

Resistance of Flexure in Wrought Iron.

Although from the fact, that in a cast-iron beam (the section being a solid rectangle) the neutral axis was found to be at the centre of gravity of the section, it might have been inferred that the same would be found in wrought iron; yet it was considered desirable to ascertain it by actual measurement. For this purpose two beams were taken, one of rolled iron, 7 feet 6 inches long, 6 inches in depth, and $1\frac{1}{4}$ inch in breadth; the other of hammered iron, 8 feet long, $7\frac{1}{4}$ inches in depth, and $1\frac{3}{4}$ inch in breadth. Holes were drilled at about 6 inches on each side of the centre, or point of application of the strain, for the insertion of the pins of the measuring instrument. The holes were six in number, and placed at equal distances from the upper to the lower side of the beam; and the experiments were conducted in the same manner as those made with the cast-iron beam, and described in my former paper.

Experiment for Determination of Neutral Axis.

Wrought Iron Beam (rolled iron).

Depth . . . . . 5·93 inches.
Breadth . . . . . 1·28 inch.
Length of bearing . . 60 inches.

Beam without weight.	Difference.	Weight applied at centre, 7840 lbs.	Difference.	Weight applied at centre, 11,200 lbs.	Difference.	Weight taken off.	Permanent set.
Micrometer readings.		Micrometer readings.		Micrometer readings.		Micrometer readings.	
1741	+25	1766	+13	1779	−38	1741	
1769	+14	1783	+ 7	1790	−22	1768	−1
1653	+ 5	1658	+ 5	1663	−10	1658	
1787	− 5	1782	+ 2	1784	+ 2	1786	−1
1706	−12	1694	− 8	1686	+22	1708	+2
1746	−23	1723	−11	1712	+38	1750	+4

The same beam with the bearing distance increased to 84 inches.

Beam without weight.	Difference.	Weight applied at centre, 8000 lbs.	Difference.	Weight taken off.	Permanent set.	Beam without weight.	Difference.	Weight applied at centre, 8000 lbs.	Difference.	Weight taken off.	Permanent set.
Microm. readings.		Microm. readings.		Microm. readings.		Microm. readings.		Microm. readings.		Microm. readings.	
1757	+36	1793	−36	1757	...	1753	−36	1717	+35	1752	−1
1787	+17	1804	−18	1786	−1	1786	−23	1763	+21	1784	−2
1682	+ 4	1686	− 3	1683	+1	1683	− 9	1674	+ 7	1681	−2
1812	−11	1801	+10	1811	−1	1813	+ 6	1819	− 5	1814	+1
1743	−26	1717	+24	1741	−2	1739	+24	1763	−21	1742	+3
1779	−39	1740	+40	1780	+1	1779	+35	1814	−38	1776	−3

Experiment for Determination of Neutral Axis.

Wrought Iron Beam (hammered iron).

Depth . . . . . 7·25 inches.
Breadth . . . . . 1·75 inch.
Length of bearing . . . . 88 inches.

Beam without weight.	Difference.	Weight applied at centre of beam, 10,266 lbs.	Difference.	Weight applied at centre of beam, 19,226 lbs.	Difference.	Weight applied at centre of beam, 23,706 lbs.	Difference.	Weight taken off.	Permanent set.
Micrometer readings.		Micrometer readings.		Micrometer readings.		Micrometer readings.		Micrometer readings.	
1560	+24	1584	+21	1605	+18	*1623	−61	1562	+ 2
1599	+18	1617	+10	1627	+ 8	1635	−34	1601	+ 2
1624	+ 7	1631	+ 4	1635	− 2	1633	−15	1618	− 6
1643	− 5	1638	− 6	1632	− 8	1624	+ 6	1630	−13
1456	−12	1444	−14	1430	−20	1410	+27	1437	−19
1401	−22	1379	−25	1354	−29	1325	+52	1377	−24

Although the extensions and compressions are only about half that of cast iron, and consequently the liability to error in the measurements is increased in proportion, yet the experiments point out that the position of the neutral axis in wrought iron, like that of cast, is at the centre of gravity of the section, and that the action is the same in both materials, excepting as to the amount of the extensions and compressions with a given strain.

The formula $2(\frac{1}{2}f+\phi\frac{1}{3})bd^2=\frac{lW}{4}$, given in the former part of the paper for cast iron, will therefore apply to wrought iron also.

The relative values of f and ϕ are not so readily ascertained in wrought iron, because the material yields by bending and not by fracture. And another point requires consideration, namely, that the ultimate compressive strain which wrought iron is capable

* Previous to these measurements being taken, a weight of 14,093 lbs. was applied on the end, equal to 28,186 lbs. on the centre of the beam, but was reduced to 23,706 lbs. in the centre. The elasticity of the beam had, however, been overcome, as shown by the permanent set and by subsequent experiments on the same beam.

of sustaining is little more than half its ultimate tensile strength. But although there exists this disproportion as regards the ultimate resistances by tension and compression, the force required to overcome the elasticity of the material is nearly the same, whether applied as a compressive or a tensile strain; the difference being, that the force which overcomes the elasticity when applied as a compressive strain, leads to the destruction or distortion of the material; while in the case of the tensile strain, the elasticity may be overcome long before the material yields by absolute rupture.

The following experiments, made in Woolwich Dockyard by Professor BARLOW, show the relative weights which overcome the elasticity of the metal when applied transversely, as compared with the weight necessary to produce the same result when applied by direct tension.

Three bars, numbered 5, 6 and 7, each 2 inches square, were tested by direct tension; and the strain which was just sufficient to overcome the elasticity of the iron, was found to be—

	Tons per square inch of section.
No. 5. Re-manufactured iron	9·5
No. 6. Re-manufactured iron, from old furnace-bars .	8·25
No. 7. New bar by Messrs. GORDON	10·00

Bars of the same quality of metal having been subjected to transverse strain in a bearing of 33 inches, the weights which overcame their elasticity, when so applied, were as follows:—

	Tons.	Mean.
No. 5	3·00	3·00
No. 5	3·00	
No. 6	2·50	2·25
No. 6	2·00	
No. 7	3·00	2·83
No. 7	2·50	
No. 7	3·00	

Using the tensile resistance in each case for f, and obtaining the value of φ from the formula

$$\varphi = \frac{lW}{4bd^2} - \frac{2}{3}f,$$

	Tons.	Tons.
No. 5 . . .	$\varphi = 6{\cdot}04$	$f = 9{\cdot}50$
No. 6 . . .	$\varphi = 3{\cdot}78$	$f = 8{\cdot}25$
No. 7 . . .	$\varphi = 5{\cdot}01$	$f = 10{\cdot}00$
Mean . .	$\varphi = 4{\cdot}94$	$f = 9{\cdot}25$

Mean ratio . . $f : \varphi :: 1 : {\cdot}53$.

In addition to these, six other experiments are given on iron, of the quality of No. 7, which are as follows:—

Number of experiment.	Length of bearing.	Breadth.	Depth.	Weight which overcame the elasticity.	Value of ϕ computed from the formula.
	in	in.	in.	tons.	tons.
8	33	1·9	2·0	2·5	3·65
9	33	1·9	2·0	2·5	3·65
10	33	1·9	2·0	2·5	3·65
11	33	1·5	3·0	4·5	4·33
12	33	1·5	3·0	4·5	4·33
13	33	1·5	2·5	3·25	4·77
				Mean	4·06

Mean ratio of f to ϕ : 1 to ·406.

The general mean of these results appears therefore to show, that the resistance of flexure in wrought iron, considered as a force acting evenly over the surface, is nearly equal to one half of the tensile resistance. It is, however, desirable that further experiments should be made with this material.

Appendix to the foregoing pages. *By* PETER BARLOW, *Esq.*, *F.R.S.*

Received March 25,—Read April 2, 1857.

Application of the preceding principles to Beams and Girders of non-symmetrical section.

IN beams of symmetrical section the neutral axis corresponds with the centre of gravity, because in that case all the direct forces above and below that point are necessarily equal. But when the section is non-symmetrical, it is requisite, in order to determine the position of the neutral axis, to find that point in the section in which this condition has place, viz. that point below which the sum of all the direct resistances to tension and curvature are equal to all those above that point due to compression and curvature; then to find the sum of the moments of these resistances separately; and finally, to equate them with the straining force.

The double-flanged girder with unequal flanges forms a good subject for testing the general application of the principles developed in the preceding pages. In such a girder, let

a denote the whole depth of the girder;
m the thickness of the middle web;
d the depth of the bottom flange;
d' the depth of the upper flange;
b the breadth of the former, *minus* m;
b' the breadth of the latter, *minus* m;
x the required distance of the neutral axis from the bottom of the girder;
x' the distance of the same from the upper face of the girder;
t the tensile resistance of the lower fibres;
c the corresponding resistance to compression of the upper fibres.

Now if we consider the centre rib as carried through the two flanges, the sum of the direct resistances due to the tension of the metal in the middle rib below x will be $\frac{1}{2}mxt$,

and the sum of those due to curvature or change of figure, $mx\varphi$; or calling $\varphi=t$*, the whole direct resistance of this central web below x will be $\frac{1}{2}mxtb$. Again, since the direct tensile resistance of the unsupported flange varies as the distance from the neutral axis, if we consider x as representing a constant quantity, and y any variable distance from the neutral axis, $bt\int\frac{ydy}{x}$ (taken between $y=x$ and $y=x-d$) becomes $\left(d+\frac{d^2}{2x}\right)bt$, the sum of all the direct tensile resistances; the resistance to change of figure being expressed simply by dbt.

The total direct resistance below the neutral axis is therefore

$$\left(\tfrac{1}{2}mx+2bd-\frac{d^2}{2x}b\right)t.$$

In like manner, the total direct resistance to compression above the neutral axis is

$$\left(\tfrac{1}{2}mx'+2b'd'-\frac{d'^2}{2x'}b'\right)c,$$

which must be made equal to the former expression.

But we must here observe, that the compression of the upper fibre c, is to the corresponding tension of the lower fibre t, as x' to x; substituting accordingly, rejecting the common factor t, and observing that $x'=a-x$, we find

$$x=\frac{3ma^2+4d'b'a+d'^2b-d'^2b'}{6ma+4(db+d'b')}.$$

Having thus determined the position of the neutral axis, we have now to take the moments of these several direct forces both above and below that line, the formula for which are however already given in the preceding pages; that for the lower part of the central web being $\frac{1}{6}m\mathrm{D}^3t$ (D now representing x, the distance above found), and that for the unsupported flange being the same as in the open beam, viz.

$$\frac{\mathrm{D}^3-\overline{\mathrm{D}-d}^3}{3\mathrm{D}}bt+\frac{d}{\mathrm{D}}(\mathrm{D}-\tfrac{1}{2}d)dbt.$$

This latter is, however, reducible to a more convenient form for numerical calculation, viz. to

$$\left(\mathrm{D}-\frac{d^2}{6\mathrm{D}}\right)dbt.$$

We have therefore

$$\tfrac{1}{6}m\mathrm{D}^2t+\left(\mathrm{D}-\frac{d^2}{6\mathrm{D}}\right)dbt=\mathrm{R},$$

the resistance below the neutral axis, and

$$\tfrac{1}{6}m\mathrm{D}'^2c+\left(\mathrm{D}'-\frac{d'^2}{6\mathrm{D}'}\right)d'b'c=\mathrm{R}',$$

the resistance above the neutral axis.

* In the preceding paper, Mr. W. H. Barlow, by obtaining from experiments a mean value of t, has, by means of his original equation for rectangular bars, *i. e.* $(\frac{2}{3}t+\frac{1}{3}\varphi)\mathrm{D}^2=\frac{1}{4}lw$, and his other equations for beams of other forms when broken transversely, endeavoured to find a mean value of φ, and he finds the latter to be to the former as about 9 : 10; but from the difficulty of obtaining the mean value of t within certain wide limits, I have not hesitated in assuming t and ϕ equal to each other in the case of cast iron.

But $c : t :: D' : D$. We have, therefore, for the whole resistance above and below the neutral axis,

$$\left(R+\frac{D'}{D}R'\right)t.$$

If now we represent by w the breaking weight in any experiment of a beam of given dimensions, and by l the length, or rather the distance between the props, there is obtained the expression

$$\left(R+\frac{D'}{D}R'\right)t=\tfrac{1}{4}lw,$$

an equation from which, when w and l are given, t may be determined. Or if t be previously experimentally determined, w may be found.

In order to submit these equations to the test of experience, we have selected from the valuable and extensive series of experiments, by EATON HODGKINSON, Esq., published in vol. v. of the Memoirs of the Literary and Philosophical Society of Manchester, Second Series, a few experiments in which the girders differ most from each other in section, dimensions, and bearing distance; and the results we obtain from the foregoing formula are given in the following pages, with the form of section and linear dimensions. It will be seen that the value of t, or the direct tensile strength of cast iron, thus obtained, falls generally between the limits of $t=14{,}000$ and $t=16{,}000$.

In the Reports of the Commissioners for inquiry into the 'Application of Iron to Railway Structures' (page 9, &c.), there are given the results of about fifty experiments on the direct tensile resistance of one-inch square cast-iron bars, under the direction of Mr. HODGKINSON; these consisted of seventeen different kinds of iron, each set of three bars being of the like quality and manufacture, and in several of these sets, which one might have expected to yield very nearly the same results, the difference is full as great as in the following Table, exhibiting in fact very nearly like numbers. This circumstance will, it is presumed, be considered satisfactory evidence of the general applicability of the principles developed in the preceding pages to cast-iron beams and girders of every variety of section.

		Distance of props.	
(1)*		4 feet 6 inches.	Depth of girder . . . . 5·125 inches. Upper flange . . . 1·75 by ·42 inch. Lower flange . . . 1·77 by ·39 inch. Thickness of centre web . . ·29 inch. Breaking weight . . . . . 6678 lbs. Computed value of $t=14578$.
(2)		4 feet 6 inches.	Depth of girder . . . . 5·125 inches. Upper flange . . . 1·74 by ·26 inch. Lower flange . . . 1·78 by ·55 inch. Thickness of centre web . . ·30 inch. Breaking weight . . . . . 7368 lbs. Computed value of $t=14128$.

* These numbers are those of the experiments selected from Mr. HODGKINSON's series.

(3)

Distance of props. 4 feet 6 inches.

Depth of girder . . . . 5·125 inches.
Upper flange . . . 1·07 by ·30 inch.
Lower flange . . . . 2·1 by ·57 inch.
Thickness of centre web . . ·32 inch.
Breaking weight . . . . . 8270 lbs.
Computed value of t=14005.

(4)

4 feet 6 inches.

Depth of girder . . . . 5·125 inches.
No upper flange.
Lower flange . . . . 2·27 by ·52 inch.
Mean thickness . . . . . ·405 inch.
Breaking weight . . . . . 8720 lbs.
Computed value of t=13868.

(9)

4 feet 6 inches.

Depth of girder . . . . 5·125 inches.
Upper flange . . . 1·05 by ·34 inch.
Lower flange . . . 3·08 by ·51 inch.
Thickness of centre web . . ·305 inch.
Breaking weight . . . . 10,727 lbs.
Computed value of t=14,765.

(11)

4 feet 6 inches.

Depth of girder . . . . 5·125 inches.
Upper flange . . . 1·60 by ·315 inch.
Lower flange . . . ·416 by ·53 inch.
Thickness of centre web . . ·38 inch.
Breaking weight . . . . 14,462 lbs.
Computed value of t=14,832.

(12)

4 feet 6 inches.

Depth of girder . . . . 5·125 inches.
Upper flange . . . 1·56 by ·315 inch.
Lower flange . . . 5·17 by ·56 inch.
Thickness of centre web . . ·34 inch.
Breaking weight . . . . 16,730 lbs.
Computed value of t=14,181.

(15)

4 feet 6 inches.

Depth of girder . . . . 5·125 inches.
Upper flange . . . 2·35 by ·29 inch.
Lower flange . . 5·43 by ·537 inches.
Thickness of centre web . . ·35 inch.
Breaking weight . . . . 16,905 lbs.
Computed value of t=13,918

(19) Distance of props. 4 feet 6 inches.

Depth of girder . . . . 5·125 inches.
Upper flange . . . 2·33 by ·31 inch.
Lower flange . . . 6·67 by ·66 inch.
Thickness of centre web . . ·266 inch.
Breaking weight . . . . 26,084 lbs.
Computed value of t=15,474.

(23) 7 feet.

Depth of girder . . . . 4·1 inches.
Upper flange . . . 2·25 by ·33 inch.
Lower flange . . . 6·00 by ·74 inch.
Thickness of centre web . . ·40 inch.
Breaking weight . . . . 13,543 lbs.
Computed value of t=16,720.

(24) 7 feet.

Depth of girder . . . . 5·2 inches.
Upper flange . . . 2·25 by ·35 inch.
Lower flange . . . 6·00 by ·77 inch.
Thickness of centre web . . ·34 inch.
Breaking weight . . . . 15,129 lbs.
Computed value of t=13,612.

(30) 9 feet.

Depth of girder . . . . 10·25 inches.
Upper flange . . . . 2·1 by ·27 inch.
Lower flange . . . 6·14 by ·77 inch.
Mean thickness of centre web ·27 inch.
Breaking weight . . . . 28,672 lbs.
Computed value of t=14,606.

(34) 4 feet 6 inches.

Depth of girder . . . . 5·125 inches.
No upper flange.
Lower flange . . . 2·27 by ·46 inch.
Mean thickness of centre web ·37 inch.
Breaking weight . . . . . 8792 lbs.
Computed value of t=15,374.

(35) 4 feet 6 inches.

Depth of girder . . . . 5·125 inches.
No upper flange.
Lower flange . . . 2·26 by ·47 inch.
Mean thickness of centre web ·352 inch.
Breaking weight . . . . 9044 lbs.
Computed value of t=15,980.

Tabulated Results.

Number of Hodgkinson's series.	Upper flange.		Lower flange.		Thickness of centre web.	Depth of girder.	Breaking distance.	Breaking weight.	Computed value of t.
	Breadth.	Depth.	Breadth.	Depth.					
	inches.	inch.	inches.	inch.	inch.	inches.	ft. in.	lbs.	lbs.
1	1·75	·42	1·77	·39	·29	$5\frac{1}{2}$	4 6	6,678	14,578
2	1·74	·26	1·78	·55	·30	$5\frac{1}{2}$	4 6	7,368	14,128
3	1·07	·30	2·10	·57	·32	$5\frac{1}{2}$	4 6	8,270	14,005
					Mean.				
4	No upper flange.		2·27	·52	·405	$5\frac{1}{2}$	4 6	8,720	13,868
9	1·05	·34	3·08	·51	·305	$5\frac{1}{2}$	4 6	10,727	14,765
11	1·60	·315	4·16	·53	·38	$5\frac{1}{2}$	4 6	14,462	14,832
12	1·56	·315	5·17	·56	·34	$5\frac{1}{2}$	4 6	16,730	14,181
15	2·35	·29	5·43	·537	·35	$5\frac{1}{2}$	4 6	16,905	13,918
19	2·33	·31	6·67	·66	·266	$5\frac{1}{2}$	4 6	26,084	15,474
23	2·25	·33	6·00	·74	·40	4·1	7 0	13,543	16,720
24	2·25	·35	6·00	·77	·34	5·2	7 0	15,129	13,612
30	2·10	·27	6·14	·77	·27	$10\frac{1}{4}$	9 0	28,672	14,606
					Mean.				
34	No upper flange.		2·27	·44	·37	$5\frac{1}{2}$	4 6	8,792	15,374
35	No upper flange.		2·26	·47	·352	$5\frac{1}{2}$	4 6	9,044	15,980

In the preceding investigation the breaking weight is given from which to determine the tensile resistance; but the usual practical question is, to find the breaking weight, having first ascertained the tensile strength, which is of course simply to reverse the last operation. In the case of small beams, of the kind employed in the foregoing experiments, a sufficiently near approximation, it will be seen, may be obtained by assuming t=14,500 or 15,000, except in peculiar kinds and mixtures of iron; these will generally require a higher number, which must be previously determined.

But it appears from the results of experiments by Mr. Hodgkinson, given at page 111 of the 'Appendix to the Report of the Commissioners on Railway Structures,' and others by Lieut.-Colonel James, R.E., page 251, &c., that a much lower value of t must be taken when the thickness of the casting becomes 2, $2\frac{1}{2}$ or 3 inches, as in large railway girders. Mr. Hodgkinson found, that bars of 1, 2 and 3 inches square, broken on props having the same relative distances, manifested a decrease of strength in the proportion of 1, ·780, ·756; and Colonel James's experiments gave a still greater decrease, viz. of 1, ·794, ·624, and which, by other experiments, he traces to an imperfect crystallization of the interior particles, in consequence probably of the more rapid cooling of the exterior parts. It appears, therefore, that in those large castings commonly employed for railway bridges, it would not be safe to assume t at more than 10,000 lbs.

The large girder broken and reported at page 94 by Mr. Hodgkinson, treated as in the preceding cases, gives t=10,533 lbs. Its length was 45 feet; depth, $29\frac{1}{2}$ inches; the thickness of the lower flange, $2\frac{9}{16}$ inches; and its weight, 18,000 lbs.

XXIII. *A Memoir on the Symmetric Functions of the Roots of an Equation.*

By Arthur Cayley, *Esq.*, *F.R.S.*

Received December 18, 1856,—Read January 8, 1857.

There are contained in a work, which is not, I think, so generally known as it deserves to be, the 'Algebra' of Meyer Hirsch, some very useful tables of the symmetric functions up to the tenth degree of the roots of an equation of any order. It seems desirable to join to these a set of tables, giving reciprocally the expressions of the powers and products of the coefficients in terms of the symmetric functions of the roots. The present memoir contains the two sets of tables, viz. the new tables distinguished by the letter (*a*), and the tables of Meyer Hirsch distinguished by the letter (*b*); the memoir contains also some remarks as to the mode of calculation of the new tables, and also as to a peculiar symmetry of the numbers in the tables of each set, a symmetry which, so far as I am aware, has not hitherto been observed, and the existence of which appears to constitute an important theorem in the subject. The theorem in question might, I think, be deduced from a very elegant formula of M. Borchardt (referred to in the sequel), which gives the generating function of any symmetric function of the roots, and contains potentially a method for the calculation of the Tables (*b*), but which, from the example I have given, would not appear to be a very convenient one for actual calculation.

Suppose in general

$$(1, b, c..)(1, x)^n = (1-\alpha x)(1-\beta x)(1-\gamma x)\ldots,$$

so that

$$-b=\Sigma\alpha,\ +c=\Sigma\alpha\beta,\ -d=\Sigma\alpha\beta\gamma,\ \&c.,$$

and if in general

$$(pqr..)=\Sigma\alpha^p\beta^q\gamma^r\ldots,$$

where as usual the summation extends only to the distinct terms, so that *e. g.* (p^2) contains only half as many terms as (pq), and so in all similar cases, then we have

$$-b=(1),\ +c=(1^2),\ -d=(1^3),\ \&c.;$$

and the two problems which arise are, first to express any combination $b^\lambda c^\mu\ldots$ in terms of the symmetric functions $(l^\lambda m^\mu..)$, and secondly, or conversely, to express any symmetric function $(l^\lambda m^\mu\ldots)$ in terms of the combinations $b^\lambda c^\mu\ldots$.

It will conduce materially to brevity if $1^\lambda 2^\mu\ldots$ be termed the partition belonging to the combination $b^\lambda c^\mu\ldots$; and in like manner if $l^\lambda m^\mu\ldots$ be termed the partition belonging to the symmetric function $(l^\lambda m^\mu\ldots)$, and if the sum of the component numbers of the partition is termed the weight.

Consider now a line of combinations corresponding to a given weight, *e. g.* the weight 4, this will be

$$\begin{array}{ccccc} e & bd & c^2 & b^2c & b^4 \quad \text{(line)} \\ 4 & 13 & 2^2 & 1^22 & 1^4, \end{array}$$

where I have written under each combination the partition which belongs to it, and in like manner a column of symmetric functions of the same weight, viz.

(4) (column)
(31)
(2^2)
(21^2)
(1^4),

where, as the partitions are obtained by simply omitting the ( ), I have not separately written down the partitions.

It is at once obvious that the different combinations of the line will be made up of numerical multiples of the symmetric functions of the column; and conversely, that the symmetric functions of the column will be made up of numerical multiples of the combinations of the line; but this requires a further examination. There are certain restrictions as to the symmetric functions which enter into the expression of the combination, and conversely, as to the combinations which enter into the expression of the symmetric function. The nature of the first restriction is most clearly seen by the following Table:—

Number of Parts.	Greatest Part.	Combinations with their several Partitions.	Contain Multiples of the Symmetric Functions.	Greatest Part does not exceed	Number of Parts not less than
1	4	e 4	(1^4),	1	4
2	3	bd 13	(1^4), (21^2),	2	3
2	2	c^2 2^2	(1^4), (21^2), (2^2),	2	2
3	2	b^2c 1^22	(1^4), (21^2), (2^2), (31),	3	2
4	1	b^4 1^4	(1^4), (21^2), (2^2), (31), (4)	4	1

Thus, for instance, the combination bd (the partition whereof is 13) contains multiples of the two symmetric functions (1^4), (21^2) only. The number of parts in the partition 13 is 2, and the greatest part is 3. And in the partitions (1^4), (21^2) the greatest part is 2, and the number of parts is not less than 3. The reason is obvious: each term of the developed expression of bd must contain at least as many roots as are contained in each term of d, that is 3 roots, and since the coefficients are linear functions in respect to each root, the combination bd cannot contain a power higher than 2 of any root. The reasoning is immediately applied to any other case, and we obtain

First Restriction.—A combination $b^\beta c^\gamma \ldots$ contains only those symmetric functions $(l^\lambda m^\mu \ldots)$, for which the greatest part does not exceed the number of parts in the partition $1^\beta 2^\gamma \ldots$, and the number of parts is not less than the greatest part in the same partition.

Consider a partition such as 1^22, then replacing each number by a line of units thus,

1
1
11,

and summing the columns, we obtain a new partition 31, which may be called the conjugate* of 1^2. It is easy to see that the expression for the combination b^2c (for which the partition is 1^2) contains with the coefficient unity, the symmetric function (31), the partition whereof is the conjugate of 1^2. In fact $b^2c=(-\Sigma\alpha)^3(\Sigma\alpha\beta)$, which obviously contains the term $+1\alpha^3\beta$, and therefore the symmetric function with its coefficient $+1(31)$; and the reasoning is general, or

Theorem. A combination $b^\beta c^\gamma..$ contains the symmetric function (partition conjugate to $1^\beta 2^\gamma\ldots$) with the coefficient unity, and sign + or − according as the weight is even or odd.

Imagine the partitions arranged as in the preceding column, viz. first the partition into one part, then the partitions into two parts, then the partitions into three parts, and so on; the partitions into the same number of parts being arranged according to the magnitude of the greatest part (the greatest magnitude first), and in case of equality according to the magnitudes of the next greatest part, and so on (for other examples, see the outside column of any one of the Tables). The order being thus completely defined, we may speak of a partition as being prior or posterior to another. We are now able to state a second restriction as follows.

Second Restriction.—The combination $b^\beta c^\gamma\ldots$ contains only those symmetric functions which are of the form (partition not prior to the conjugate of $1^\beta 2^\gamma\ldots$).

The terms excluded by the two restrictions are many of them the same, and it might at first sight appear as if the two restrictions were identical; but this is not so: for instance, for the combination bd^2, see Table VII(a), the term (41^3) is excluded by the first restriction, but not by the second; and on the other hand, the term (3^31), which is not excluded by the first restriction, is excluded by the second restriction, as containing a partition 3^31 prior in order to 32^2, which is the partition conjugate to 13^3, the partition of bd^3. It is easy to see why bd^3 does not contain the symmetric function (3^31); in fact, a term of (3^31) is $(\alpha^3\beta^3\gamma)$, which is obviously not a term of $bd^3=(-\Sigma\alpha)(\Sigma\alpha\beta\gamma)^3$; but I have not investigated the general proof.

I proceed to explain the construction of the Tables (a). The outside column contains the symmetric functions arranged in the order before explained; the outside or top line contains the combinations of the same weight arranged as follows, viz. the partitions taken in order from right to left are respectively conjugate to the partitions in the outside column, taken in order from top to bottom; in other words, each square of the sinister diagonal corresponds to two partitions which are conjugate to each other. It is to be noticed that the combinations taken in order, from left to right, are *not* in the order in which they would be obtained by ARBOGAST'S Method of Derivations from an operand a^x, a being ultimately replaced by unity. The squares above the sinister diagonal are empty (*i. e.* the coefficients are zero), the greater part of them in virtue of both restrictions, and the remainder in virtue of the second restriction; the empty squares below the sinister diagonal are empty in virtue of the second restriction; but the property was not assumed in the calculation.

* The notion of Conjugate Partitions is, I believe, due to Professor SYLVESTER or Mr. FERRERS.

The greater part of the numbers in the Tables (a) were calculated, those of each table from the numbers in the next preceding table by the following method, depending on the derivation of the expression for $b^{p+1}c^q\ldots$ from the expression for $b^pc^q\ldots$. Suppose, for example, the column cd of Table V(a) is known, and we wish to calculate the column bcd of Table VI(a). The process is as follows:—

Given

2^21	21^3	1^5
1	3	10

we obtain

321	2^3	31^3	2^21^2	21^4	1^6
1	3		2		
		3	6	12	
				10	60
1	3	3	8	22	60

where the numbers in the last line are the numbers in the column bcd of Table VI(a). The partition 2^21, considered as containing a part zero, gives, when the parts are successively increased by 1, the partitions 321, 2^3, 2^21^2, in which the indices of the increased part (*i. e.* the original part plus unity) are 1, 3, 2; these numbers are taken as multipliers of the coefficient 1 of the partition 2^21, and we thus have the new coefficients 1, 3, 2 of the partitions 321, 2^3, 2^21^2. In like manner the coefficient 3 of the partition 21^3 gives the new coefficients 3, 6, 12 of the partitions 31^3, 2^21^2, 21^4, and the coefficient 10 of the partition 1^5 gives the new coefficients 10, 60 of the partitions 21^4 and 1^6, and finally, the last line is obtained by addition. The process in fact amounts to the multiplication separately of each term of $cd=$

$$1(2^21)+3(21^3)+10(1^5)$$

by $b=(1)$. It would perhaps have been proper to employ an analogous rule for the calculation of the combinations $c^qd^r\ldots$ not containing b, but instead of doing so I availed myself of the existing Tables (b). But the comparison of the last line of each Table (a) (which as corresponding to a combination b^p was always calculated independently of the Tables (b)) with such last line as calculated from the corresponding Table (b), seems to afford a complete verification of both the Tables; and my process has in fact enabled me to detect several numerical errors in the Tables (b), as given in the English translation of the work above referred to. It is not desirable, as regards facility of calculation and independently of the want of verification, to calculate either set of Tables wholly from the other; the rules for the independent calculation of the Tables (b) are fully and clearly explained in the work referred to, and I have nothing to add upon this subject.

The relation of symmetry, alluded to in the introductory paragraph of the present memoir, exists in each Table of either set, and is as follows: viz. the number in the Table corresponding to any two partitions in the outside column and the outside line respectively, is equal to the number corresponding to the same two partitions in the outside line and the outside column respectively. Or, calling the two partitions P, Q, and writing for shortness, combination (P) for the combination represented by the partition

P, and for greater clearness, symmetric function (P) (instead of merely (P)) to denote the symmetric function represented by the partition P, we have the following two theorems, viz.

Theorem. The coefficient in combination (P) of symmetric function (Q) is equal to the coefficient in combination (Q) of symmetric function (P);
and conversely,

Theorem. The coefficient in symmetric function (P) of combination (Q) is equal to the coefficient in symmetric function (Q) of combination (P).

M. BORCHARDT's formula, before referred to, is given in the 'Monatsbericht' of the Berlin Academy (March 5, 1855)*, and may be thus stated: viz. consideringt he case of n roots, write

$$(1, b, c, \dots k^n)(1, x)^n = (1-\alpha x)(1-\beta x)..(1-\varkappa x) = fx,$$

then

$$\Sigma\left(\frac{1}{1-\alpha x}\frac{1}{1-\beta y}\cdots\frac{1}{1-\varkappa u}\right) = \frac{1}{k}(-)^n \frac{fxfy..fu}{\Pi(x,y,..u)}\frac{d}{dx}\frac{d}{dy}\cdots\frac{d}{du}\frac{\Pi(x,y,..u)}{fxfy\dots fu},$$

where $\Pi(x, y, .. u)$ denotes the product of the differences of the quantities $x, y, \dots u$, and on the left-hand side the summation extends to all the different permutations of $\alpha, \beta, \dots \varkappa$, or what is the same thing, of $x, y, .. u$.

Suppose for a moment that there are only two roots, so that

$$(1, b, c)(1, x)^2 = (1-\alpha x)(1-\beta x),$$

then the left-hand side is

$$\frac{1}{(1-\alpha x)(1-\beta y)} + \frac{1}{(1-\alpha y)(1-\beta x)},$$

which is equal to

$$2+(\alpha+\beta)(x+y)+(\alpha^2+\beta^2)(x^2+y^2)+2\alpha\beta xy+(\alpha^3+\beta^3)(x^3+y^3)+(\alpha^2\beta+\alpha\beta^2)(x^2y+xy^2)+\&c.,$$

and the right-hand side is

$$\frac{1}{c}\cdot\frac{fxfy}{x-y}\frac{d}{dx}\frac{d}{dy}\frac{x-y}{fxfy},$$

which is equal to

$$\frac{1}{c}\frac{fxfy}{x-y}\left\{\frac{f'xfy-f'yfx+(x-y)f'xf'y}{(fx)^2(fy)^2}\right\},$$

and therefore to

$$\frac{1}{c}\cdot\frac{1}{fxfy}\left\{\frac{f'xfy-f'yfx}{x-y}+f'xf'y\right\};$$

or substituting for fx, fy their values,

$$\frac{f'xfy-f'yfx}{x-y}$$

becomes equal to

$$2c-b^2-bc(x+y)-2c^2xy,$$

and $f'xf'y$ is equal to

$$b^2+2bc(x+y)+4c^2xy.$$

The right-hand side is therefore equal to

$$\frac{2+b(x+y)+2cxy}{(1+bx+cx^2)(1+by+cy^2)}.$$

* And in *Crelle*, t. liii. p. 195.—Note added 4th Dec. 1857, A. C.

And comparing with the value of the left-hand side, we see that this expression may be considered as the generating function of the symmetric functions of (α, β), viz. the expression in question is developable in a series of the symmetric functions of (x, y), the coefficients being of course functions of b and c, and these coefficients are (to given numerical factors *près*) the symmetric functions of the roots (α, β).

And in general it is easy to see that the left-hand side of M. BORCHARDT's formula is equal to

$$[0]+[1](1)(1)'+[2](2)(2)'+[1^2](1^2)(1^2)'+\&c.,$$

where (1), (2), (1^2), &c. are the symmetric functions of the roots $(\alpha, \beta, \ldots \varkappa)$, $(1)'$, $(2)'$, $(1^2)'$, &c. are the corresponding symmetric functions of $(x, y, \ldots u)$, and [0], [1], [2], $[1^2]$, &c. are mere numerical coefficients; viz. [0] is equal to $1.2.3\ldots n$, and [1], [2], $[1^2]$, &c. are such that the product of one of these factors into the number of terms in the corresponding symmetric function (1), (2), (1^2), &c. may be equal to $1.2.3\ldots n$. The right-hand side of M. BORCHARDT's formula is therefore, as in the particular case, the generating function of the symmetric functions of the roots $(\alpha, \beta, \ldots \varkappa)$, and if a convenient expression of such right-hand side could be obtained, we might by means of it express all the symmetric functions of the roots in terms of the coefficients.

Tables relating to the Symmetric Functions of the Roots of an Equation.

The outside line of letters contains the combinations (powers and products) of the coefficients, the coefficients being all with the positive sign, and the coefficient of the highest power being unity; thus in the case of a cubic equation the equation is

$$x^3+bx^2+cx+d=(x-\alpha)(x-\beta)(x-\gamma)=0.$$

The outside line of numbers is obtained from that of letters merely by writing 1, 2, 3.. for b, c, d.., and may be considered simply as a different notation for the combinations. The outside column contains the different symmetric functions in the notation above explained, viz. (1) denotes $\Sigma\alpha$, (2) denotes $\Sigma\alpha^2$, (1^2) denotes $\Sigma\alpha\beta$, and so on. The Tables (a) are to be read according to the columns; thus Table II(a) means $b^2=1(2)+2(1)^2$, $c=(1^2)$. The Tables (b) are to be read according to the lines; thus Table II(b) means $(2)=-2c+1b^2$, $(1^2)=+1c$.

I(a).

	1
	b
(1)	−1

II(a).

	2	1^2
	c	b^2
(2)		+1
(1^2)	+1	+2

III(a).

	3	12	1^3
	d	bc	b^3
(3)			−1
(21)		−1	−3
(1^3)	−1	−3	−6

I(b).

=	1
	b
(1)	−1

II(b).

=	2	1^2
	c	b^2
(2)	−2	+1
(1^2)	+1	

III(b).

=	3	12	1^3
	d	bc	b^3
(3)	−3	+3	−1
(21)	+3	−1	
(1^3)	−1		

IV(a).

‖	4 e	13 bd	2^2 c^2	$1^2 2$ b^2c	1^4 b^4
(4)					+ 1
(31)				+ 1	+ 4
(2^2)			+1	+ 2	+ 6
(21^2)		+1	+2	+ 5	+12
(1^4)	+1	+4	+6	+12	+24

IV(b).

=	4 e	13 bd	2^2 c^2	$1^2 2$ b^2c	1^4 b^4
(4)	−4	+4	+2	−4	+1
(31)	+4	−1	−2	+1	
(2^2)	+2	−2	+1		
(21^2)	−4	+1			
(1^4)	+1				

V(a).

‖	5 f	14 be	23 cd	$1^2 3$ b^2d	12^2 bc^2	$1^3 2$ b^3c	1^5 b^5
(5)							− 1
(41)						− 1	− 5
(32)					− 1	− 3	− 10
(31^2)				− 1	− 2	− 7	− 20
($2^2 1$)			− 1	− 2	− 5	−12	− 30
(21^3)		−1	− 3	− 7	−12	−27	− 60
(1^5)	−1	−5	−10	−20	−30	−60	−120

V(b).

=	5 f	14 be	23 cd	$1^2 3$ b^2d	12^2 bc^2	$1^3 2$ b^3c	1^5 b^5
(5)	−5	+5	+5	−5	−5	+5	−1
(41)	+5	−1	−5	+1	+3	−1	
(32)	+5	−5	+1	+2	−1		
(31^2)	−5	+1	+2	−1			
($2^2 1$)	−5	+3	−1				
(21^3)	+5	−1					
(1^5)	−1						

VI(a).

‖	6 g	15 bf	24 ce	$1^2 4$ b^2e	3^2 d^2	123 bcd	$1^3 3$ b^3d	2^3 c^3	$1^3 2^2$ b^2c^2	$1^4 2$ b^4c	1^6 b^6
(6)											+ 1
(51)										+ 1	+ 6
(42)									+ 1	+ 4	+ 15
(3^2)								+ 1	+ 2	+ 6	+ 20
(41^2)							+ 1	∽	+ 2	+ 9	+ 30
(321)						+ 1	+ 3	+ 3	+ 8	+ 22	+ 60
(2^3)					+ 1	+ 3	+ 6	+ 6	+ 15	+ 36	+ 90
(31^3)				+ 1	∽	+ 3	+ 10	+ 6	+ 18	+ 48	+120
($2^2 1^2$)			+ 1	+ 2	+ 2	+ 8	+ 18	+15	+ 34	+ 78	+180
(21^4)		+1	+ 4	+ 9	+ 6	+22	+ 48	+36	+ 78	+168	+360
(1^6)	+1	+6	+15	+30	+20	+60	+120	+90	+180	+360	+720

VI(b).

=	6 g	15 bf	24 ce	$1^2 4$ b^2e	3^2 d^2	123 bcd	$1^3 3$ b^3d	2^3 c^3	$1^3 2^2$ b^2c^2	$1^4 2$ b^4c	1^6 b^6
(6)	− 6	+6	+6	−6	+3	−12	+6	−2	+9	−6	+1
(51)	+ 6	−1	−6	+1	−3	+ 7	−1	+2	−4	+1	
(42)	+ 6	−6	+2	+2	−3	+ 4	−2	−2	+1		
(3^2)	+ 3	−3	−3	+3	+3	− 3	∽	+1			
(41^2)	− 6	+1	+2	−1	+3	− 3	+1				
(321)	−12	+7	+4	−3	−3	+ 1					
(2^3)	− 2	+2	−2	∽	+1						
(31^3)	+ 6	−1	−2	+1							
($2^2 1^2$)	+ 9	−4	+1								
(21^4)	− 6	+1									
(1^6)	+ 1										

VII(a).

‖	7 A	16 bg	25 cf	1^25 b^2f	34 de	124 bce	1^24 b^2e	12^3 bd^2	2^23 c^2d	1^223 b^2cd	1^43 b^4d	12^2 bc^2	1^32^2 b^3c^2	1^52 b^5c	1^7 b^7
(7)															− 1
(61)														− 1	− 7
(52)													− 1	− 5	− 21
(43)												− 1	− 3	− 10	− 35
(51^2)											− 1	∽	− 2	− 11	− 42
(421)										− 1	− 4	− 3	− 11	− 35	− 105
(3^{2}1)									− 1	− 2	− 6	− 7	− 18	− 50	− 140
(32^2)								− 1	− 2	− 5	− 12	− 12	− 31	− 80	− 210
(41^3)							− 1	∽	∽	− 3	− 13	− 6	− 24	− 75	− 210
(321^2)						− 1	− 3	− 2	− 5	− 13	− 34	− 27	− 68	− 170	− 420
(2^{3}1)					− 1	− 3	− 6	− 7	− 12	− 27	− 60	− 51	− 117	− 270	− 630
(31^4)				− 1	∽	− 4	− 13	− 6	− 12	− 34	− 88	− 60	− 150	− 360	− 840
(2^{2}1^3)			− 1	− 2	− 3	− 11	− 24	− 18	− 31	− 68	−150	−117	− 258	− 570	−1260
(21^5)		−1	− 5	−11	−10	− 35	− 75	− 50	− 80	−170	−360	−270	− 570	−1200	−2520
(1^7)	−1	−7	−21	−42	−35	−105	−210	−140	−210	−420	−840	−630	−1260	−2520	−5040

VII(b).

=	7 A	16 bg	25 cf	1^25 b^2f	34 de	124 bce	1^24 b^2e	12^2 bd^2	2^23 c^2d	1^223 b^2cd	1^43 b^4d	12^3 bc^3	1^32^2 b^3c^2	1^52 b^5c	1^7 b^7
(7)	− 7	+7	+7	−7	+7	−14	+7	−7	−7	+21	−7	+7	−14	+7	−1
(61)	+ 7	−1	−7	+1	−7	+ 8	−1	+4	+7	− 9	+1	−5	+ 5	−1	
(52)	+ 7	−7	+3	+2	−7	+ 4	−2	+7	−3	− 6	+2	+3	− 1		
(43)	+ 7	−7	−7	+7	+5	+ 2	−3	−5	+1	+ 3	∽	−1			
(51^2)	− 7	+1	+2	−1	+7	− 3	+1	−4	−2	+ 4	−1				
(421)	−14	+8	+4	−3	+2	− 8	+3	+1	+2	− 1					
(3^{2}1)	− 7	+4	+7	−4	−5	+ 1	∽	+2	−1						
(32^2)	− 7	+7	−3	−2	+1	+ 2	∽	−1							
(41^3)	+ 7	−1	−2	+1	−3	+ 3	−1								
(321^2)	+21	−9	−6	+4	+3	− 1									
(2^{3}1)	+ 7	−5	+3	∽	−1										
(31^4)	− 7	+1	+2	−1											
(2^{2}1^3)	−14	+5	−1												
(21^5)	+ 7	−1													
(1^7)	− 1														

‖	8 i	17 bh	26 cg	1^26 b^2g	35 df	125 bcf	1^35 b^3f	4^3 e^3	134 bde	2^24 c^2e	1^324 b^2ce	$1^$ b
(8)												
(71)												
(62)												
(53)												
(4^2)												
(61^2)												
(521)												
(431)												
(42^2)												
(3^22)												
(51^3)												+
(421^3)											+ 1	+
(3^21^3)										+ 1	+ 2	+
(32^21)									+ 1	+ 2	+ 5	+
(2^4)								+ 1	+ 4	+ 6	+ 12	+
(41^4)							+ 1	∽	∽	∽	+ 4	+
(321^4)						+ 1	+ 3	∽	+ 3	+ 7	+ 18	+
(2^21^4)					+ 1	+ 3	+ 6	+ 2	+ 11	+ 18	+ 39	+
(31^5)				+ 1	∽	+ 5	+ 16	∽	+ 10	+ 20	+ 55	+
(2^21^5)			+ 1	+ 2	+ 4	+ 14	+ 30	+ 6	+ 32	+ 53	+114	+
(21^6)		+ 1	+ 6	+13	+15	+ 51	+108	+20	+ 95	+150	+315	+
(1^8)	+1	+8	+28	+56	+56	+168	+336	+70	+280	+420	+840	+

=	8 i	17 bh	26 cg	1^26 b^2g	35 df	125 bcf	1^35 b^3f	4^3 e^3	134 bde	2^24 c^2
(8)	− 8	+ 8	+ 8	−8	+8	−16	+8	+4	−16	−
(71)	+ 8	− 1	− 8	+1	−8	+ 9	−1	−4	+ 9	+
(62)	+ 8	− 8	+ 4	+2	−8	+ 4	−2	−4	+16	−
(53)	+ 8	− 8	− 8	+8	+7	+ 1	−3	−4	+ 1	+
(4^2)	+ 4	− 4	− 4	+4	−4	+ 8	−4	+6	− 8	−
(61^2)	− 8	+ 1	+ 2	−1	+8	− 3	+1	+4	− 9	−
(521)	−16	+ 9	+ 4	−3	+1	− 8	+3	+8	−10	−
(431)	−16	+ 9	+16	−9	+1	−10	+4	−8	+10	∽
(42^2)	− 8	+ 8	− 4	−2	+8	− 4	+2	−4	∽	+
(3^22)	− 8	+ 8	+ 2	−5	−7	+ 5	∽	+4	− 1	−
(51^3)	+ 8	− 1	− 2	+1	−3	+ 3	−1	−4	+ 4	+
(421^3)	+24	−10	− 6	+4	−9	+11	−4	+4	− 1	−
(3^21^3)	+12	− 5	− 9	+5	+3	− 1	∽	+2	− 2	+
(32^21)	+24	−17	∽	+5	+6	− 3	∽	−4	+ 1	
(2^4)	+ 2	− 2	+ 2	∽	−2	∽	∽	+1		
(41^4)	− 8	+ 1	+ 2	−1	+3	− 3	+1			
(321^4)	−32	+11	+ 8	−5	−3	+ 1				
(2^21^4)	−16	+ 9	− 4	∽	+1					
(31^5)	+ 8	− 1	− 2	+1						
(2^21^4)	+20	− 6	+ 1							
(21^6)	− 8	+ 1								
(1^8)	+ 1									

VIII(a).

$1^4 4$ b^4e	$2^3 2^2$ cd^2	$1^3 3^2$ b^3d^2	$1^2 3^2 3$ bc^2d	$1^2 23$ b^2cd	$1^3 3$ b^3d	2^4 c^4	$1^2 2^2$ b^2c^2	$1^4 2^2$ b^4c^2	$1^6 2$ b^6c	1^8 b^8
										+ 1
									+ 1	+ 8
								+ 1	+ 6	+ 28
							+ 1	+ 4	+ 15	+ 56
						+ 1	+ 2	+ 6	+ 20	+ 70
					+ 1	∽	∽	+ 2	+ 13	+ 56
				+ 1	+ 5	∽	+ 3	+ 14	+ 51	+ 168
			+ 1	+ 3	+ 10	+ 4	+ 11	+ 32	+ 95	+ 280
		+ 1	+ 2	+ 7	+ 20	+ 6	+ 18	+ 53	+ 150	+ 420
	+ 1	+ 2	+ 5	+ 12	+ 30	+ 12	+ 31	+ 80	+ 210	+ 560
+ 1	∽	∽	∽	+ 3	+ 16	∽	+ 6	+ 30	+ 108	+ 336
+ 4	∽	+ 2	+ 5	+ 18	+ 55	+ 12	+ 39	+ 114	+ 315	+ 840
+ 6	+ 2	+ 4	+ 12	+ 30	+ 80	+ 28	+ 68	+ 172	+ 440	+ 1120
+ 12	+ 5	+ 12	+ 24	+ 58	+ 140	+ 48	+ 117	+ 284	+ 690	+ 1680
+ 24	+ 12	+ 28	+ 48	+ 108	+ 240	+ 90	+ 204	+ 468	+ 1080	+ 2520
+ 17	∽	+ 6	+ 12	+ 46	+ 140	+ 24	+ 84	+ 246	+ 660	+ 1680
+ 46	+ 12	+ 30	+ 58	+ 141	+ 340	+ 108	+ 258	+ 612	+ 1440	+ 3360
+ 84	+ 31	+ 68	+ 117	+ 258	+ 570	+ 204	+ 453	+ 1008	+ 2250	+ 5040
+ 140	+ 30	+ 80	+ 140	+ 340	+ 800	+ 240	+ 570	+ 1320	+ 3000	+ 6720
+ 246	+ 80	+ 172	+ 284	+ 612	+ 1320	+ 468	+ 1008	+ 2172	+ 4680	+ 10080
+ 660	+ 210	+ 440	+ 690	+ 1440	+ 3000	+ 1080	+ 2250	+ 4680	+ 9720	+ 20160
+ 1680	+ 560	+ 1120	+ 1680	+ 3360	+ 6720	+ 2520	+ 5040	+ 10080	+ 20160	+ 40320

VIII(b).

$2^2 4$ c^2e	$1^2 24$ b^2ce	$1^4 4$ b^4e	$2^3 2^2$ cd^2	$1^2 3^2$ b^2d^2	$1^2 3^2 3$ bc^2d	$1^2 23$ b^2cd	$1^3 3$ b^3d	2^4 c^4	$1^2 2^2$ b^2c^2	$1^4 2^2$ b^4c^2	$1^6 2$ b^6c	1^8 b^8
−8	+ 24	−8	−8	+ 12	+ 24	− 32	+ 8	+ 2	− 16	+ 20	− 8	+ 1
+ 8	− 10	+ 1	+ 8	− 5	− 17	+ 11	− 1	− 2	+ 9	− 6	+ 1	
− 4	− 6	+ 2	+ 2	− 9	∽	+ 8	− 2	+ 2	− 4	+ 1		
+ 8	− 9	+ 3	− 7	+ 3	+ 6	− 3	∽	− 2	+ 1			
− 4	+ 4	∽	+ 4	+ 2	− 4	∽	∽	+ 1				
− 2	+ 4	− 1	− 5	+ 5	+ 5	− 5	+ 1					
− 4	+ 11	− 3	+ 5	− 1	− 3	+ 1						
∽	− 1	∽	− 1	− 2	+ 1							
+ 4	− 2	∽	− 2	+ 1								
− 2	∽	∽	+ 1									
+ 2	− 4	+ 1										
− 2	+ 1											
+ 1												

IX(a

ll	9 j	18 bi	27 ch	1^27 b^2h	35 dg	125 bcy	1^36 b^3g	45 ef	135 bdf	2^25 c^2f	1^325 b^2cf	1^45 b^2f	14^3 be^3	234 cde	1^234 b^2de	12^34 bc^2e	1^324 b^2ce
(9)																	
(81)																	
(72)																	
(63)																	
(54)																	
(71^2)																	
(621)																	
(531)																	
(4^21)																	
(52^2)																	
(432)																	
(3^3)																	
(61^3)																	
(521^3)																	−
(431^3)																− 1	−
(42^31)															− 1	− 2	−
(3^221)														− 1	− 2	− 5	−
(32^3)													− 1	− 3	− 7	− 12	−
(51^4)												− 1	≶	≶	≶	≶	−
(421^4)											− 1	− 4	≶	≶	− 3	− 7	−
(3^21^4)										− 1	− 2	− 6	≶	− 3	− 6	− 17	−
(32^21^2)									− 1	− 2	− 5	− 12	− 2	− 8	− 19	− 36	−
(2^41)								− 1	− 4	− 6	− 12	− 24	− 9	− 22	− 48	− 78	− 1
(41^6)							− 1	≶	≶	≶	− 5	− 21	≶	≶	− 10	− 20	−
(321^4)						− 1	− 3	≶	− 4	− 9	− 23	− 58	− 6	− 22	− 54	− 101	− 2
(2^31^4)					− 1	− 3	− 6	− 3	− 15	− 24	− 51	− 108	− 24	− 60	− 129	− 213	− 4
(31^6)				− 1	≶	− 6	− 19	≶	− 15	− 30	− 81	− 204	− 20	− 60	− 155	− 270	− 6
(2^21^2)			− 1	− 2	− 5	− 17	− 36	− 10	− 50	− 81	− 172	− 366	− 70	− 165	− 350	− 565	−12
(21^7)		−1	− 7	−15	−21	− 70	−147	− 35	−161	−252	− 525	−1092	−210	− 455	− 945	−1470	−30
(1^9)	−1	−9	−36	−72	−84	−252	−504	−126	−504	−756	−1512	−3024	−630	−1260	−2520	−3780	−75

X(a).

b^{24} b^3ce	b^{14} b^5e	b^3 d^3	12^3 bcd^2	$1^3 3^2$ b^3d^2	$2^3 3$ c^3d	$1^2 2^2 3$ b^2c^2d	$1^4 23$ b^4cd	$1^6 3$ b^6d	12^4 bc^4	$1^3 2^3$ b^3c^3	$1^4 2^2$ b^4c^2	$1^7 2$ b^7c	1^9 b^9
													— 1
												— 1	— 9
											— 1	— 7	— 36
										— 1	— 5	— 21	— 84
									— 1	— 3	— 10	— 35	— 126
								— 1	∽	∽	— 2	— 15	— 72
							— 1	— 6	∽	— 3	— 17	— 70	— 252
						— 1	— 4	— 15	— 4	— 15	— 50	— 161	— 504
					— 1	— 2	— 6	— 20	— 9	— 24	— 70	— 210	— 630
				— 1	— 3	— 3	— 8	— 22	— 6	— 24	— 81	— 252	— 756
			— 1	— 3	— 6	— 6	— 15	— 36	— 36	— 93	— 240	— 630	— 1680
	— 1	— 1	∽	∽	∽	∽	∽	— 3	— 19	∽	— 6	— 36	— 147	— 504
— 1	— 5	∽	∽	∽	— 2	∽	— 5	— 23	— 81	— 12	— 51	— 172	— 525	— 1512
— 3	— 10	∽	— 2	— 6	— 7	— 19	— 54	— 155	— 48	— 129	— 350	— 945	— 2520
— 12	— 30	— 3	— 13	— 30	— 27	— 65	— 158	— 390	— 136	— 333	— 820	— 2030	— 5040
— 27	— 60	— 6	— 27	— 64	— 51	— 120	— 282	— 660	— 234	— 555	— 1320	— 3150	— 7560
— 4	— 21	∽	∽	— 6	∽	— 12	— 58	— 204	— 24	— 108	— 366	— 1092	— 3024
— 25	— 75	∽	— 12	— 42	— 27	— 85	— 241	— 645	— 168	— 459	— 1200	— 3045	— 7560
— 42	— 110	— 6	— 30	— 72	— 64	— 152	— 372	— 920	— 300	— 720	— 1740	— 4200	— 10080
— 85	— 200	— 15	— 65	— 152	— 120	— 281	— 656	— 1530	— 516	— 1203	— 2800	— 6510	— 15120
— 168	— 360	— 36	— 136	— 300	— 234	— 516	— 1140	— 2520	— 906	— 2016	— 4500	— 10080	— 22680
— 75	— 225	∽	— 30	— 110	— 60	— 200	— 570	— 1500	— 360	— 990	— 2550	— 6300	— 15120
— 241	— 570	— 36	— 158	— 372	— 282	— 656	— 1516	— 3480	— 1140	— 2610	— 5940	— 13440	— 30240
— 459	— 990	— 93	— 333	— 720	— 555	— 1203	— 2610	— 5670	— 2016	— 4383	— 9540	— 20790	— 45360
— 645	— 1500	— 90	— 390	— 920	— 660	— 1530	— 3480	— 7800	— 2520	— 5670	— 12600	— 27720	— 60480
—1200	— 2550	— 240	— 820	— 1740	—1320	— 2800	— 5940	—12600	— 4500	— 9540	—20220	— 42840	— 90720
—3045	— 6300	— 630	—2030	— 4200	—3150	— 6510	—13440	—27720	—10080	—20790	—42840	— 88200	—181440
—7560	—15120	—1680	—5040	—10080	—7560	—15120	—30240	—60480	—22680	—45360	—90720	—181440	—362880

IX(b).

	9 j	18 bi	27 ch	1^{7} $b^{3}h$	35 dy	125 bcy	$1^{6}5$ $b^{2}y$	45 ef	135 bif	$2^{2}5$ $c^{2}f$	$1^{3}25$ $b^{2}cf$	$1^{5}5$ $b^{2}f$	14^{2} bc^{2}	234 cde	$1^{2}34$ $b^{2}de$	1 b
(9)	− 9	+ 9	+ 9	− 9	+ 9	−18	+9	+ 9	−18	−9	+27	−9	− 9	−18	+27	+
(81)	+ 9	− 1	− 9	+ 1	− 9	+10	−1	− 9	+10	+9	−11	+1	+ 5	+18	−11	−
(72)	+ 9	− 9	+ 5	+ 2	− 9	+ 4	−2	− 9	+18	−5	− 6	+2	+ 9	+ 4	−20	+
(63)	+ 9	− 9	− 9	+ 9	+ 9	∽	−3	− 9	∽	+9	− 9	+3	+ 9	∽	− 9	−
(54)	+ 9	− 9	− 9	+ 9	− 9	+18	−9	+11	− 2	−1	− 7	+4	−11	− 2	+13	+
(71^{2})	− 9	+ 1	+ 2	− 1	+ 9	− 3	+1	+ 9	−10	−2	+ 4	−1	− 5	−11	+11	+
(621)	−18	+10	+ 4	− 3	∽	− 8	+3	+18	−10	−4	+11	−3	−14	− 4	+13	+
(531)	−18	+10	+18	−10	∽	−10	+4	− 2	− 5	−8	+15	−4	+ 6	+ 2	− 5	−
$(4^{1}1)$	− 9	+ 5	+ 9	− 5	+ 9	−14	+5	−11	+ 6	+1	− 1	∽	+ 5	+ 2	− 5	+
(52^{2})	− 9	+ 9	− 5	− 2	+ 9	− 4	+2	− 1	− 8	∽	+ 6	−2	+ 1	+ 6	∽	−
(432)	−18	+18	+ 4	−11	∽	− 4	+5	− 2	+ 2	+6	− 5	∽	+ 2	− 8	+ 1	+
(3^{3})	− 3	+ 3	+ 3	− 3	− 6	+ 3	∽	+ 3	+ 3	−3	∽	∽	− 3	+ 3	∽	
(61^{3})	+ 9	− 1	− 2	+ 1	− 3	+ 3	−1	− 9	+ 4	+2	− 4	+1	+ 5	+ 5	− 5	−
(521^{2})	+27	−11	− 6	+ 4	− 9	+11	−4	− 7	+15	+6	−15	+4	− 1	− 5	+ 1	+
(431^{2})	+27	−11	−20	+11	− 9	+13	−5	+13	− 5	∽	+ 1	∽	− 5	+ 1	+ 2	−
$(42^{2}1)$	+27	−19	+ 1	+ 5	− 9	+12	−5	+ 3	− 2	−6	+ 3	∽	+ 1	+ 2	− 1	
$(3^{2}21)$	+27	−19	−13	+12	+18	− 7	∽	− 7	− 4	+3	∽	∽	+ 3	− 1		
(32^{3})	+ 9	− 9	+ 5	+ 2	− 3	− 2	∽	+ 1	+ 2	∽	∽	∽	− 1			
(51^{4})	− 9	+ 1	+ 2	− 1	+ 3	− 3	+1	+ 4	− 4	−2	+ 4	−1				
(421^{3})	−36	+12	+ 8	− 5	+12	−14	+5	− 4	+ 1	+2	− 1					
$(3^{2}1^{3})$	−18	+ 6	+11	− 6	− 3	+ 1	∽	− 2	+ 2	−1						
$(32^{2}1^{2})$	−54	+30	+ 5	− 9	− 9	+ 4	∽	+ 4	− 1							
$(2^{3}1)$	− 9	+ 7	− 5	∽	+ 3	∽	∽	− 1								
(41^{5})	+ 9	− 1	− 2	+ 1	− 3	+ 3	−1									
(321^{4})	+45	−13	−10	+ 6	+ 3	− 1										
$(2^{2}1^{5})$	+30	−14	+ 5	∽	− 1											
(31^{6})	− 9	+ 1	+ 2	− 1												
$(2^{1}1^{5})$	−27	+ 7	− 1													
(21^{7})	+ 9	− 1														
(1^{9})	− 1															

$1^2 2^4$ bc^2e	$1^3 2^4$ b^2ce	$1^4 4$ b^2e	3^3 d^3	$1^2 3^3$ bcd^3	$1^3 3^3$ b^2d^3	$2^3 3$ c^3d	$1^2 2^3 3$ b^2c^3d	$1^2 3^2 3$ b^3cd	$1^4 3$ b^3d	$1^2 4^2$ be^2	$1^3 2^2$ b^2c^2	$1^4 2^2$ b^3c^2	$1^3 2$ b^2c	1^4 b^3
+27	−36	+9	−3	+27	−18	+9	−54	+45	−9	−9	+30	−27	+9	−1
−19	+12	−1	+3	−19	+ 6	−9	+30	−13	+1	+7	−14	+ 7	−1	
+ 1	+ 8	−2	+3	−13	+11	+5	+ 5	−10	+2	−5	+ 5	− 1		
− 9	+12	−3	−6	+18	− 3	−3	− 9	+ 3	∽	+3	− 1			
+ 3	− 4	∽	+3	− 7	− 2	+1	+ 4	∽	∽	−1				
+ 5	− 5	+1	−3	+12	− 6	+2	− 9	+ 6	−1					
+12	−14	+3	+3	− 7	+ 1	−2	+ 4	− 1						
− 2	+ 1	∽	+3	− 4	+ 2	+2	− 1							
+ 1	∽	∽	−3	+ 3	∽	−1								
− 6	+ 2	∽	−3	+ 3	− 1									
+ 2	∽	∽	+3	− 1										
∽	∽	∽	−1											
− 5	+ 5	−1												
+ 3	− 1													
− 1														

X(a).

||	10 k	19 hj	28 ci	1^28 b^2i	37 dh	127 bch	1^27 b^2h	46 eg	136 bdg	2^26 c^2g	1^226 b^2cg	1^46 b^4g	5^3 f^3	145 bef	235 cdf	1^235 b^2df	12^25 bc^2f	1^325 b^3cf	1^45 b^4f	24^3 ce^2	1^242 b^2e^2	3^24 d^2e	1234 $bcde$	1^234 b^2de
(10)																								
(91)																								
(82)																								
(73)																								
(64)																								
(5²)																								
(81²)																								
(721)																								
(631)																								
(541)																								
(62²)																								
(532)																								
(4²2)																								
(43²)																								
(71³)																								
(621³)																								
(531³)																								
(4³1³)																								
(52³1)																								+ 1
(4321)																							+ 1	+ 3
(3³1)																						+ 1	+ 3	+ 6
(42²)																					+ 1	↬	+ 3	+ 10
(3²2²)																				+ 1	+ 2	+ 2	+ 8	+ 18
(61⁴)																			+ 1	↬	↬	↬	↬	↬
(521³)																		+ 1	+ 5	↬	↬	↬	↬	+ 3
(431³)																	+ 1	+ 3	+ 10	↬	↬	↬	+ 3	+ 9
(42³1²)																+ 1	+ 2	+ 7	+ 20	↬	+ 2	↬	+ 8	+ 27
(3²21²)															+ 1	+ 2	+ 5	+ 12	+ 30	+ 2	+ 4	+ 5	+ 21	+ 48
(32³1)														+ 1	+ 3	+ 7	+ 12	+ 27	+ 60	+ 7	+ 16	+ 12	+ 49	+ 112
(2⁵)													+ 1	+ 5	+ 10	+ 20	+ 30	+ 60	+ 120	+ 20	+ 45	+ 30	+ 110	+ 240
(51⁴)												+ 1	↬	↬	↬	↬	↬	+ 5	+ 26	↬	↬	↬	↬	+ 10
(421⁴)											+ 1	+ 4	↬	↬	↬	+ 4	+ 9	+ 32	+ 95	↬	+ 6	↬	+ 22	+ 76
(3²1⁴)										+ 1	+ 2	+ 6	↬	↬	+ 4	+ 8	+ 22	+ 54	+ 140	+ 6	+ 12	+ 12	+ 56	+ 132
(32²1³)									+ 1	+ 2	+ 5	+ 12	↬	+ 3	+ 11	+ 26	+ 48	+ 112	+ 260	+ 18	+ 42	+ 31	+ 128	+ 294
(2²1⁵)								+ 1	+ 4	+ 6	+ 12	+ 24	+ 2	+ 14	+ 32	+ 68	+ 108	+ 228	+ 480	+ 53	+ 114	+ 80	+ 284	+ 612
(41⁵)							+ 1	↬	↬	↬	+ 6	+ 25	↬	↬	↬	+ 15	+ 30	+ 111	+ 330	↬	+ 20	↬	+ 60	+ 215
(321⁴)						+ 1	+ 3	↬	+ 5	+ 11	+ 28	+ 70	↬	+ 10	+ 35	+ 85	+ 156	+ 368	+ 860	+ 50	+ 120	+ 80	+ 335	+ 775
(3³1³)					+ 1	+ 3	+ 6	+ 4	+ 19	+ 30	+ 63	+ 132	+ 6	+ 42	+ 99	+ 210	+ 339	+ 720	+ 1530	+ 144	+ 306	+ 213	+ 735	+ 1566
(31⁷)				+ 1	↬	+ 7	+ 22	↬	+ 21	+ 42	+ 112	+ 280	↬	+ 35	+ 105	+ 266	+ 462	+ 1092	+ 2520	+ 140	+ 350	+ 210	+ 875	+ 2030
(2⁴1⁴)			+ 1	+ 2	+ 6	+ 20	+ 42	+ 15	+ 72	+ 115	+ 242	+ 510	+ 20	+ 130	+ 296	+ 622	+ 990	+ 2082	+ 4380	+ 400	+ 840	+ 570	+ 1900	+ 3990
(21⁶)		+ 1	+ 8	+17	+ 28	+ 92	+192	+ 56	+252	+ 392	+ 812	+1680	+ 70	+ 406	+ 868	+1792	+2772	+ 5712	+11760	+1120	+2310	+1540	+ 4900	+10080
(1²⁸)	+1	+10	+45	+90	+120	+360	+720	+210	+840	+1260	+2520	+5040	+252	+1260	+2520	+5040	+7560	+15120	+30240	+3150	+6300	+4200	+12600	+25200

	2^24 c^2e	1^22^24 b^2c^2e	1^24 b^2ce	1^4 b^4e	13^3 bd^3	2^23^3 c^2d^3	1^23^3 b^2cd^3	1^43^3 b^4d^3	12^23 bc^2d	1^22^23 b^2c^2d	1^23 b^2cd	1^33 b^3d	2^5 c^5	2^23^2 b^2c^4	1^23^2 b^2c^3	1^22^2 b^2c^2	1^2 b^2c	1^{10} b^{10}
																		+ 1
																	+ 1	+ 10
																+ 1	+ 8	+ 45
															+ 1	+ 6	+ 28	+ 120
														+ 1	+ 4	+ 15	+ 56	+ 210
													+ 1	+ 2	+ 6	+ 20	+ 70	+ 252
												+ 1	∽	∽	∽	+ 2	+ 17	+ 90
											+ 1	+ 7	∽	∽	+ 3	+ 20	+ 92	+ 360
										+ 1	+ 5	+ 21	∽	+ 4	+ 19	+ 72	+ 252	+ 840
									+ 1	+ 3	+ 10	+ 35	+ 5	+ 14	+ 42	+ 130	+ 406	+ 1260
								+ 1	∽	+ 2	+ 11	+ 42	∽	+ 6	+ 30	+ 115	+ 392	+ 1260
							+ 1	+ 4	+ 3	+ 11	+ 35	+ 105	+ 10	+ 32	+ 99	+ 296	+ 868	+ 2520
						+ 1	+ 2	+ 6	+ 7	+ 18	+ 50	+ 140	+ 20	+ 53	+ 144	+ 400	+ 1120	+ 3150
					+ 1	+ 2	+ 5	+ 12	+ 12	+ 31	+ 80	+ 210	+ 30	+ 80	+ 213	+ 570	+ 1540	+ 4200
				+ 1	∽	∽	∽	∽	∽	∽	+ 3	+ 22	∽	∽	+ 6	+ 42	+ 192	+ 720
			+ 1	+ 6	∽	∽	∽	∽	∽	+ 5	+ 28	+ 112	∽	+ 12	+ 63	+ 242	+ 812	+ 2520
		+ 1	+ 4	+ 15	∽	∽	+ 2	+ 8	+ 7	+ 26	+ 85	+ 266	+ 20	+ 68	+ 210	+ 622	+ 1792	+ 5040
	+ 1	+ 2	+ 6	+ 20	∽	+ 2	+ 4	+ 12	+ 16	+ 42	+ 120	+ 350	+ 45	+ 114	+ 306	+ 840	+ 2310	+ 6300
1	∽	+ 2	+ 9	+ 30	∽	∽	+ 5	+ 22	+ 12	+ 48	+ 156	+ 462	+ 30	+ 108	+ 339	+ 990	+ 2772	+ 7560
3	+ 3	+ 8	+ 22	+ 60	+ 3	+ 8	+ 21	+ 56	+ 49	+ 128	+ 335	+ 875	+ 110	+ 284	+ 735	+ 1900	+ 4900	+ 12600
6	+ 6	+ 15	+ 36	+ 90	+ 10	+ 18	+ 42	+ 96	+ 87	+ 210	+ 510	+ 1260	+ 180	+ 444	+ 1092	+ 2700	+ 6720	+ 16800
10	+ 6	+ 18	+ 48	+ 120	+ 6	+ 15	+ 42	+ 115	+ 87	+ 228	+ 585	+ 1470	+ 180	+ 468	+ 1194	+ 3015	+ 7560	+ 18900
18	+ 15	+ 34	+ 78	+ 180	+ 18	+ 34	+ 80	+ 188	+ 156	+ 370	+ 880	+ 2100	+ 310	+ 740	+ 1776	+ 4280	+ 10360	+ 25200
	∽	∽	+ 4	+ 25	∽	∽	∽	+ 6	∽	+ 12	+ 70	+ 280	∽	+ 24	+ 132	+ 510	+ 1680	+ 5040
3	∽	+ 7	+ 32	+ 111	∽	∽	+ 12	+ 54	+ 27	+ 112	+ 368	+ 1092	+ 60	+ 228	+ 720	+ 2082	+ 5712	+ 15120
9	+ 10	+ 27	+ 76	+ 215	+ 6	+ 18	+ 48	+ 132	+ 112	+ 294	+ 775	+ 2030	+ 240	+ 612	+ 1566	+ 3990	+ 10080	+ 25200
27	+ 18	+ 54	+ 149	+ 390	+ 15	+ 34	+ 99	+ 270	+ 198	+ 523	+ 1340	+ 3360	+ 390	+ 1008	+ 2547	+ 6330	+ 15540	+ 37800
48	+ 42	+ 99	+ 236	+ 570	+ 42	+ 80	+ 186	+ 436	+ 358	+ 844	+ 2000	+ 4760	+ 680	+ 1604	+ 3792	+ 8980	+ 21280	+ 50400
12	+ 87	+ 198	+ 450	+ 1020	+ 87	+ 156	+ 358	+ 820	+ 645	+ 1479	+ 3390	+ 7770	+ 1170	+ 2688	+ 6180	+ 14220	+ 32760	+ 75600
40	+ 180	+ 390	+ 840	+ 1800	+ 180	+ 310	+ 680	+ 1560	+ 1170	+ 2580	+ 5700	+ 12600	+ 2040	+ 4530	+ 10080	+ 22500	+ 50400	+ 113400
10	∽	+ 20	+ 95	+ 330	∽	∽	+ 30	+ 140	+ 60	+ 260	+ 860	+ 2520	+ 120	+ 480	+ 1530	+ 4380	+ 11760	+ 30240
76	+ 48	+ 149	+ 416	+ 1095	+ 36	+ 78	+ 236	+ 650	+ 450	+ 1196	+ 3050	+ 7560	+ 840	+ 2172	+ 5436	+ 13290	+ 31920	+ 75600
32	+ 115	+ 270	+ 650	+ 1580	+ 96	+ 188	+ 436	+ 1032	+ 820	+ 1920	+ 4520	+ 10640	+ 1500	+ 3480	+ 8100	+ 18840	+ 43680	+ 100800
94	+ 228	+ 523	+ 1196	+ 2730	+ 210	+ 370	+ 844	+ 1920	+ 1479	+ 3358	+ 7610	+ 17220	+ 2580	+ 5844	+ 13212	+ 29820	+ 67200	+ 151200
12	+ 468	+ 1008	+ 2172	+ 4680	+ 444	+ 740	+ 1604	+ 3480	+ 2688	+ 5844	+ 12720	+ 27720	+ 4530	+ 9876	+ 21564	+ 47160	+ 103320	+ 226800
15	+ 120	+ 390	+ 1095	+ 2850	+ 90	+ 180	+ 570	+ 1580	+ 1020	+ 2730	+ 6900	+ 16800	+ 1800	+ 4680	+ 11610	+ 27900	+ 65520	+ 151200
75	+ 585	+ 1340	+ 3050	+ 6900	+ 510	+ 880	+ 2000	+ 4520	+ 3390	+ 7610	+ 17000	+ 37800	+ 5700	+ 12720	+ 28260	+ 62520	+ 137760	+ 302400
56	+ 1194	+ 2547	+ 5436	+ 11610	+ 1092	+ 1776	+ 3792	+ 8100	+ 6180	+ 13212	+ 28260	+ 60480	+ 10080	+ 21564	+ 46152	+ 98820	+ 211680	+ 453600
30	+ 1470	+ 3360	+ 7560	+ 16800	+ 1260	+ 2100	+ 4760	+ 10640	+ 7770	+ 17220	+ 37800	+ 82320	+ 12600	+ 27720	+ 60480	+ 131040	+ 282240	+ 604800
90	+ 3015	+ 6330	+ 13290	+ 27900	+ 2700	+ 4280	+ 8980	+ 18840	+ 14220	+ 29820	+ 62520	+ 131040	+ 22500	+ 47160	+ 98820	+ 207000	+ 433440	+ 907200
80	+ 7560	+ 15540	+ 31920	+ 65520	+ 6720	+ 10360	+ 21280	+ 43680	+ 32760	+ 67200	+ 137760	+ 282240	+ 50400	+ 103320	+ 211680	+ 433440	+ 887040	+ 1814400
00	+ 18900	+ 37800	+ 75600	+ 151200	+ 16800	+ 25200	+ 50400	+ 100800	+ 75600	+ 151200	+ 302400	+ 604800	+ 113400	+ 226800	+ 453600	+ 907200	+ 1814400	+ 3628800

=	10 k	19 bj	28 ci	$1^{3}8$ $b^{2}i$	37 dh	127 bch	$1^{3}7$ $b^{3}h$	46 eg	136 bdg	$2^{2}6$ $c^{2}g$	$1^{2}26$ $b^{2}cg$	$1^{4}6$ $b^{4}g$	5^{3} f^{2}	145 bef	235 cdf	$1^{2}35$ $b^{2}df$	$12^{2}5$ $bc^{2}f$	$1^{3}25$ $b^{3}cf$	$1^{5}5$ $b^{5}f$	24^{3} ce^{2}
(10)	− 10	+10	+10	−10	+10	−20	+10	+10	−20	−10	+30	−10	+ 5	−20	−20	+30	+30	−40	+10	−1
(91)	+ 10	− 1	−10	+ 1	−10	+11	− 1	−10	+11	+10	−12	+ 1	− 5	+11	+20	−12	−21	+13	− 1	+1
(82)	+ 10	−10	+ 6	+ 2	−10	+ 4	− 2	−10	+20	− 6	− 6	+ 2	− 5	+20	+ 4	−22	+ 2	+ 8	− 2	+
(73)	+ 10	−10	−10	+10	+11	− 1	− 3	−10	− 1	+10	− 9	+ 3	− 5	+20	− 1	− 9	− 9	+12	− 3	+1
(64)	+ 10	−10	−10	+10	−10	+20	−10	+14	− 4	− 2	− 6	+ 4	− 5	− 4	+20	− 6	−18	+16	− 4	−1
(5^{2})	+ 5	− 5	− 5	+ 5	− 5	+10	− 5	− 5	+10	+ 5	−15	+ 5	+10	−15	−15	+10	+10	− 5	≎	+
(81^{2})	− 10	+ 1	+ 2	− 1	+10	− 3	+ 1	+10	−11	− 2	+ 4	− 1	+ 5	−11	−12	+12	+ 5	− 5	+ 1	−
(721)	− 20	+11	+ 4	− 3	− 1	− 8	+ 3	+20	−10	− 4	+11	− 3	+10	−31	− 3	+13	+12	−14	+ 3	−1
(631)	− 20	+11	+20	−11	− 1	−10	+ 4	− 4	− 4	− 8	+15	− 4	+10	− 7	−19	+15	+18	−19	+ 4	+
(541)	− 20	+11	+20	−11	+20	−31	+11	− 4	− 7	− 8	+18	− 5	−15	+23	+10	−12	− 1	+ 1	≎	+
(62^{2})	− 10	+10	− 6	− 2	+10	− 4	+ 2	− 2	− 8	≎	+ 6	− 2	+ 5	− 8	− 4	+10	+ 4	− 8	+ 2	+1
(532)	− 20	+20	+ 4	−12	− 1	− 3	+ 5	+20	−19	− 4	+15	− 5	−15	+10	+17	− 4	−13	+ 5	≎	−1
$(4^{2}2)$	− 10	+10	+ 2	− 6	+10	−12	+ 6	−14	+ 4	+10	− 6	≎	+ 5	+ 4	−12	+ 2	+ 2	≎	≎	+
(43^{2})	− 10	+10	+10	−10	−11	+ 1	+ 3	− 2	+13	− 4	− 3	≎	+ 5	− 8	+ 1	− 3	+ 3	≎	≎	+
(71^{2})	+ 10	− 1	− 2	+ 1	− 3	+ 3	− 1	−10	+ 4	+ 2	− 4	+ 1	− 5	+11	+ 5	− 5	− 5	+ 5	− 1	+
(621^{2})	+ 30	−12	− 6	+ 4	− 9	+11	− 4	− 6	+15	+ 6	−15	+ 4	−15	+18	+15	−19	−17	+19	− 4	−
(531^{2})	+ 30	−12	−22	+12	− 9	+13	− 5	− 6	+15	+10	−19	+ 5	+10	−12	− 4	+ 3	+ 2	− 1	≎	+
$(4^{2}1^{2})$	+ 15	− 6	−11	+ 6	−15	+17	− 6	+ 9	− 3	− 1	+ 1	≎	+ 5	− 8	+ 1	+ 2	− 1	≎	≎	−
$(52^{2}1)$	+ 30	−21	+ 2	+ 5	− 9	+12	− 5	−18	+18	+ 4	−17	+ 5	+10	− 1	−13	+ 2	+ 9	− 3	≎	+
(4321)	+ 60	−42	−28	+26	+ 3	+21	−12	+12	−15	− 8	+ 7	≎	− 5	≎	+ 5	+ 4	− 3	≎	≎	+
$(3^{2}1)$	+ 10	− 7	−10	+ 7	+11	− 4	≎	+ 2	− 7	+ 4	≎	≎	− 5	+ 5	− 1	≎	≎	≎	≎	−
(42^{3})	+ 10	−10	+ 6	+ 2	−10	+ 4	− 2	+10	≎	− 4	+ 2	≎	− 5	≎	+ 4	− 2	≎	≎	≎	−
$(3^{2}2^{2})$	+ 15	−15	+ 1	+ 7	+ 6	− 7	≎	− 9	+ 3	+ 2	≎	≎	+ 5	− 1	− 2	≎	≎	≎	≎	+
(61^{4})	− 10	+ 1	+ 2	− 1	+ 3	− 3	+ 1	+ 4	− 4	− 2	+ 4	− 1	+ 5	− 5	− 5	+ 5	+ 5	− 5	+ 1	
(521^{3})	− 40	+13	+ 8	− 5	+12	−14	+ 5	+16	−19	− 8	+19	− 5	− 5	+ 1	+ 5	− 1	− 3	+ 1		
(431^{3})	− 40	+13	+24	−13	+12	−16	+ 6	− 8	+ 5	≎	− 1	≎	− 5	+ 5	− 1	− 2	+ 1			
$(42^{3}1^{2})$	− 60	+33	+ 4	− 9	+18	−23	+ 9	−12	+ 3	+ 8	− 4	≎	+ 5	− 1	− 2	+ 1				
$(3^{2}21^{3})$	− 60	+33	+28	−21	−24	+ 9	≎	≎	+ 6	− 4	≎	≎	+ 5	− 3	+ 1					
$(32^{2}1)$	− 40	+31	− 8	− 7	− 2	+ 5	≎	+ 8	− 3	≎	≎	≎	− 5	+ 1						
(2^{5})	− 2	+ 2	− 2	≎	+ 2	≎	≎	− 2	≎	≎	≎	≎	+ 1							
(51^{4})	+ 10	− 1	− 2	+ 1	− 3	+ 3	− 1	− 4	+ 4	+ 2	− 4	+1								
(421^{4})	+ 50	−14	−10	+ 6	−15	+17	− 6	+ 4	− 1	− 2	+ 1									
$(3^{2}1^{4})$	+ 25	− 7	−13	+ 7	+ 3	− 1	≎	+ 2	− 2	+ 1										
$(32^{2}1^{3})$	+100	−46	−12	+14	+12	− 5	≎	− 4	+ 1											
$(2^{4}1^{2})$	+ 25	−16	+ 9	≎	− 4	≎	≎	+ 1												
(41^{5})	− 10	+ 1	+ 2	− 1	+ 3	− 3	+ 1													
(321^{5})	− 60	+15	+12	− 7	− 3	+ 1														
$(2^{3}1^{4})$	− 50	+20	− 6	≎	+ 1															
(31^{7})	+ 10	− 1	− 2	+ 1																
$(2^{3}1^{6})$	+ 35	− 8	+ 1																	
(21^{8})	− 10	+ 1																		
(1^{10})	+ 1																			

X(*b*).

$2^4 3^1$ ce^3	$1^2 4^1$ b^2e^2	$3^2 4$ d^2e	1234 $bcde$	$1^2 34$ b^2de	$2^2 4$ c^2e	$1^2 2^2 4$ b^2c^2e	$1^2 24$ b^2ce	$1^2 4$ b^2e	13^2 bd^2	$2^2 3^2$ c^2d^2	$1^2 23^2$ b^2cd^2	$1^2 3^3$ b^2d^3	$12^2 3$ bc^2d	$1^2 2^2 3$ b^2c^2d	$1^2 23$ b^2cd	$1^2 3$ b^2d	2^3 c^3	$2^2 3^2$ b^2c^2	$1^2 2^3$ b^2c^3	$1^2 2^2$ b^2c^2	$1^2 2$ b^2c	1^{10} b^{10}
−10	+15	−10	+60	−40	+10	−60	+50	−10	+10	+15	−60	+25	−40	+100	−60	+10	− 2	+25	−50	+35	−10	+1
+10	− 6	+10	−42	+13	−10	+33	−14	+ 1	− 7	−15	+33	− 7	+31	− 46	+15	− 1	+ 2	−16	+20	− 8	+ 1	
+ 2	−11	+10	−28	+24	+ 6	+ 4	−10	+ 2	−10	+ 1	+28	−13	− 8	− 12	+12	− 2	− 2	+ 9	− 6	+ 1		
+10	−15	−11	+ 3	+12	−10	+18	−15	+ 3	+11	+ 6	−24	+ 3	− 2	+ 12	− 3	∽	+ 2	− 4	+ 1			
−14	+ 9	− 2	+12	− 8	+10	−12	+ 4	∽	+ 2	− 9	∽	+ 2	+ 8	− 4	∽	∽	− 2	+ 1				
+ 5	+ 5	+ 5	− 5	− 5	− 5	+ 5	∽	∽	− 5	+ 5	+ 5	∽	− 5	∽	∽	∽	+ 1					
− 6	+ 6	−10	+26	−13	+ 2	− 9	+ 6	− 1	+ 7	+ 7	−21	+ 7	− 7	+ 14	− 7	+ 1						
−12	+17	+ 1	+21	−16	+ 4	−23	+17	− 3	− 4	− 7	+ 9	− 1	+ 5	− 5	+ 1							
+ 4	− 3	+13	−15	+ 5	∽	+ 3	− 1	∽	− 7	+ 3	+ 6	− 2	− 3	+ 1								
+ 4	− 8	− 8	∽	+ 5	∽	− 1	∽	∽	+ 5	− 1	− 3	∽	+ 1									
+10	− 1	− 4	− 8	∽	− 4	+ 8	− 2	∽	+ 4	+ 2	− 4	+ 1										
−12	+ 1	+ 1	+ 5	− 1	+ 4	− 2	∽	∽	− 1	− 2	+ 1											
+ 2	− 3	+ 2	+ 4	∽	− 2	∽	∽	∽	− 2	+ 1												
+ 2	+ 3	− 1	− 3	∽	∽	∽	∽	∽	+ 1													
+ 6	− 6	+ 3	−12	+ 6	− 2	+ 9	− 6	+ 1														
− 6	+ 1	− 3	+ 7	− 1	+ 2	− 4	+ 1															
+ 2	+ 2	− 3	+ 4	− 2	− 2	+ 1																
− 3	+ 3	+ 3	− 3	∽	+ 1																	
+ 2	− 1	+ 3	− 3	+ 1																		
+ 4	− 3	− 3	+ 1																			
− 2	∽	+ 1																				
− 2	+ 1																					
+ 1																						

XXIV. *On Hourly Observations of the Magnetic Declination made by Captain* Rochfort Maguire, *R.N., and the Officers of H.M.S. 'Plover,' in* 1852, 1853 *and* 1854, *at Point Barrow, on the shores of the Polar Sea. By Major-General* Edward Sabine, *R.A., D.C.L., Treas. and Vice-President.*

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Amongst the measures adopted for the relief of Sir John Franklin's expedition, H.M.S. 'Plover,' commanded by Captain T. E. L. Moore, R.N., was despatched from England in 1848, and stationed until 1852 in Behring Strait, having on board supplies of provision for the 'Erebus' and 'Terror,' in the event of their succeeding in effecting a passage, or for their crews in case they should be obliged to abandon the ships and should attempt a retreat by the western route. The 'Plover' was annually communicated with and provisioned afresh by one of the vessels of the Pacific squadron. In 1852 Captain Moore was succeeded in the command of the 'Plover' by Captain Rochfort Maguire, R.N., and several other changes in the officers and seamen were made. It was also directed that the 'Plover' should take up a more advanced position than she had occupied in the preceding years. She was accordingly moved in September 1852 to a small harbour adjoining Point Barrow, the most northerly point of the American Continent between Behring Strait and Mackenzie River, in lat. 71° 21′ N., long. 156° 15′ W., where she was made secure for the winter by the middle of October, and where she remained until the summer of 1854. A more uninviting and apparently uninteresting situation for a two years' residence cannot well be imagined. It is thus described by Mr. Thomas A. Hull, the second master, in a letter which I received from him at the end of the first year, dated August 1853:—"Point Barrow might well be called the World's End; the coast for more than 100 miles both east and west of it has not a rise above 40 feet, and the Spit of Point Barrow is but 15 feet above the sea in the highest part, and in most places it is not above 5. The islands to the eastward are still lower, some of them being overflowed by a rise of 3 feet occasioned by a gale in December." Point Barrow seems, however, to have been admirably chosen for the objects for which the 'Plover' was employed. It would have been scarcely possible for boats or parties of men retreating along the coast from the eastward, to have passed without being seen; whilst by a judicious treatment of the native Esquimaux, a friendly communication was throughout maintained with them, and their good offices, in case they had been needed, secured along a considerable extent of coast. In this most dreary situation, in which the monotony of existence during the long period of a year and three quarters had but one relief,—that of an excursion to Port Clarence in Behring Strait, in the summer of 1853,

to communicate with the ship conveying the annual supplies,—Captain MAGUIRE and his officers found occupation and interest in the observations which form the subject of this Paper, and which, if I mistake not, deserve to be ranked amongst the most important contributions yet made to our knowledge of the magnetic variations of short period.

The instruments which Captain MAGUIRE possessed (supplied from Woolwich) were two dip circles, one of 9½ inches, the other of 6 inches, each with two needles of BARROW'S construction, and a portable declinometer, such as is described in Captain RIDDELL'S useful 'Manual,' pp. 15 and 16, having a perforated magnet of 3 inches in length, carrying a collimator scale in the interior, the divisions of which were read by a detached telescope. The observatory is thus described by Mr. HULL, whose zeal and assiduity in conducting the observations under Captain MAGUIRE'S direction, and accuracy in tabulating them, are spoken of by Captain MAGUIRE in terms of the highest praise:—"The observatory was composed of an outer house of ice, 12 feet square and 7 feet high, within which was another one of seal skin, 7 feet by 6. Two posts, 23 inches apart, were then sunk, and being firmly frozen into the earth, served as supports for the declinometer and reading telescope; whilst another post was placed in the north-western corner for the chronometer, and a pedestal was placed outside the observatory for the dip circle, which was afterwards removed into a separate house, about 15 feet north-west of the observatory. The dip was observed twice a week: we had the misfortune to break two dip needles through awkward handling, owing to the extreme cold. The hourly observations with the declinometer commenced on the 5th of November. We soon discovered that the aurora was connected with the movements of the magnet; the brighter the aurora, the quicker the magnetic changes became; and from repeated observations I came to the conclusion, that the appearance of the aurora in the *South* was connected with the motion of the magnet to the *east* of the magnetic North, and if in the *North*, to the *west* of the same. In addition to these disturbances, considerable irregularities took place in the daytime, generally in the forenoon, and always in cloudy and misty weather. During these disturbances it was no uncommon occurrence for the magnet to go out of the field of the telescope, not returning again for several minutes, and it was generally to the eastward on these occasions. A heavy gale from the S.W. set in on the 17th December, 1852, and occasioned the loss of five days' observations. It had the effect of raising the water 3 feet above the usual level, and at 3 P.M., on going to take the observations, the sea was found washing the wall of the observatory. The northern part of the Spit was at this time entirely free from ice, as was also the channel as far as we could see, which, owing to the heavy drift, could not be above 100 yards. The gale not having moderated, and the water being still on the rise, it was thought advisable to remove the instruments, which was fortunately accomplished without injury. At about 10 P.M. the gale moderated, the sky clearing and showing a most beautiful aurora, when we found that the ice had gone out of the channel. The next day was clear and fine, displaying to us the extraordinary view of open water, with only a few pieces of ice in the horizon. From that time not a week passed without a well-defined water-sky

being visible to seaward. On the 21st of December the instruments were replaced: the water had come within 6 inches of the top of the pedestal."

On the 1st of June, 1853, the observatory was unroofed, on account of the increasing heat of the weather, and on the 22nd of June the hourly observations were discontinued, all hands being required to assist in the preparations to meet the vessel at Port Clarence which brought the annual supply of provisions. The observations were again resumed on the 12th of October, 1853, with the same instruments, placed as in the preceding year; and were continued to the 30th of June, 1854, when the 'Plover' finally quitted Point Barrow on her return to England.

The hourly observations of the declination at Point Barrow consist therefore of two series: the first of eight months, from November 1852 to June 1853; and the second of nine months, from October 1853 to June 1854; making, together, seventeen months. The observations were arranged in monthly tables, of the same form as that adopted at the Colonial Observatories, having the monthly, daily and hourly means computed from the whole of the observations without any omissions on account of excessive disturbance. The record was kept in scale-divisions, the value of a scale-division being 2′·287. In this state the Tables were received at Woolwich.

For the purpose of separating the disturbances of largest amount from the rest of the observations, ten scale-divisions above or below the normal at the same hour in the same month were adopted as a suitable measure of what should be regarded as a large disturbance; the corresponding arc value is 22′·87. The number of disturbed observations thus separated was 907 in 1852–53, and 914 in 1853–54. The whole number of observations was 4659 in 1852–53, and 5413 in 1853–54: making, together, 10,072, of which 1821 differed from their respective normals by an amount equalling or exceeding 22′·87; being between one-fifth and one-sixth of the whole number. The aggregate amount of disturbance, counted from the respective normals, was 44156′·0 in 1852–53, and 47388′·2 in 1853–54; of which, in 1852–53, 24823′·0 was easterly, and 19333′·0 westerly disturbance; and in 1853–54, 31935′·1 easterly, and 15453′·1 westerly disturbance. The sum of the easterly values in the seventeen months was 56758′·1, and of the westerly values, 34786′·1. In both series, therefore, the easterly disturbances preponderated; in 1852–53 in the proportion of 1·28 to 1, and in 1853–54 of 2·07 to 1; in both, conjointly, in the proportion of 1·63 to 1. Consequently the influence of the disturbances at Point Barrow was to occasion a small mean deflection of the needle towards the east, thus slightly increasing the easterly declination.

In the following Table the aggregate values of the disturbances in the seventeen months are distributed into the *hours* of their occurrence; and the ratios are given which the values at the different hours bear to the mean of all the hours, or, in other words, to the aggregate values divided by 24.

TABLE I.

Aggregate values of the Disturbances distributed into the different hours of their occurrence.

Local astronomical hours.	Periods ending June 30.		Sums in the 17 months.	Ratios.	Local civil hours.
	1852–53 (8 months).	1853–54 (9 months).			
18	1974·4	2937·2	4911·6	1·29	6 A.M.
19	3046·7	4580·2	7626·9	2·00	7 A.M.
20	3329·8	4752·4	8082·2	2·12	8 A.M.
21	3564·7	4248·9	7813·6	2·05	9 A.M.
22	2934·4	2921·2	5855·6	1·53	10 A.M.
23	2105·9	1482·8	3588·7	0·94	11 A.M.
0	1328·4	924·3	2252·7	0·59	Noon.
1	1042·4	680·3	1722·7	0·44	1 P.M.
2	769·3	692·7	1462·0	0·38	2 P.M.
3	445·3	490·0	935·3	0·24	3 P.M.
4	643·5	207·9	851·4	0·22	4 P.M.
5	290·6	562·8	853·4	0·22	5 P.M.
6	330·2	414·6	744·8	0·19	6 P.M.
7	1111·8	272·4	1384·2	0·36	7 P.M.
8	1199·2	760·6	1959·8	0·51	8 P.M.
9	1740·5	1046·9	2787·4	0·73	9 P.M.
10	1500·3	1755·3	3255·6	0·85	10 P.M.
11	2682·4	2898·5	5580·9	1·46	11 P.M.
12	2537·6	3262·1	5799·7	1·52	Midnight.
13	2983·8	3037·6	6021·4	1·58	1 A.M.
14	2373·5	2782·5	5156·0	1·35	2 A.M.
15	1852·1	2001·9	3854·0	1·00	3 A.M.
16	1976·5	2114·1	4090·6	1·07	4 A.M.
17	2392·7	2561·0	4953·7	1·29	5 A.M.
		Total	91544·2		
		Mean $\frac{91544·2}{24}$ =	3814·3 =	1·00	

We find in this Table, as at Toronto and elsewhere, unmistakeable evidence of the existence of a law regulating the occurrence of the larger disturbances according to the hours of solar time. From 11 A.M. to 10 P.M. inclusive, the ratios are throughout less than unity; from 11 P.M. to 10 A.M. inclusive, above unity. There are two epochs of maximum and two of minimum disturbance in the twenty-four hours: the principal maximum is from 7 to 9 A.M., when the proportion reaches twice the average amount; the lesser maximum is from 11 P.M. to 1 A.M., when the proportion is about one and a half to one. The principal minimum is from 3 to 6 P.M., when the proportion is less than a quarter of the average; and there is a minor minimum at 3 and 4 A.M.

The accordance in systematic character of the variation at the different hours at Point Barrow with that of the corresponding variation at Toronto, may be seen by means of the following Table (II.), in which the ratios at Point Barrow are exhibited in comparison with those at the same hours of local time at Toronto. There is the same unmistakeable evidence at both stations that the amount of disturbance at the different hours

is regulated by a law which has reference to the hours of solar time: and there are similar double epochs of maxima and minima, although the order in which these occur, and the particular times of their occurrence, are not strictly the same; but the differences in these respects will be better discussed and more clearly appreciated when we pass from the consideration of the occurrence of the disturbances in the *aggregate* to that of their distribution into their *easterly and westerly constituents*.

TABLE II.

Ratios of the aggregate Disturbances at the different hours at Toronto and at Point Barrow.

Local astronomical hours.	Ratios.		Local civil hours.
	Point Barrow.	Toronto.	
18	1·29	1·05	6 A.M.
19	2·00	1·17	7 A.M.
20	2·12	1·27	8 A.M.
21	2·05	1·11	9 A.M.
22	1·53	0·87	10 A.M.
23	0·94	0·66	11 A.M.
0	0·59	0·49	Noon.
1	0·44	0·30	1 P.M.
2	0·38	0·40	2 P.M.
3	0·24	0·40	3 P.M.
4	0·22	0·53	4 P.M.
5	0·22	0·56	5 P.M.
6	0·19	0·84	6 P.M.
7	0·36	0·98	7 P.M.
8	0·51	1·22	8 P.M.
9	0·73	1·82	9 P.M.
10	0·85	1·55	10 P.M.
11	1·46	1·25	11 P.M.
12	1·52	1·35	Midnight.
13	1·58	1·52	1 A.M.
14	1·35	1·21	2 A.M.
15	1·00	1·13	3 A.M.
16	1·07	1·34	4 A.M.
17	1·29	1·05	5 A.M.

In Table III. we have the aggregate values of the disturbances at Point Barrow separated into their easterly and westerly constituents, with the ratios at each hour to the mean hourly easterly or westerly value, or to the sums of all the hours divided by 24.

TABLE III.

Local astronomical hours.	Disturbance-values.		Ratios.		Local civil hours.
	Easterly.	Westerly.	Easterly.	Westerly.	
18	3891·9	1019·7	1·65	0·70	6 A.M.
19	6679·5	947·4	2·82	0·65	7 A.M.
20	7610·2	472·0	3·22	0·33	8 A.M.
21	6818·1	995·5	2·88	0·68	9 A.M.
22	4563·6	1292·0	1·93	0·89	10 A.M.
23	2438·2	1150·5	1·03	0·79	11 A.M.
0	920·0	1332·7	0·40	0·92	Noon.
1	797·9	924·8	0·34	0·64	1 P.M.
2	646·9	815·0	0·27	0·56	2 P.M.
3	347·4	587·9	0·15	0·40	3 P.M.
4	341·4	510·0	0·14	0·35	4 P.M.
5	185·0	668·4	0·08	0·46	5 P.M.
6	207·4	537·4	0·09	0·36	6 P.M.
7	315·1	1069·1	0·13	0·74	7 P.M.
8	352·8	1607·0	0·15	1·11	8 P.M.
9	1128·3	1659·1	0·48	1·14	9 P.M.
10	1366·2	1895·5	0·57	1·30	10 P.M.
11	1931·2	3649·7	0·81	2·52	11 P.M.
12	2280·4	3519·3	0·96	2·43	Midnight.
13	2289·1	3732·3	0·96	2·57	1 A.M.
14	2540·4	2615·6	1·07	1·80	2 A.M.
15	2202·2	1651·8	0·93	1·14	3 A.M.
16	2892·3	1198·3	1·22	0·82	4 A.M.
17	4018·6	935·1	1·69	0·64	5 A.M.
Total in the 17 months	56758·1	34786·1			
Mean hourly values...	Easterly = 2364·9 = 1·00 Westerly = 1449·4 = 1·00				

On inspecting the ratios in this Table, it is obvious that the easterly and westerly disturbances have distinct laws in respect to the times of their occurrence. When examining the disturbances in the aggregate in Table II. (in which the easterly and westerly values are combined), we had occasion to observe, as one of the most notable features, the existence of *two* epochs of maximum in the twenty-four hours, one in the morning at 7, 8, and 9 A.M., and one in the evening, or more properly in the night, at 11 P.M., midnight, and 1 A.M.; the morning maximum being considerably the greater. We now perceive by means of the separation which is shown in Table III., that the morning maximum of the aggregate values is occasioned chiefly by the easterly disturbances, and their evening or night maximum chiefly by the westerly disturbances. The easterly values have not even a secondary maximum at 11 P.M. to 1 A.M.; nor have the westerly values a secondary maximum at 7, 8, and 9 A.M. Both the easterly and the westerly have their minimum nearly at the same hours, viz. from 3 to 6 P.M.; and if allowance be made for such small irregularities as we should be prepared to expect in the results of a series of only a few months' continuance, there would appear a strong probability that both easterly and westerly are single progressions, having their respective maxima

the easterly about 8 A.M. and the westerly about midnight. In the easterly values in particular the feature of a double progression, or of two maxima or two minima in the twenty-four hours, is almost wholly obliterated; whilst in the westerly, the distinctive character of the second maximum is softened, and the time of its occurrence removed to between 10 A.M. and noon. It may be remarked generally of the easterly disturbances, that they have in all respects their characteristic features more strongly marked than is the case in the westerly disturbances; thus, 8 A.M. is distinctly the hour of the easterly maximum, the values at 8 A.M. exceeding considerably those at 7 or 9 A.M.; and between 5 and 6 P.M. is as distinctly the time of minimum, whether we regard the particular values at those hours, or the low values from 3 P.M. to 8 P.M., between which hours 5½ P.M. is intermediate. The *range* of the easterly variation, or the proportion which the values at the hours of maximum bear to those at the hours of minimum, is also much greater in the easterly than in the westerly disturbances, being nearly as twenty-five to one in the easterly, and not more than about seven to one in the westerly. In the less distinctly marked features of the westerly disturbances, it may be well imagined that a greater continuance of observation may be required to give equally assured conclusions; and that the second minimum, which appears in the westerly at 8 A.M., may be rather an accidental than a persistent feature, and might disappear in a series of longer duration. Admitting this possibility, the westerly variation would in such case approximate even more nearly than it does in Table III. to a single progression in the twenty-four hours, having its maximum about midnight and its minimum about 4 or 5 P.M.: and thus the double progression shown by the aggregate values in Table II. would be resolved, as in some other cases, into two single progressions (easterly and westerly), having different hours of maximum and minimum.

The distinction which has been thus shown to exist between the phenomena of the easterly and westerly disturbances at Point Barrow, is a fitting preparation for the comparison which the next Table presents with the analogous phenomena at Toronto. For reasons, which will subsequently appear, the easterly ratios at Point Barrow have been placed in Table IV. by the side of the westerly ratios at Toronto, and the westerly ratios at Point Barrow by the side of the easterly at Toronto.

Table IV.

Exhibiting the ratios at the different hours, at Point Barrow and Toronto, of the values of the easterly and westerly portions of the larger disturbances to their respective mean hourly values.

Local astronomical hours.	Point Barrow, easterly disturbances.	Toronto, westerly disturbances.	Point Barrow, westerly disturbances.	Toronto, easterly disturbances.	Local civil hours.
18	1·65	1·82	0·70	0·45	6 A.M.
19	2·82	2·23	0·65	0·35	7 A.M.
20	3·22	2·58	0·33	0·26	8 A.M.
21	3·88	2·25	0·68	0·21	9 A.M.
22	1·93	1·62	0·89	0·28	10 A.M.
23	1·03	1·00	0·79	0·39	11 A.M.
0	0·40	0·80	0·92	0·24	Noon.
1	0·34	0·41	0·64	0·21	1 P.M.
2	0·27	0·65	0·56	0·20	2 P.M.
3	0·15	0·62	0·40	0·22	3 P.M.
4	0·14	0·80	0·35	0·32	4 P.M.
5	0·08	0·71	0·46	0·44	5 P.M.
6	0·09	0·57	0·36	1·05	6 P.M.
7	0·13	0·39	0·74	1·44	7 P.M.
8	0·15	0·28	1·11	1·95	8 P.M.
9	0·48	0·22	1·14	3·09	9 P.M.
10	0·57	0·45	1·30	2·41	10 P.M.
11	0·81	0·27	2·52	2·02	11 P.M.
12	0·96	0·82	2·43	1·76	Midnight.
13	0·96	1·19	2·57	1·79	1 A.M.
14	1·07	1·00	1·80	1·37	2 A.M.
15	0·93	0·94	1·14	1·28	3 A.M.
16	1·22	1·21	0·82	1·45	4 A.M.
17	1·69	1·23	0·64	0·91	5 A.M.

It is obvious on the mere inspection of Table IV., that the principal analogy is not between the disturbances which deflect the needle in the *same* direction at the two stations, but between those which deflect the needle in *opposite* directions; between the easterly disturbances at Point Barrow and the westerly at Toronto, and between the westerly disturbances at Point Barrow and the easterly at Toronto. Nor is this analogy manifested in slight and unimportant particulars, but on the contrary in the most marked characteristics of both classes of phenomena. Thus, when the average disturbance in the twenty-four hours is taken as the unit, the easterly disturbances at Point Barrow and westerly at Toronto are seen to have the same characteristic distribution, viz. the amount of disturbance at the several hours from noon to about midnight is below unity, and at the hours from a little after midnight to just before noon above unity; whilst on the other hand, the westerly disturbances at Point Barrow and easterly at Toronto accord with each other in presenting a very different distribution, viz. their amount exceeds the average at the four or five hours on either side of midnight, and is below unity at all the other hours of the day and night. The analogy thus manifested in the general distribution of the disturbances is equally strongly marked when particular points are regarded; as, for example, it has been shown, in the earlier part of this communication,

that the disturbances both at Point Barrow and at Toronto, when viewed in the *aggregate*, have at each station two very marked and decided maxima in the twenty-four hours, one about 8 A.M., and the other towards midnight; and that when the disturbances at Point Barrow are separated into their easterly and westerly constituents, the night maximum disappears in the easterly and the morning maximum in the westerly values, whilst the morning maximum of the aggregate values is found to consist almost wholly of deflections towards the east, and the night maximum almost wholly of deflections towards the west. Now we may perceive by Table IV., that when the separation is made into easterly and westerly values, and the ratios of each class at the several hours to the respective units are computed, the phenomena at Toronto present an almost precisely corresponding arrangement, but with this essential difference, that the morning maximum of the aggregate values is due at Point Barrow to the great preponderance of easterly values, and at Toronto to the great preponderance of westerly values; and that the evening maximum of the aggregate values is, conversely, due to the great preponderance of westerly values at Point Barrow and of easterly at Toronto.

The hours of *least* disturbance show also a much greater accordance when the easterly ratios at Point Barrow and westerly at Toronto are viewed together, than when any other comparison is made. The amount of disturbance diminishes as soon as the hour of noon is past, and continues comparatively small until much later in the evening than is the case in either the westerly disturbances at Point Barrow or the easterly at Toronto.

Whilst thus endeavouring to bring into view the several circumstances which seem to indicate a connexion subsisting between the disturbances of opposite character at Point Barrow and Toronto, it is well that we should also notice one or two points in which, on the contrary, the easterly disturbances appear to agree with the easterly, and the westerly with the westerly better than either with the opposite. Thus,—1st, the values of greatest comparative magnitude, viz. those in which the proportion exceeds three times the average value, are easterly deflections at both stations, occurring however several hours apart, viz. at 8 A.M. at Point Barrow, and at 9 P.M. at Toronto. It is a consequence of this preponderance of easterly deflection at the two stations, (in the morning at the one station and in the evening at the other,) that the mean influence of the disturbances on the general direction of the needle is the same at both stations, viz. a small increase of east declination at Point Barrow, and a still smaller diminution of west declination at Toronto. 2nd, the highest values of the westerly disturbances at both stations show nearly an equal proportion to their respective units, viz. about $2\frac{1}{2}$ times the mean hourly disturbance-value. The times of occurrence are also in this case very different, viz. at Toronto in the forenoon and at Point Barrow about midnight. At both stations the highest hourly values of the westerly deflections bear about an equal proportion in respect of magnitude to the highest easterly values.

It would be obviously premature to attempt to connect the phenomena thus brought into view, and into comparison with each other, with such physical relations as might be imagined suitable to produce or to explain their accordances and differences. The points

on the earth's surface where the periodical laws of the occasional disturbances have hitherto been examined, are too few to furnish a basis for prudent generalization. The *primary* dependence of the disturbances upon the sun seems to be established by the universality with which, wherever they have been examined, their mean effects have exhibited a variation regulated by the hours of solar time; as well as by the evidence which has been uniformly furnished, wherever the investigation has been made, of their participation in the same solar period which appears to govern the increase and decrease in the frequency and amount of the solar spots. Perhaps the next most important questions which present themselves for solution concern the *mode of operation* of the primary or exciting cause; whether, for instance, the phenomena which we observe are the results of the sun's influence, acting independently of the magnetism of the earth,—or of a reaction, in which the specialities of the latter will require to be taken into the account; or whether, as may possibly be the case, the action of the primary cause is modified in different localities by physical considerations of other kinds, such as particular conditions of the earth or of its atmosphere. It may be possible that a further knowledge of the phenomena at a very few additional stations, supposing them to be judiciously distributed, might be sufficient for the solution of these or similar questions; particularly if the observations were to include variations of the magnetic *force* as well as of the magnetic *direction*. The instrumental means and the processes of observation are sufficiently simple. Instruments similar to those which have been so usefully, and so honourably to themselves, employed by Captain MAGUIRE and his officers, have been sent with nearly all the expeditions which, in the last twelve years, have wintered within the Arctic circle; but the maintenance of a routine of hourly observation during several months of compulsory detention, in the absence of authoritative direction or professional encouragement, requires perhaps a greater amount of private zeal and devotion than can be expected, unless in such exceptional cases as the one which has supplied the materials for this communication. In one of these expeditions in particular (the only one that unhappily has not returned in safety), the well-known zeal of its commander Sir JOHN FRANKLIN in the cause of science, and the anxiety of his officers to cooperate with him in every useful and honourable work, gave reason for hopes of the highest promise. In the letters written by the commanders of the 'Erebus' and 'Terror' after their arrival at Baffin's Bay, the full purpose of establishing their magnetic observatories on shore or on the ice, wherever the ships should be detained during a winter, is prominently dwelt upon. They were provided with instruments for the variation of the force as well as of the direction, prepared at Woolwich under the superintendence of my then assistant, Captain RIDDELL. That observations *were* made with these instruments at the first winter station, in 1845–1846, no reasonable doubt can be entertained; or that the observations were repeated at one at least of the subsequent occasions of similar detention in a new locality. The records of these observations, too voluminous to have been brought away by parties setting forth with the prospect of having to make their way across the continent to the Hudson's Bay Stations, could scarcely fail to possess a high scientific

value; especially those of the second year, when the situation of the ships was probably intermediate between the meridians of Toronto and Point Barrow. In the Royal Society at least the hope is still cherished that the recovery of these records, amongst the other journals and papers of the 'Erebus' and 'Terror,' may yet recompense the devotion and perseverance with which the research has been pursued. By the aid of the Royal Society, the distinguished officer who has recently quitted our shores for the Polar Sea in command of the Fox Yacht, has been supplied from the Kew Observatory with the necessary instruments for observations of both the magnetic direction and force, to be used at the station where the ship will remain whilst the sledge parties are absent. All that the most zealous and earnest purposes can accomplish may be confidently expected from Captain M'Clintock and his associates in this honourable enterprise; but the number of persons who can be considered as competent to take part in scientific observations is unfortunately but barely sufficient for the conduct of the travelling parties for the primary object for which the enterprise was undertaken. Such opportunities of special scientific research are of rare occurrence; and it is greatly to be regretted that Mr. Gray, Mate in the Royal Navy, who having served under Captain Maguire in the 'Plover' had been well trained in magnetic observations, and was most desirous to have accompanied Captain M'Clintock, was unable to procure from the Admiralty leave of absence from his employment on the Coast Guard Service.

But, to return to the Point Barrow observations:—It is well known that apparent anomalies have been found in the diurnal variation of the declination in the high magnetic latitudes of the northern hemisphere, when compared with the regular and consistent diurnal march of the phenomena in other parts of the hemisphere. It is usually a principal feature in these anomalies, that the westerly extreme of the diurnal range, which most commonly elsewhere occurs between 1 and 2 P.M., is found to take place later in the afternoon, and even in some cases nearly as late as midnight. A leading step towards the explanation of these apparent anomalies was made, when it was shown that the diurnal variation derived from a mean of *all* the observations must in all instances be regarded as the joint effect of two distinct variations superimposed upon each other, proceeding from different causes and having different laws, viz. the occasional disturbances, and the regular solar variation. Hence it was natural to imagine that in passing from the parts of the hemisphere where the regular solar variation predominated, by reason of its greater amount, to other parts where the disturbance-variation should predominate, differences, of the nature of the so-called anomalies, would present themselves in the results which should comprehend their joint effects. Still, however, in attempting, with the very limited knowledge that was then possessed of the differences that might be supposed to take place in the disturbance-variation in different localities, to apply this mode of explanation to particular cases, doubts have been entertained of its sufficiency to meet the phenomena in all cases. The fact which has now been established by the comparison of the disturbances at Point Barrow and Toronto, that the prevailing disturbance-

deflection at certain hours of the day may be easterly at one station whilst it is westerly at another, augments the probability, that, whenever the effects of the two classes of phenomena are separated so that the approximate laws of each may be known, the apparent anomalies will disappear, and the solar variation will be found, after the disturbance-variation has been eliminated, to correspond in its times of easterly and westerly elongation with the order of the phenomena as generally observed in the same hemisphere: whilst on the other hand we may derive from the great dissimilarity in the laws of the disturbance-variation and those of the regular solar variation the salutary caution, that until the disturbance-variation at any particular station has become known by actual investigation, no *certain* inference can be drawn as to its direction or amount at different hours; or as to the modification of the turning-hours which its junction with the solar-diurnal variation may produce.

The subjoined Table (V.) exhibits in parallel columns the diurnal variation at Point Barrow as derived,—1, in column A, from the whole of the observations including the disturbances; it is here the joint effect of both classes of phenomena; and 2, in column B, from the observations when the disturbed observations have been separated and omitted. On comparing these two columns, it is seen that the time of westerly extreme, which, when the whole of the observations are taken into account, falls as late as 11 P.M., is restored in column B, which represents approximately the regular solar-diurnal variation, to the same hour at which it occurs at Toronto, viz. between 1 and 2 P.M. In column D of the same Table, the solar-diurnal variation at Toronto has been reproduced from the third volume of the 'Toronto Observations' (Table LXV. Column 4, p. lxxxviii), for the purpose of facilitating its comparison with the corresponding values in column B. In pursuing this comparison, we find that the characteristics of this variation at the two stations are as nearly the same as can reasonably be expected, considering that it is only the *larger* disturbances that have been separated, and that some minor effects of the same class must still remain in the body of the observations after the removal of the disturbances of greatest magnitude. Thus we find at both stations that the north end of the magnet is at its easterly extreme about 7 or 8 A.M.; that in returning from thence towards the west its motion is more rapid than at any other part of the twenty-four hours; and that it passes through its mean direction about 11 A.M., and reaches its westerly extreme a little after 1 P.M. Its subsequent motion, towards the east, is comparatively slower at Point Barrow than at Toronto, and is checked at both stations by a small retrogression towards the west, which is greatly diminished both in continuance and amount by the omission of the larger disturbances, marking thereby the source to which it is most probably due. The most notable difference which we find in the comparison of columns B and D is, as already noticed, that the return towards the east during the hours of the afternoon and night takes place comparatively more slowly at Point Barrow than at Toronto. At the latter station the magnet passes through its mean direction between 7 and 8 P.M., whereas at Point Barrow it does not do so until 1 or 2 A.M. Now this is precisely what should take place on the supposition that a portion of the easterly

disturbance-variation at Toronto, and of the westerly disturbance-variation at Point Barrow at those hours remains (as is undoubtedly the case) in the body of the observations after the separation of the *larger* disturbances. The prevailing disturbances are easterly at Toronto and westerly at Point Barrow from 6 to 7 P.M. until 3 to 4 A.M.; and the influence of that portion of them which is not separated must necessarily have the effect of augmenting the rapidity of the easterly motion at Toronto, and of retarding it at Point Barrow.

TABLE V.

Diurnal Variation of the Declination at Point Barrow, derived from the mean of seventeen months of hourly observation. Column A shows the results when the large disturbances are retained; Column B the results when the large disturbances are omitted; and Column C the differences, or the diurnal variation due to the large disturbances. Column D shows (for the purpose of comparison with B) the diurnal variation at Toronto omitting the large disturbances.

Local astronomical hours.	Point Barrow. Retaining the large disturbances.	Point Barrow. Omitting the large disturbances.	Point Barrow. Diurnal variation due to the large disturbances.	Toronto. Omitting the large disturbances.	Local civil hours.
	A.	B.	C.	D.	
18	19·3 E.	14·6 E.	4·7 E.	3·0 E.	6 A.M.
19	27·1 E.	15·2 E.	11·9 E.	4·0 E.	7 A.M.
20	27·0 E.	12·7 E.	14·3 E.	4·4 E.	8 A.M.
21	19·9 E.	8·2 E.	11·7 E.	3·6 E.	9 A.M.
22	9·3 E.	3·8 E.	5·5 E.	1·2 E.	10 A.M.
23	0·4 W.	1·4 W.	1·0 E.	1·7 W.	11 A.M.
0	8·2 W.	4·8 W.	3·4 W.	4·0 W.	Noon.
1	10·7 W.	8·2 W.	2·5 W.	5·1 W.	1 P.M.
2	9·8 W.	7·5 W.	2·3 W.	4·9 W.	2 P.M.
3	9·9 W.	7·2 W.	2·7 W.	3·8 W.	3 P.M.
4	9·8 W.	7·2 W.	2·6 W.	2·5 W.	4 P.M.
5	10·2 W.	7·0 W.	3·2 W.	1·3 W.	5 P.M.
6	9·7 W.	6·7 W.	3·0 W.	0·5 W.	6 P.M.
7	8·4 W.	4·4 W.	4·0 W.	0·1 W.	7 P.M.
8	9·0 W.	3·8 W.	5·2 W.	0·2 E.	8 P.M.
9	7·5 W.	3·9 W.	3·6 W.	0·5 E.	9 P.M.
10	7·9 W.	4·4 W.	3·5 W.	0·7 E.	10 P.M.
11	11·5 W.	5·2 W.	6·3 W.	0·7 E.	11 P.M.
12	10·8 W.	5·3 W.	5·5 W.	0·6 E.	Midnight.
13	8·0 W.	2·8 W.	5·2 W.	0·5 E.	1 A.M.
14	1·9 W.	0·6 E.	2·5 W.	0·5 E.	2 A.M.
15	3·6 E.	4·4 E.	0·8 W.	0·7 E.	3 A.M.
16	10·9 E.	9·0 E.	1·9 E.	1·1 E.	4 A.M.
17	16·6 E.	11·4 E.	5·2 E.	1·9 E.	5 A.M.

It must be regarded as a strong confirmation of the propriety of considering the regular solar-diurnal variation and the disturbance-variation as distinct classes of phenomena, requiring their respective laws to be separately investigated, when we see the correspondence between the principal features of the one variation (the regular solar-diurnal), and the antagonism of those of the other (the disturbance-variation), exemplified as they are at Point Barrow and Toronto; the epochs of extreme easterly and of

extreme westerly deflection being the same in the solar-diurnal variations, and nearly opposite to each other in the disturbance-variations. The primary cause of the phenomena is doubtless the same in both variations; they manifest alike the influence of the solar hours, and are subject alike to the same alternate periods of increase and decrease as the solar spots; but the mode of action, or the processes by which the effects are produced in the two classes, can scarcely be otherwise than dissimilar. It seems probable that (omitting for the present the consideration of the magnitudes) the general characters of the solar-diurnal variation will be found to correspond,—*i. e.* the hours of the easterly and westerly extremes to be the same,—in all the extra-tropical parts of the same hemisphere; whilst in the case of the disturbance-variation it seems also probable that every diversity in the times of occurrence of the turning-hours may be found in different localities.

In the hypotheses which physicists have framed to account for the times at which the solar-diurnal variation changes its direction from west to east and from east to west, it has been sometimes supposed that the variable *amount of the declination* in different localities is one of the determining circumstances. The observations at Toronto and Point Barrow do not favour this supposition: the declination at Point Barrow (taken to the nearest degree) is 41° East, and at Toronto 2° West; there is consequently an angular difference in the magnetic direction at the two stations of about 43°, which is nearly equivalent to three hours of time in the sun's azimuth relatively to the direction of the needle, whilst the conclusion from the observations in regard to the turning-hours is, that they are the same at Toronto and Point Barrow, or at most have only such small differences as may reasonably be ascribed to the presence of the small portion of the disturbances which we are unable to eliminate.

With respect to the comparative *magnitudes* of the range of the solar-diurnal variation at Toronto and Point Barrow, and to the proportion which the difference in magnitude bears to the difference in the horizontal force of the earth at the two stations, we may obtain an approximate conclusion, by comparing the amounts of deflection at 1 P.M., when the solar-diurnal variation at both stations has nearly reached its western extreme, and when the influence of the disturbances is nearly at a minimum. The deflections at 1 P.M. are 5′·1 at Toronto, and 8′·2 at Point Barrow. The absolute values of the horizontal force of the earth (which is the antagonistic force to all the minor variations, tending to retain the magnet in its mean position) are about 3·53 at Toronto and 1·79 at Point Barrow. In accordance with the inverse ratio of these values, we should have a deflection of 10′, instead of 8′·2, at Point Barrow, as the corresponding deflection to 5′·1 at Toronto. The range observed at Point Barrow is therefore a little less than the range which might be computed from the amount of deflection at Toronto. But the data upon which the comparison is based, particularly the amount at Point Barrow, can only be regarded as approximate, and will by no means justify a more pre-

cise conclusion, than that the observed ranges appear to be *nearly* in the inverse proportion of the respective values of the horizontal force.

But when we turn our attention to a comparison of the magnitudes of the range of the *disturbance-variation* at the two stations, we cannot fail to be impressed with the excessive disproportion which we perceive between the decrease of the earth's horizontal force and the increase in the range of the disturbance-variation at Point Barrow, when compared with Toronto. At the latter station the whole range of the mean diurnal effect of the disturbances scarcely exceeds 2′·0, whilst at Point Barrow it is not less than nine or ten times that amount. The disproportion would have been still greater if the collimator scale of the Point Barrow declinometer had permitted the disturbances of largest amount to have been registered at their full value. During the seventeen months of observation the magnet was forty times* deflected beyond the limits of the scale; in such cases the extreme cognisable division on the side of the deflection was the amount entered in the Table, although, of course, the actual deflection must have been greater. The collimator scale comprised 160 divisions, each of the value of 2′·287; and as the mean position corresponded nearly with the 80th scale-division, it follows that when a deflection exceeded about 180′ (or 3°) on either side of the mean position, the excess, could it have been registered, must have increased, *pro tanto*, the sum of the disturbances. The disproportion would also have been considerably greater, if the amount adopted as the standard value of a large disturbance at Point Barrow, and employed in separating the observations belonging to that category, had been more nearly proportional to the inverse ratio of the horizontal force there and at Toronto. In such case, all the observations at Point Barrow, which differed more than about 10′ from the normal at the same month and at the same hour, would have been classed amongst the large disturbances. But it was soon found that the disturbances at Toronto and Point Barrow bore no such analogy to each other. Even with 22′·86 (instead of 10′) as the measure of a large disturbance, the proportion of disturbed observations to the whole number was still much greater than at Toronto, where 5′·0 was taken as the measure of a large disturbance. It seems scarcely possible, therefore, to avoid the conclusion, that the average amount of deflection caused by the occasional disturbances at Point Barrow is *very considerably* greater, when compared with the same at Toronto, than can be explained by the difference of the horizontal force of the earth at the two stations.

A circumstance which strikingly accords with the excessive amount of the occasional disturbances at Point Barrow, is the great frequency with which appearances of the

* Of the 40 instances in which the deflection exceeded the limits of the recording scale, 30 were deflections to the east and 10 to the west; 24 occurred between the hours of 5 A.M. and noon, of which 22 were to the east and 2 to the west; 22 of the easterly deflections occurred from 19^h to 21^h inclusive, and 6 of the westerly from 11 P.M. to 1 A.M. inclusive. Referring to Table V., it will be seen that, if the instrumental means had permitted these deflections to have been registered at their full values, the principal effect must have been to have increased the ratios of easterly disturbance at Point Barrow at the three hours of 19, 20 and 21, which were previously the hours of its maximum; and in like manner to have increased the ratios of westerly disturbance at 11, 12 and 13 hours, which were the hours of its maximum.

aurora are registered, and which much exceeds, I believe, any previous record of the kind. It was the custom of Captain MAGUIRE and his officers to make a distinguishing mark opposite to the entry of every hourly observation of the declinometer when aurora was visible; and in the Tables received at Woolwich, the observations so distinguished are written in red ink, and those when no aurora was visible, in black ink. In the months of December, January and February, in which months there was scarcely any daylight, and the sky was generally freest from clouds, we find, in the first year, the whole number of hourly observations in the three months to have been 1786; the number at which aurora was seen, 464; and the number at which it was either not present or obscured by clouds, 1322; and in the same three months of the second year, the numbers were respectively 1837, 615 and 1222. Thus, in the six months, there were 1079 hours when aurora was seen contemporaneously with the declinometer observation, and 2544 hours at which it was either not present or not visible; in other words, aurora was actually seen contemporaneously with little less than one-third the number of hourly observations in the six months. The hours when aurora was most common were from 7 P.M. to 7 A.M.; and if we examine on how many days in the above-named six months the aurora was seen, and on how many days it was not seen, we find that it was seen between 7 P.M. and 7 A.M. on 131 days, and that it was not seen (during the whole twenty-four hours) on 22 days; or, as it may be otherwise stated, during the months of December, January and February, in two successive years, the aurora was actually seen six days out of every seven.

When the 1079 observations of the aurora are distributed into the different hours of their occurrence, we find them to have been as follows:—

TABLE VI.

Showing the number of times that the Aurora is recorded to have been seen at the several observation-hours in the months of December, January, and February 1852–1853, and in the same months in the following year.

Hours of local civil time.	Number of auroras observed.	Hours of local civil time.	Number of auroras observed.	Hours of local civil time.	Number of auroras observed.	Hours of local civil time.	Number of auroras observed.
6 A.M.	66	Noon.	0	6 P.M.	30	Midnight.	85
7 A.M.	54	1 P.M.	0	7 P.M.	56	1 A.M.	103
8 A.M.	28	2 P.M.	0	8 P.M.	56	2 A.M.	98
9 A.M.	10	3 P.M.	0	9 P.M.	60	3 A.M.	95
10 A.M.	2	4 P.M.	5	10 P.M.	77	4 A.M.	80
11 A.M.	0	5 P.M.	15	11 P.M.	88	5 A.M.	71

We perceive by this Table, that there is not a single instance recorded of the aurora having been seen between 11 A.M. and 3 P.M., both hours included; and as there is little or no daylight at these hours during the months in question, the *general absence* of aurora (and not merely its *invisibility*) in this part of the twenty-four hours may fairly be inferred. The most frequent hour of its appearance is 1 A.M.; and the great degree

of regularity in the decrease of the numbers on either side of the maximum is a very remarkable feature. The progression bears a very close resemblance and analogy to that of the *westerly* disturbances at Point Barrow, as seen in Table IV., and appears to have little or nothing in common with the progression of the easterly disturbances at Point Barrow in the same Table.

The presence of aurora is however by no means to be regarded as an indication of the contemporaneous existence of a disturbance of sufficient magnitude to be admitted into the category of the "disturbed" observations of this paper. In the six months referred to, in which the aurora was recorded at 1079 hours, only 272 of the contemporaneous observations were, in this sense, "disturbed," and 807 were not so; *i. e.* 272 out of 1079, or only about one-fourth of the observations made during the recorded presence of the aurora, differed as much as 22'·86 from the normal at the same hour and in the same month.

Of the fact, that the aggregate amount of disturbance at Point Barrow, when compared with Toronto, greatly exceeds the calculation founded on the inverse proportion of the horizontal force at the two stations, there can be no reasonable doubt. To what is this disproportion owing? Is it, as some may suppose, that Point Barrow is situated nearer than Toronto to the locality where the greatest intensity of disturbance is manifested? If it be so,—and this, be it remembered, is a question which admits of being ascertained by future investigations, similar to the present, at stations suitably chosen,—the question which will immediately present itself will be, by what peculiar physical or other conditions the locality of the greatest intensity of disturbance is distinguished. In the mean time, the widely dissimilar proportions of the disturbance-variation, and of the solar-diurnal variation, at Point Barrow and Toronto, supply additional evidence of the distinct laws by which the two classes of phenomena are governed, and of the necessity of their being separately studied *.

* The disturbances at Point Barrow do not appear to give support to a conjecture that has been recently expressed, that the "occasional disturbances may possibly include two distinct classes of changes, obeying separate laws, one of them being strictly periodic and constituting a part of the regular diurnal change, whilst the other is strictly abnormal and simultaneous at different stations." The regular diurnal change at Point Barrow has its greatest *easterly* elongation about 7 A.M., which is nearly the hour of the greatest *westerly* disturbance-variation; and its greatest *westerly* elongation about 1 P.M., at which time the disturbances, both easterly and westerly, are nearly a minimum. The indications of the phenomena seem therefore to be as nearly opposed as can well be to the conjecture referred to. It is quite true that if the attempt be made to obtain the laws of the disturbances from observations *from which the effects of the regular diurnal variation have not been previously eliminated*, the laws so obtained will include "two distinct classes of changes, obeying separate laws, one of them constituting a part of the regular diurnal variation." But when suitable processes have been employed, by which the mean influence of the regular diurnal variation has been eliminated from the observations which are used in examining the periodic laws of the disturbances, and when the characteristic value of a large disturbance has been taken in proper proportion to the ordinary range of the diurnal variation, I have not found in any case an indication of an admixture in the disturbance laws of an element corresponding in its turning-hours, &c. to the periodic laws of the regular solar-diurnal variation.—Sept. 5, 1857.

The dip of the needle was observed usually twice in each week, sometimes with the 9½-inch circle, and sometimes with the 6-inch. From October 1852 to June 1853 inclusive, 74 observations of the dip were made, and from October 1853 to June 1854 inclusive, 71. The abstract of the monthly results obtained by these observations is contained in the following Table.

TABLE VII.

Mean Dip of the Needle obtained in each Month.

Months.	Circle employed.	No. of observations.	Dip.	Months.	Circle employed.	No. of observations.	Dip.
	in.		° ′		in.		° ′
1852. October	6	6	81 38·7 N.	1853. October	9½	5	81 32·8 N.
„ November ...	9½	11	81 30·0	„ November ...	9½	10	81 34·5
„ December ...	9½	8	81 32·8	„ December ...	9½	9	81 32·3
1853. January	9½	9	81 33·4	1854. January	6	8	81 31·8
„ February.....	6	9	81 39·9	„ February.....	6	6	81 36·3
„ March	9½	8	81 39·0	„ March.....	6	9	81 38·3
„ April	9½	8	81 42·4	„ April	9½	8	81 34·0
„ May	9½	9	81 33·9	„ May	9½	10	81 37·8
„ June	6	6	81 34·3	„ June	9½	6	81 42·4
Mean 1852–1853		74	81 36·0	Mean 1853–1854		71	81 35·6

Of the 18 monthly determinations, 12 were observed with the 9½-inch circle and 6 with the 6-inch. The mean of the 12 with the 9½-inch is 81° 35′·4 N., and the mean of the 6 with the 6-inch is 81° 36′·6 N.

There are also Term Observations of the Declination at 5′ intervals on twenty term days, *i. e.* from November 1852 to July 1854 inclusive, omitting August 1853. These are reserved at Woolwich, in the hope of finding that corresponding observations were made elsewhere. It is much to be feared that few, if any, corresponding observations will be found to have been made over the whole extent of the North American Continent.

The hourly observations at Point Barrow which have furnished the results discussed in this paper, are printed in the following pages. The observations recorded during the presence of aurora are distinguished by an asterisk prefixed.

HOURLY OBSERVATIONS

OF THE

MAGNETIC DECLINATION AT POINT BARROW,

1852, 1853, 1854.

Point Barrow, 1852; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

	Mean Point Barrow Time.	0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.
	D.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.
November.	1												
	2												
	3												
	4												
	5	.....	.....	82·5	75·0	77·0	74·2	78·0	70·0	75·0	*73·0	*78·0	*85·0
	6	73·5	77·5	78·0	79·0	77·0	75·5	.....	.....	.....	.....	.....	.....
	7	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	75·0	74·0
	8	77·5	77·7	76·5	77·2	70·5	78·0	77·5	74·5	79·0	79·0	78·0	*76·0
	9	78·2	78·0	78·5	77·7	77·0	76·0	65·5	75·0	*62·0	*73·6	70·0	78·2
	10	80·0	79·0	80·0	77·5	75·0	78·0	*74·0	*71·3	*80·0	*57·0	*74·0	*51·0
	11	73·0	78·0	72·0	76·0	62·0	85·0	87·2	86·0	85·5	53·0	*50·0	58·0
	12	87·0	73·2	70·8	78·0	89·2	86·0	81·0	74·0	72·0	88·0	68·0	*(a)
	13	73·0	72·0	73·0	77·2	79·0	79·0	65·0	*74·5	*(a)	*15·0	81·5	71·0
	14	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	15	80·0	72·0	73·0	75·0	76·0	75·0	77·2	77·0	77·0	72·2	91·7	76·5
	16	72·0	59·2	64·7	68·0	70·0	70·0	73·0	*68·0	*48·0	*69·0	*75·0	*46·0
	17	66·2	73·5	75·0	76·5	75·0	75·5	83·0	73·0	76·0	59·0	72·0	15·2
	18	71·0	72·0	72·0	72·0	75·0	65·0	74·5	69·8	76·0	*73·5	*71·0	*77·0
	19	76·0	78·0	69·0	72·0	74·0	75·0	70·0	70·0	75·0	67·0	76·0	70·0
	20	75·0	74·0	73·7	77·7	77·0	73·0	75·0	74·0	75·0	75·0	75·0	75·0
	21	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	22	79·0	77·0	79·0	72·0	73·0	75·8	77·5	75·2	78·0	78·0	*75·0	*76·0
	23	74·0	76·0	74·0	71·0	68·0	75·0	72·0	78·0	76·0	81·0	75·2	72·0
	24	65·0	78·0	77·5	78·5	82·0	74·0	75·0	76·0	74·0	76·0	79·0	76·0
	25	74·0	76·0	72·0	67·5	70·0	79·0	75·3	80·0	75·0	75·5	79·0	75·0
	26	79·0	66·5	53·5	72·1	*76·0	*88·5	*80·0	*74·0	76·5	94·0	*102·0	80·2
	27	80·0	76·0	71·0	83·0	84·0	76·0	76·0	76·0	77·0	77·0	75·0	75·2
	28	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	29	71·2	72·0	70·0	71·5	69·0	75·0	73·0	78·0	74·0	74·0	74·0	73·0
	30	69·0	72·0	72·0	69·0	74·0	70·0	62·0	63·5	33·5	71·8	*66·0	85·0
December.	1	64·7	73·0	74·0	71·0	71·0	73·0	69·0	73·0	65·0	69·0	*82·0	*108·0
	2	71·0	71·7	71·0	70·9	72·0	74·0	76·5	*62·0	*95·0	*62·0	*100·0	*37·0
	3	75·0	60·5	67·5	67·0	72·0	71·0	69·0	72·0	71·5	72·5	73·0	72·0
	4	66·0	74·0	70·0	70·0	72·0	72·0	71·0	*62·4	77·5	69·0	69·5	66·0
	5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	6	81·0	75·3	76·2	76·5	79·6	80·0	80·5	79·0	78·0	76·0	*78·0	*60·0
	7	73·0	77·0	75·0	76·0	71·5	77·0	*75·5	*86·0	*82·0	*73·2	*89·0	*72·0
	8	73·0	65·0	68·0	72·0	76·0	78·0	*74·0	*88·0	76·7	*56·5	*69·0	*75·0
	9	65·0	66·0	78·0	76·0	72·0	75·0	73·0	*66·5	*70·0	*71·7	*79·7	71·1
	10	75·5	68·0	75·0	73·0	72·7	78·5	78·0	*83·0	81·0	*72·5	*85·0	*88·0
	11	75·0	75·0	78·5	79·0	77·5	87·0	73·0	*78·0	*75·0	*76·0	*75·0	*110·0
	12	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	13	72·3	73·0	79·5	59·0	75·0	70·0	81·0	107·0	90·0	*87·0	*76·0	*100·0
	14	68·0	79·6	79·0	72·0	52·0	*79·0	85·0	76·0	99·0	*81·0	*77·0	*82·0
	15	79·0	73·0	77·0	79·0	78·0	80·0	80·0	*80·0	*76·0	*89·0	*86·3	*77·0
	16	81·0	82·0	81·0	86·0	81·0	79·5	76·5	80·0	81·0	80·0	81·5	83·0
	17	83·0	85·3	82·0	.....	.....	.....	.....	.....	.....	.....	.....	.....
	18												
	19												
	20												
	21	.....	.....	.....	82·0	84·0	89·0	86·0	86·0	87·0	85·0	86·5	82·3
	22	77·5	77·0	85·3	81·3	82·0	85·7	77·0	*82·3	*85·5	87·7	80·0	92·0
	23	89·0	72·0	89·0	92·0	86·0	81·0	87·2	89·0	91·5	87·7	91·2	90·0
	24	81·0	82·0	77·0	82·0	89·0	75·0	80·0	86·7	84·0	85·0	87·5	84·0
	25	75·0	75·5	85·5	83·3	84·5	84·0	87·0	84·5	80·0	*145·0	74·0	81·0
	26	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	27	83·0	82·0	86·0	82·0	83·0	81·0	85·0	86·0	88·0	77·0	86·0	90·0
	28	94·0	82·0	.....	.....	75·0	*69·0	*78·0	*65·7	*43·0	106·0	73·0	*67·0
	29	80·0	82·0	86·0	86·0	*79·5	*78·0	*80·2	*80·0	88·0	83·0	90·0	82·7
	30	86·0	84·0	87·0	86·0	85·7	85·0	83·0	86·0	87·0	85·0	*86·0	87·0
	31	85·7	84·0	83·0	86·0	87·0	83·0	86·0	*84·0	81·5	83·0	83·0	81·0

Observations noted with a * were made when the Aurora was visible. (a) Off Scale West.

Point Barrow, 1853; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.	Mean Point Barrow Time.	
sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	D.	
												1	November.
												2	
												3	
												4	
*105·0	73·0	82·7	81·2	82·0	83·0	92·3	96·0	79·3	69·0	87·2	75·0	5	
.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	6	
77·0	75·0	87·0	81·0	88·0	81·0	85·0	86·0	88·0	85·0	80·2	76·5	7	
*84·0	92·8	78·0	89·0	81·7	80·0	83·2	91·0	95·0	86·2	84·5	78·0	8	
77·5	85·0	*75·0	.....	.....	.....	.....	.....	.....	.....	82·0	81·0	9	
*71·0	72·0	*76·0	85·2	89·0	80·0	89·0	92·0	156·5	118·7	116·5	61·0	10	
78·7	79·0	81·0	85·0	86·0	115·0	94·0	89·0	94·0	106·0	86·0	107·0	11	
60·0	53·0	68·0	63·7	*132·0	23·0	*(a)	*93·5	88·2	1·7	84·2	76·0	12	
89·0	78·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	13	
.....	.....	*81·0	84·0	82·0	79·5	97·0	94·0	78·5	74·0	81·7	80·0	14	
76·0	73·0	80·0	83·0	92·0	76·0	80·2	90·0	75·0	87·5	61·5	85·0	15	
78·5	77·0	73·5	92·2	85·6	82·0	92·0	95·0	79·0	76·0	84·0	78·0	16	
89·0	84·0	79·0	104·0	86·5	94·0	.....	83·5	89·5	68·0	68·5	71·0	17	
76·0	78·0	54·0	78·0	84·0	81·0	86·0	93·0	85·5	82·5	77·0	74·0	18	
78·0	75·0	68·0	86·5	76·0	76·0	76·0	75·0	78·0	75·0	75·0	75·0	19	
75·7	76·7	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	20	
.....	80·0	80·0	78·5	79·5	80·0	81·3	78·0	77·7	85·0	84·0	75·0	21	
73·2	*84·0	78·0	65·0	82·0	84·0	95·0	84·0	87·5	89·0	82·0	74·7	22	
75·0	73·0	75·0	86·0	82·0	85·0	81·2	88·0	95·0	90·0	98·0	63·0	23	
82·5	85·5	55·2	78·0	76·3	79·0	82·0	83·0	87·0	86·0	82·0	72·0	24	
79·0	79·0	76·5	78·5	79·0	70·0	84·0	135·1	80·0	95·0	62·0	82·0	25	
96·0	*76·4	*53·0	84·3	106·5	*66·5	82·8	93·4	71·6	96·0	92·7	77·5	26	
73·7	73·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	27	
.....	79·5	72·3	77·0	74·3	77·0	79·0	78·0	77·5	77·2	79·0	76·0	28	
74·0	75·0	73·7	*65·0	69·0	87·0	81·0	70·7	75·5	73·0	78·0	77·0	29	
78·5	87·2	76·0	*78·0	72·0	83·5	70·0	72·5	75·0	78·0	72·0	69·5	30	
*114·5	*(a)	*78·0	*82·0	*89·0	105·0	68·2	135·0	73·0	58·0	70·0	67·5	1	December.
*20·0	*90·0	86·0	70·3	*79·0	69·0	79·0	76·0	73·0	77·0	61·0	80·0	2	
73·0	70·0	71·0	*75·5	75·0	74·0	81·0	88·8	*108·0	89·0	82·0	65·0	3	
69·0	67·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	4	
.....	*74·0	*83·0	*82·0	*78·0	*95·0	*83·0	*66·0	84·0	83·0	91·0	82·0	5	
*83·0	*88·6	*69·0	*87·0	*82·5	*85·0	*91·5	*94·0	85·0	125·0	92·0	78·0	6	
*67·2	*80·0	*78·0	*83·0	*82·0	*76·9	*84·0	*112·2	*119·8	95·0	96·0	80·0	7	
*77·0	*66·0	*89·0	*86·0	*99·0	*83·0	*97·0	*79·0	*92·0	81·0	86·0	66·5	8	
152·5	*55·0	83·5	89·0	*103·0	77·0	76·0	104·0	105·0	76·5	73·5	61·0	9	
*88·0	*102·7	*53·5	*69·7	*84·0	95·0	105·0	81·5	81·0	78·0	83·0	86·0	10	
67·3	*50·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	11	
.....	*95·2	*87·0	*84·0	62·0	*107·0	*105·0	*109·0	*120·0	50·3	94·0	76·0	12	
83·0	86·0	78·0	*67·5	84·5	79·0	83·0	108·0	76·2	82·5	81·5	85·0	13	
*78·0	*77·5	*82·0	*77·5	*84·5	82·0	89·0	*87·0	86·5	76·0	80·0	80·0	14	
*81·0	*80·5	*81·0	86·5	92·0	88·7	83·3	91·4	80·0	83·5	83·0	81·0	15	
80·0	90·0	91·0	96·0	89·0	88·5	93·0	87·5	103·0	84·0	79·7	82·0	16	
.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	17	
												18	
												19	
												20	
87·0	87·5	87·0	85·0	84·0	90·0	91·2	93·0	90·0	103·0	98·0	101·0	21	
85·0	87·3	73·3	*77·5	96·8	105·3	107·9	125·0	100·9	99·7	110·0	69·0	22	
70·0	67·0	72·0	89·0	83·0	91·0	80·0	100·7	133·5	91·0	77·5	80·0	23	
82·5	93 5	100·0	92·0	89·0	92·0	112·0	82·0	87·0	96·0	80·0	88·5	24	
88·0	86·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	25	
.....	87·7	87·0	84·0	88·0	85·5	86·0	85·0	87·0	90·0	(a)	80·0	26	
88·0	61·5	90·0	91·0	97·0	117·5	92·0	104·0	92·0	92·0	90·0	95·0	27	
*78·5	81·0	83·0	96·0	110·0	*104·0	*111·0	*102·0	122·5	119·0	102·0	114·0	28	
70·0	70·5	19·5	91·0	85·0	85·0	85·0	85·2	85·0	83·0	87·5	85·0	29	
*104·0	*83·0	*92·0	*90·5	86·0	90·0	91·5	88·0	86·0	87·0	87·0	86·8	30	
87·0	*92·0	*96·0	*84·0	*92·0	85·0	91·0	100·3	98·0	95·0	82·0	85·0	31	

(a) Off Scale West.

Point Barrow, 1853; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

	Mean Point Barrow Time.	0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.
	D.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.
January.	1	77·3	82·3	81·0	80·0	*85·0	*91·0	*86·0	*80·0	*84·0	*71·0	*73·0	*72·0
	2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	3	92·0	83·0	82·1	85·0	85·0	86·0	87·0	87·0	89·0	89·0	*90·5	*92·5
	4	84·0	85·0	86·0	89·0	88·0	89·5	86·0	88·0	88·5	89·0	89·0	88·0
	5	92·5	95·0	96·3	88·0	86·0	89·0	*82·5	*91·0	*93·7	87·0	*89·0	*92·0
	6	95·0	89·0	85·0	88·0	89·0	*85·0	*86·0	*86·0	*85·0	*79·0	*121·0	*94·0
	7	94·0	86·0	84·5	85·0	87·0	89·5	82·0	89·0	*86·0	*91·0	*86·0	*62·0
	8	89·0	78·0	83·0	87·0	91·0	88·0	83·0	92·0	103·5	105·5	82·0	75·0
	9	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	10	96·0	92·0	86·0	91·0	93·0	82·0	90·0	87·0	91·0	92·0	94·0	91·0
	11	99·0	91·0	92·0	97·7	92·0	92·0	92·5	91·0	92·0	91·0	90·5	*88·0
	12	92·0	87·0	89·0	90·0	93·0	85·0	92·5	85·0	92·0	92·0	86·0	*90·0
	13	89·0	84·0	83·0	87·0	82·5	84·0	*86·0	*91·0	89·0	89·0	*91·0	*94·5
	14	93·0	80·0	85·0	86·0	88·0	*86·0	*84·0	*95·0	*90·0	*101·0	*93·0	*91·0
	15	91·0	88·0	90·0	85·0	92·0	81·0	92·0	*86·0	*92·5	*93·0	*94·0	*92·0
	16	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	17	88·0	89·0	86·8	84·0	82·0	92·0	*90·0	*88·0	*89·7	*89·0	*85·0	*92·5
	18	80·0	74·0	63·0	78·0	83·0	76·0	76·0	*70·0	*81·0	*86·5	*80·0	*70·0
	19	77·0	75·0	80·3	70·0	79·0	80·0	81·0	81·0	83·2	81·5	81·7	80·7
	20	84·0	78·0	80·0	86·0	82·0	80·0	77·0	80·0	81·0	80·0	*82·0	*103·0
	21	82·5	81·0	82·0	81·0	81·0	86·0	81·0	80·5	85·0	83·0	82·0	72·0
	22	82·3	86·0	82·0	81·0	81·0	82·0	83·0	83·0	86·0	83·0	83·3	81·0
	23	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	24	80·0	80·0	82·0	81·0	79·0	80·0	80·0	73·5	76·0	92·0	82·0	81·0
	25	78·0	79·0	79·0	82·0	84·0	83·0	86·0	82·0	81·0	84·2	84·0	83·0
	26	83·0	83·0	82·0	76·0	79·0	83·0	82·0	84·0	82·0	81·0	82·5	81·0
	27	85·0	81·0	82·5	81·5	80·0	79·0	*82·0	*78·5	*79·0	*80·0	*78·0	*81·0
	28	82·0	79·0	81·0	83·0	85·0	79·0	80·0	83·0	84·0	82·2	81·0	82·0
	29	82·0	84·0	80·5	79·0	78·0	77·0	78·5	80·0	86·0	82·0	79·5	95·0
	30	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	31	98·5	83·5	83·5	87·0	87·0	88·0	85·0	85·0	85·3	88·0	*84·0	*87·0
February.	1	75·0	74·0	83·0	87·0	81·0	84·5	*69·0	*82·5	*79·5	*88·2	*86·0	*81·0
	2	92·2	84·0	83·0	84·0	87·0	85·0	88·0	86·0	86·0	86·0	*86·0	*81·5
	3	87·0	85·5	80·5	87·0	86·5	83·0	82·0	*87·0	*89·0	*76·0	*91·0	*83·0
	4	90·0	84·3	84·7	81·0	87·0	85·0	*87·5	*88·5	*82·0	*90·0	*86·0	86·0
	5	95·0	94·0	87·0	81·0	79·0	76·0	84·0	88·0	85·0	*92·0	*91·0	*85·0
	6	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	7	85·0	80·0	78·5	78·0	79·5	85·0	82·0	81·0	78·0	*92·0	*99·7	*63·5
	8	81·0	80·0	78·3	81·0	83·0	81·0	81·0	83·0	81·0	79·0	*75·0	*81·0
	9	82·0	79·0	82·0	86·0	87·0	82·0	86·0	82·0	82·5	82·0	81·3	*80·5
	10	86·0	76·0	62·0	68·0	74·0	67·0	71·0	*79·0	*84·0	83·0	*82·0	*82·0
	11	82·0	86·0	80·0	82·0	82·0	80·0	85·0	*79·0	*83·0	*81·0	*84·0	*82·5
	12	81·0	78·0	82·0	83·0	87·0	85·0	83·0	80·0	82·5	84·0	81·0	*82·5
	13	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	14	82·0	73·0	76·2	88·0	80·0	88·3	80·0	*67·0	*85·0	*83·0	*108·0	*103·7
	15	89·0	86·0	89·0	86·0	79·0	83·0	84·0	79·5	85·0	81·0	*89·0	*85·0
	16	89·0	81·0	78·0	80·0	76·0	72·0	83·0	70·0	70·0	68·0	65·0	89·0
	17	80·0	87·0	76·0	81·0	86·0	80·0	89·0	80·0	81·2	86·5	81·0	75·0
	18	83·0	72·0	77·0	78·0	81·0	82·0	85·0	85·0	83·0	83·0	82·0	81·0
	19	84·0	84·0	82·0	81·5	82·0	81·0	77·0	80·0	84·0	80·0	*69·5	*62·0
	20	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	21	88·0	85·0	74·0	81·0	79·0	77·5	81·0	75·0	76·0	69·0	82·0	77·0
	22	78·0	82·0	76·0	79·0	85·0	89·0	74·0	77·0	76·5	78·0	72·0	41·5
	23	81·5	69·0	73·0	75·0	78·0	76·0	78·0	75·0	79·0	78·0	76·0	83·3
	24	80·0	82·0	85·0	78·0	79·0	70·8	77·0	74·2	*73·0	*79·0	*75·0	*92·0
	25	74·9	78·0	77·0	77·0	78·8	77·6	77·8	81·0	80·9	81·6	81·3	80·0
	26	78·0	78·0	77·0	76·0	82·0	81·0	79·0	82·0	82·0	84·0	85·0	82·5
	27	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	28	79·0	83·0	76·0	79·0	71·0	76·0	83·5	84·0	83·0	81·0	*76·0	*109·0

Observations noted with a * were made when the Aurora was visible.

Point Barrow, 1853; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.	Mean Point Barrow Time.	
sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	D.	
*92·3	91·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	1	January.
.....	*80·0	*83·0	*89·0	*90·0	89·0	84·0	88·0	86·0	87·0	93·0	87·0	2	
*51·0	*94·0	*91·0	*87·0	*90·0	*101·5	99·0	98·0	99·0	104·0	100·0	80·2	3	
*84·0	88·0	*89·0	*91·0	*91·0	*90·0	87·0	88·0	91·5	91·0	93·0	93·0	4	
*102·0	65·0	*99·5	*68·0	*95·0	*110·0	*99·0	*91·0	*100·0	86·0	93 0	92·0	5	
*95·0	*89·0	*93·5	86·0	86·0	84·0	88·0	92·0	84·0	93·0	106·0	90·0	6	
59·0	77·0	101·0	154·0	44·0	93·0	95·0	103·5	102·0	90·3	90·0	90·0	7	
93·0	84·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	8	
.....	*108·0	*74·0	96·5	97·2	94·5	101·0	101·0	102·0	157·0	112·0	58·5	9	
84·0	*(a)	*102·0	104·0	103·0	102·0	94·0	*97·2	93·0	91·0	86·0	95·0	10	
95·0	*97·0	*96·0	93·5	102·0	94·0	96·3	99·0	102·5	108·0	86·0	90·0	11	
*92·0	89·0	78·0	*90·0	105·0	132·0	109·0	141·0	115·0	80·0	87·5	102·0	12	
*88·0	*97·0	*93·2	*96·0	*90·0	*114·0	*100·0	*85·5	*95·5	96·5	108·5	85·0	13	
*84·0	*72·0	*107·0	*95·0	*96·0	*95·0	*99·0	*97·0	98·0	95·0	97·0	92·0	14	
*90·0	*92·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	15	
.....	*76·0	*64·0	*86·0	*87·0	*145·0	*99·0	*115·0	*89·0	*39·0	81·0	95·0	16	
*83·3	*85·2	*92·0	*92·0	*93·2	*95·0	*89·0	*99·0	*114·0	*103·0	84·0	82·0	17	
*84·0	*52·5	*102·0	*63·0	84·0	80·0	102·0	98·0	96·2	100·0	97·5	82·0	18	
*82·2	*82·3	*62·0	*71·0	84·0	83·8	84·0	84·7	84·0	84·5	89·9	82·0	19	
*92·0	*90·0	*86·0	*75·0	*104·0	*109·0	*121·0	*78·5	79·0	82·0	86·0	83·5	20	
*83·5	80·0	*94·0	84·0	86·0	88·0	89·0	83·0	84·0	85·0	85·0	82·0	21	
81·1	82·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	22	
.....	75·0	82·0	82·5	84·0	86·0	92·0	87·0	89·0	88·0	89·0	81·0	23	
81·0	*89·0	*91·0	*96·0	*88·0	*94·0	*95·0	95·0	83·5	89·0	82·0	75·0	24	
82·5	83·0	84·0	84·2	84·0	86·0	87·5	85·0	84·0	82·5	86·0	82·0	25	
80·0	*73·5	85·0	84·0	83·0	84·0	89·0	85·0	85·0	81·0	82·0	84·0	26	
*72·0	*88·0	*78·5	*103·0	*108·5	94·0	93·0	88·0	92·0	88·0	85·0	82·5	27	
*83·0	*82·0	*85·0	*80·0	*84·0	*86·0	*89·0	*87·0	87·0	94·0	85·0	86·0	28	
69·0	82·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	29	
.....	87·0	87·0	*91·0	*88·0	107·0	91·0	99·0	88·2	87·0	92·0	95·0	30	
*86·0	86·0	112·0	89·0	106·5	89·0	90·0	95·0	90·0	89·5	92·0	89·0	31	
*73·5	*88·0	*73·0	*92·0	*86·0	88·0	127·0	95·0	87·0	79·0	80·0	84·0	1	February.
*72·0	*94·0	*98·2	*97·5	94·0	96·0	93·0	90·0	86·0	88·0	86·0	91·0	2	
*94·0	*88·0	*88·0	*116·0	*141·0	*95·0	*95·0	113·0	104·0	94·5	96·0	83·0	3	
86·5	83·0	79·0	76·5	88·0	87·0	95·0	105·0	102·0	130·0	72·0	82·0	4	
*85·0	*63·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	5	
.....	*88·0	*86·0	*92·0	*89·0	*95·0	*93·0	99·0	89·0	88·0	87·5	86·0	6	
73·0	90·0	*83·0	*81·5	*95·0	*87·0	82·0	88·0	106·0	80·0	83·0	84·0	7	
*65·0	*72·0	*69·0	*66·5	*60·0	*83·0	*85·0	87·0	85·0	86·0	82·0	81·0	8	
*105·0	*86·0	*82·0	*85·0	*84·0	*87·0	*75·0	81·0	86·0	95·0	92·0	89·0	9	
*81·0	*80·0	*86·0	*78·0	*89·0	*91·0	*88·0	*96·0	78·0	83·0	85·0	80·0	10	
*72·5	83·0	*78·0	*87·0	*96·0	*98·0	*91·0	84·0	85·0	89·0	87·0	85·0	11	
*86·0	*84·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	12	
.....	*74·0	*110·0	*81·0	85·0	*67·5	*68·0	*126·0	90·0	93·0	104·0	59·0	13	
*80·0	*94·0	*78·0	*86·0	84·0	86·0	89·0	87·0	85·5	86·5	85·0	81·0	14	
*87·0	88·0	83·0	*73·5	97·0	90·0	95·0	102·0	115·0	102·0	86·0	87·0	15	
81·0	94·0	103·0	114·5	99·0	96·0	97·0	93·0	102·0	111·0	115·0	83·0	16	
86·0	85·0	*89·0	*86·0	83·0	88·5	79·0	73·0	91·0	83·0	80·0	82·0	17	
85·0	78·0	*85·0	*92·0	*94·0	93·0	85·0	87·0	83·0	81·0	86·0	80·0	18	
*65·0	*78·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	19	
.....	99·5	*114·0	*84·0	87·2	91·0	98·0	102·0	108·0	69·5	98·0	100·0	20	
75·0	86·0	82·0	81·0	95·0	98·7	91·0	96·0	94·0	102·0	104·0	98·0	21	
103·5	70·0	74·0	86·0	92·0	95·0	88·0	89·0	76·0	75·0	73·0	74·5	22	
80·0	78·0	83·0	79·0	78·5	79·0	84·0	78·0	80·0	74·0	78·0	78·0	23	
78·0	82·0	*76·0	*64·0	*67·0	*96·0	93·0	88·0	95·0	96·2	84·0	84·0	24	
80·2	*80·0	*84·0	*80·0	*91·0	90·0	82·0	84·5	94·1	100·8	78·0	78·0	25	
83·0	82·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	26	
.....	104·0	91·0	93·0	96·0	92·0	96·0	110·0	100·0	105·0	85·0	85·0	27	
*89·0	*89·0	*89·0	*87·0	85·0	102·0	94·0	92·0	92·0	70·0	86·0	75·0	28	

(a) Off Scale East.

Point Barrow, 1853; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

Mean Point Barrow Time.	D.	0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.
		sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.
March.	1	83·0	79·0	82·5	80·0	78·0	83·0	81·5	88·0	*82·0	*80·0	*70·5	*80·2
	2	80·0	82·0	76·2	81·0	80·0	82·0	84·0	83·0	82·5	83·0	75·0	73·0
	3	82·0	80·0	78·0	82·0	82·0	82·2	79·0	84·0	81·0	82·0	82·0	85·0
	4	89·0	84·0	71·5	81·5	80·0	82·5	84·2	84·0	77·0	81·0	*82·0	*84·0
	5	77·0	82·0	78·0	81·0	83·0	81·0	82·0	85·0	83·0	*82·0	*81·0	*84·0
	6	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	7	54·0	62·0	87·0	83·0	75·0	82·0	80·0	86·0	52·5	92·5	72·5	72·0
	8	62·0	53·0	97·5	80·0	90·0	83·0	79·0	71·5	*88·2	*89·0	*76·0	*23·0
	9	75·0	73·0	85·3	88·0	70·0	88·0	82·0	79 2	74·0	94·0	95·0	69·0
	10	76·0	79·0	66·0	77·0	68·0	84·0	79·0	88·5	73·0	*74·5	*75·0	*75·0
	11	109·0	63·0	71·0	72·0	83·0	79·0	85·0	123·0	*80·0	*110·0	*69·5	*72·0
	12	89·0	62·5	71·5	79·2	46·5	76·0	72·5	67·2	87·0	*95·5	*94·0	*97·0
	13	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	14	88·0	86·0	79·5	84·2	80·0	89·0	78·0	85 0	83·3	84·5	*89·0	*86·5
	15	77·7	83·0	70·3	80·0	83·0	84·0	76·0	87·5	84·0	82·0	83·5	82·0
	16	95·5	85·5	88·5	83·0	78·0	89·0	85·5	88·5	102·5	86·5	106·0	84·5
	17	89·0	73·5	80·0	85·0	78·5	81·2	82·5	90·5	76·5	*83·0	*80·7	*76·0
	18	75·0	78·0	83·2	83·5	94·0	81·0	86·0	78·5	83·5	83·5	86·0	84·5
	19	91·0	105·0	85·0	87·0	84·0	83·5	76·5	81·5	*87·3	85·5	88·7	86·0
	20	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	21	85·0	82·0	84·0	83·0	81·0	105·0	93·5	85·0	82·7	84·3	84·0	84·0
	22	82·0	97·0	97·0	85·0	87·0	83·0	85·0	86·0	84·0	85·0	96·0	96·0
	23	86·2	79·5	82·4	80·7	79·9	82·0	81·3	76·5	79·3	77·3	*68·5	*68·5
	24	83·5	78·5	81·0	79·0	84·0	82·0	91·0	94·0	78·5	84·0	79·0	79·5
	25	83·0	84·0	80·5	75·0	83·0	85·0	85·0	84·0	86·0	86·2	82·0	78·2
	26	82·0	78·7	86·0	83·0	84·5	84·0	85·5	84·5	78·0	75·3	*64·5	*102·5
	27	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	28	83·0	86·0	84·0	85·0	81·0	85·3	80·0	82·0	80·5	89·2	89 0	*85·0
	29	87·0	76·5	85·0	84·7	81·5	83·0	82·5	87·5	84·0	83·8	78·0	*97·0
	30	71·7	93·0	86·0	84·5	81·5	82·0	85·0	85·0	83·7	92·0	*81·5	*85·0
	31	82·0	89·0	88·0	86·0	83·0	85·0	85·0	85·5	88·5	95·0	94·0	*103·0
April.	1	83·0	86·0	90·0	82·7	85·0	77·0	77·5	77·0	82·0	147·5	*56·0	*60·5
	2	87 0	84·0	88·0	83·5	84·0	84·0	84·0	86·7	86·0	86·5	86·0	89·0
	3	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	4	86·0	83·0	83·5	82·0	82·0	83·5	83·0	85·0	75·0	45·0	*79·0	*76·0
	5	143·0	73·5	81·5	74·0	76·2	83·0	86·0	61·0	64·0	92·0	*63·0	*44·0
	6	82·0	80·0	75·0	84·0	77·0	82·0	81·0	84·5	85·5	77·5	94·0	*76·0
	7	81·0	81·2	79·0	81·0	91·0	84·5	84·0	86·0	82·0	75·0	78·5	84·5
	8	70·0	89·0	74·5	84·5	72·5	79·0	84·0	65·0	68·0	69·0	80·5	*92·7
	9	74·0	95·5	83·5	76·5	76·2	84·0	70·5	80·0	78·0	81·0	*104·0	*96·0
	10	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	11	89·0	80·0	78·0	81·0	81·2	81·2	86·0	85·0	93·0	68·0	32·5	74·5
	12	95·0	93·0	76·0	78·0	85·5	82·0	85·0	85·0	86·0	82·5	80·0	86·7
	13	85·0	86·0	85·5	87·0	85·0	85·0	84·5	85·0	86·0	86·0	84·0	87·0
	14	90·0	85·0	79·0	81·0	83·0	84·5	87·0	84·0	80·5	81·5	72·0	76·0
	15	88 5	86·0	83·3	88·0	96·0	83·0	79·0	82·5	86·8	90·0	87·5	86·0
	16	96·7	76·0	82·5	82·0	86·5	84·0	87·0	85·0	86·0	88·7	83·7	76·5
	17	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	18	87·0	91·0	78·0	80·0	83·0	79·0	87·0	80·5	84·0	82·0	84·2	88·0
	19	85·0	79·0	78 5	82·0	119·0	82·0	87·0	81·0	80·0	79·5	81·5	85·0
	20	94·0	88·2	74·5	76·8	77·8	80·8	77·0	.....	.....	83·0	83·0	84·5
	21	85·0	87·0	84·7	79·0	81·0	80·0	81·0	84·0	87·0	87·0	80·0	76·0
	22	76·0	75·0	75·0	84·0	86·0	85·7	70·0	87·0	56·5	64·0	67·5	36·0
	23	86·5	89·0	84·0	85·0	85·0	82·0	80·0	84·0	88·5	84·0	81·5	78·5
	24	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	25	91·0	91·4	83·0	78·0	76·5	76·0	85·2	81·0	84·5	83·0	83·0	83·0
	26	97·0	95·0	88·0	86·0	81·0	87·0	87·0	89·0	88·0	92·5	88·2	89·0
	27	91·0	92·0	84·0	88·0	88·0	89·0	87·0	87·0	88·3	87·5	90·5	91·5
	28	91·0	91·0	88·0	86·0	88·0	86·0	85·0	89·0	87·0	76·0	73·0	91·2
	29	92·0	85·0	91·0	89·0	94·0	88·5	75·0	84·0	84·0	95·0	89·0	82·0
	30	98·0	88·0	93·0	86·0	84·0	83·0	90·0	85·0	86·0	90·0	82·0	89·0

Observations noted with a * were made when the Aurora was visible.

Point Barrow, 1853; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.	Mean Point Barrow Time.	
sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	D.	
*92·0	*95·0	*77·0	*89·0	*86·0	90·0	90·0	89·0	94·0	82·0	88·3	82·0	1	March.
74·5	78·0	84·0	86·0	89·0	109·0	78·0	87·0	85·0	97·0	80·0	83·0	2	
83·0	84·0	81·0	86·5	107·5	96·0	90·0	86·0	86·2	86·0	91·0	72·0	3	
*81·0	84·0	92·0	87·5	88·0	98·0	98·0	98·0	96·0	82·0	91·0	76·0	4	
*88·0	*78·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	5	
.....	*127·6	*82·5	*87·0	*104·5	99·0	85·0	104·0	(a)	(a)	(a)	138·0	6	
121·0	73·0	16·0	43·0	64·0	(b)	82·0	146·0	120·5	113·5	111·0	122·0	7	
*130·0	*85·0	*117·0	*35·3	99·0	105·0	101·5	116·0	103·0	78·0	78·0	76·0	8	
*54·0	*59·0	*88·0	*92·0	*96·0	90·0	94·0	110·0	94·0	93·0	128·0	95·0	9	
83·5	*80·0	*94·0	*93·0	106·0	120·0	110·0	98·0	124·0	92·0	142·0	110·5	10	
*71·5	*103·0	*75·0	*91·0	101·5	86·0	94·0	102·0	120·0	72·5	106·0	103·0	11	
*77·0	*65·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	12	
.....	*115·0	*75·0	*100·0	*85·5	89·0	90·0	92·5	88·0	89·0	86·0	88·0	13	
*82·0	*75·0	*82·0	*74·0	*76·0	93·5	90·0	102·3	93·0	97·0	92·5	92·3	14	
83·0	78·0	*59·0	*80·0	*97·5	86·0	91·0	95·0	94·0	95·0	96·5	82·0	15	
85·0	84·0	88·0	84·0	76·0	84·0	85·0	93·0	96·0	90·0	95·0	67·0	16	
*70·5	*69·0	*119·0	*103·0	91·0	88·0	84·0	110·0	81·0	78·5	76·5	84·0	17	
83·0	99·5	84·0	97·0	87·5	86·5	75·5	94·0	88·5	94·0	84·0	85·0	18	
102·5	82·2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	19	
.....	94·0	88·0	*89·5	95·0	105·0	95·0	93·5	88·0	90·0	84·0	84·5	20	
87·0	91·5	94·5	107·0	78·0	85·0	90·5	89·0	95·0	88·0	81·0	83·0	21	
103·5	81·0	*90·5	87·0	94·0	86·0	84·0	86·0	90·0	91·0	84·0	78·0	22	
72·5	74·2	83·2	90·1	102·3	100·0	83·1	88·0	85·5	87·0	84·2	86·5	23	
81·0	81·0	85·0	82·2	77·0	102·5	97·5	110·5	111·5	120·0	103·0	103·0	24	
84·0	91·7	92·0	87·2	87·0	89·0	88·0	96·0	124·5	99·0	80·0	84·0	25	
*97·5	86·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	26	
.....	*58·5	*85·0	*88·2	125·0	115·0	100·0	95·5	103·0	115·0	77·0	79·0	27	
84·1	88·7	87·0	86·0	85·7	147·0	85·5	88·0	88·0	92·0	89·0	87·0	28	
*84·0	*91·0	*103·5	71·0	104·5	85·0	94·0	115·0	106·0	83·8	96·5	75·5	29	
*87·0	*127·5	*133·0	84·0	96·0	83·0	83·2	106·3	76·0	78·0	85·0	82·5	30	
85·0	*111·5	90·0	110·5	91·5	86·5	85·0	80·5	95·0	87·0	88·0	83·0	31	
101·3	87·3	*93·0	88·5	90·0	90·0	86·0	87·5	94·0	89·0	90·0	87·6	1	April.
*84·0	*86·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	2	
.....	.....	92·0	92·0	83·0	87·0	92·0	84·0	86·0	89·0	90·7	87·0	3	
*85·5	*81·0	96·0	104·5	87·5	90·2	98·0	97·0	98·0	(a)	150·0	112·0	4	
*75·0	*91·7	116·5	85·5	89·2	92·5	21·5	25·5	98·0	135·0	102·0	107·0	5	
*104·0	*94·0	96·0	91·0	95·0	102·5	93·0	97·0	117·5	98·5	113·0	75·0	6	
33·0	75·0	86·0	95·0	106·0	96·0	115·0	100·0	108·0	80·5	120·7	97·0	7	
65·0	*124·0	75·5	78·0	81·0	117·0	112·5	84·0	96·0	135·0	94·0	67·0	8	
80·0	50·2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	9	
.....	151·0	102·0	89·0	90·0	99·7	87·0	91·0	129·5	91·0	96·2	94·0	10	
85·0	*90·5	88·0	85·5	97·0	81·0	91·0	99·0	87·0	113·0	84·0	98·5	11	
*89·6	*90·5	94·0	81·5	97·5	106·0	108·0	97·5	115·0	105·0	91·0	83·0	12	
87·5	81·0	82·0	83·5	91·0	98·0	96·0	115·0	109·0	102·0	97·0	94·0	13	
82·0	75·0	84·0	76·0	91·0	89·7	87·0	79·0	95·5	92·5	89·5	88·0	14	
86·0	89·0	92·0	90·0	96·0	93·0	96·0	89·0	92·0	82·3	95·0	92·3	15	
82·0	78·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	16	
.....	87·0	97·5	94·0	85·5	92·0	92·6	96·0	99·0	(c)	.....	95·0	17	
85·0	82·0	84·5	85·0	97·0	97·7	92·3	114·0	118·0	86·0	125·0	91·0	18	
89·0	90·0	98·0	92·5	98·0	94·0	104·0	98·0	101·0	113·0	108·6	112·0	19	
70·5	83·0	80·2	89·2	97·2	90·0	92·0	97·5	96·8	93·8	92·7	84·0	20	
84·0	83·0	81·0	87·0	92·0	91·0	107·5	105·0	112·0	95·0	92·0	100·4	21	
80·0	83·0	91·0	92·0	87·0	102·0	97·0	94·0	122·0	118·5	105·5	74·5	22	
67·3	82·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	23	
.....	86·5	87·0	89·0	89·5	91·0	108·0	95·5	90·0	94·0	92·0	96·0	24	
82·0	86·0	77·5	70·0	79·5	82·0	82·5	82·0	83·0	82·0	93·2	95·0	25	
91·0	91·0	89·0	86·0	85·0	102·0	100·5	96·7	93·0	92·0	91·0	90·0	26	
90·0	88·0	89·0	93·0	95·0	95·0	93·0	91·0	94·0	92·0	91·0	91·5	27	
91·5	86·0	91·0	91·0	92·5	105·0	95·2	97·0	94·0	91·0	92·0	96·0	28	
81·0	85·0	89·3	101·0	98·0	93·0	94·0	106·0	102·0	99·0	92·0	98·0	29	
75·0	87·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	30	

(a) Off Scale East. (b) Off Scale West. (c) Instrument removed.

Point Barrow, 1853; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

	Mean Point Barrow Time.	0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.
	D.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.
May.	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	2	100·0	93·0	95·0	98·0	81·3	75·0	70·0	121·0	68·0	75·5	94·5	51·0
	3	100·0	95·0	96·5	115·0	121·0	75·9	75·0	98·0	94·0	93·0	79·0	82·0
	4	91·0	112·0	87·0	87·0	89·0	80·5	83·0	101·0	72·0	132·0	92·0	75·0
	5	91·0	96·0	94·0	89·0	94·0	87·0	80·0	84·0	85·3	94·5	85·0	71·0
	6	97·0	88·0	100·0	116·0	91·0	94·0	93·0	89·0	85·0	86·0	81·3	98·0
	7	96·0	111·0	96·0	84·0	79·0	95·2	90·0	84·0	87·0	83·0	99·0	90·0
	8	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	9	96·0	90·0	94·0	88·0	95·0	85·0	85·0	87·0	86·0	93·0	87·0	89·5
	10	89·0	87·0	82·0	86·5	87·0	80·0	86·0	89·0	90·0	90·3	90·3	89·3
	11	87·0	86·0	81·0	83·0	84·0	87·0	86·0	88·0	89·2	89·0	86·0	84·0
	12	83·0	84·0	83·0	84·0	86·0	87·0	87·7	86·0	89·0	89·0	87·0	85·0
	13	86·0	87·0	85·0	87·0	85·2	88·0	87·0	87·0	89·0	88·2	88·2	88·5
	14	93·0	86·0	88·0	89·0	90·0	97·5	85·0	85·0	89·0	87·0	88·0	88·7
	15	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	16	87·0	87·0	88·0	87·0	84·0	82·0	78·0	76·0	79·0	98·2	89·0	88·0
	17	74·5	87·0	87·0	86·0	82·0	79·5	78·0	76·0	78·0	83·0	93·5	89·5
	18	84·0	91·0	88·2	81·0	83·0	85·0	89·0	88·0	91·0	88·0	87·5	90·0
	19	72·0	72·0	82·0	84·7	82·0	86·5	86·0	80·0	95·0	86·0	85·0	89·0
	20	89·0	84·0	87·0	91·0	89·0	87·0	84·0	88·0	89·0	90·0	87·2	89·0
	21	90·0	86·0	82·0	85·6	80·0	86·0	89·0	87·0	89·0	87·2	89·0	87·2
	22	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	23	104·0	111·0	69·0	81·0	85·0	84·0	82·0	85·0	87·0	87·0	86·0	85·0
	24	89·0	92·0	89·0	94·0	92·0	91·0	96·0	84·0	88·0	86·0	65·0	78·0
	25	74·0	87·2	94·2	81·5	87·0	86·0	87·0	85·7	86·0	84·0	89·0	86·0
	26	89·7	87·0	83·0	88·5	91·0	86·0	89·0	89·0	91·0	90·0	89·0	87·0
	27	87·9	88·0	91·0	87·5	92·0	84·7	83·3	72·0	81·3	57·5	64·2	87·4
	28	104·0	89·0	84·0	86·0	78·0	79·5	85·0	90·2	85·1	91·0	86·5	96·0
	29	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	30	102·0	102·0	98·0	96·0	82·0	84·0	76·0	71·0	76·0	78·5	82·5	77·3
	31	79·0	84·5	93·7	88·0	88·5	83·0	77·2	84·0	89·0	85·0	79·0	73·0
June.	1	(a)				92·0	85·0	44·7	44·5	58·0	70·0	56·0	52·0
	2	89·0	73·0	69·0	74·0	82·0	74·0	73·0	84·0	76·5	82·5	81·0	81·2
	3	90·0	86·2	71·2	76·0	72·0	75·0	75·0	77·0	83·0	82·0	72·5	78·0
	4	84·0	70·0	79·0	76·0	74·0	67·0	83·2	81·0	72·0	85·0	82·0	81·0
	5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	6	72·0	76·0	80·0	76·0	70·0	66·0	73·0	38·0	65·0	63·2	71·0	82·0
	7	87·0	69·3	79·2	78·0	75·0	77·2	78·0	78·0	79·0	79·0	75·0	74·0
	8	77·0	73·3	76·0	85·0	74·0	77·0	78·0	76·0	78·0	78·0	78·0	77·0
	9	89·0	91·0	83·0	80·0	79·0	69·7	71·0	69·5	57·0	57·0	79·5	84·0
	10	75·2	77·0	76·5	75·0	75·0	75·0	76·0	77·0	72·0	78·0	78·0	74·0
	11	78·0	73·0	71·0	74·0	74·0	71·0	71·0	81·5	70·0	79·0	71·0	58·0
	12	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	13	67·2	69·0	78·0	80·0	85·0	71·0	75·0	78·0	76·0	81·5	76·0	67·0
	14	85·0	44·3	65·5	80·0	64·0	73·0	73·0	63·0	76·5	78·0	60·5	44·2
	15	103·0	54·0	161·0	64·4	78·0	77·5	77·5	72·0	73·0	71·0	78·0	71·3
	16	85·0	77·0	71·0	73·0	78·5	74·0	74·0	75·0	76·0	76·0	77·0	73·0
	17	79·5	85·0	84·2	78·0	75·0	74·0	74·0	81·0	85·0	75·0	74·5	77·2
	18	82·0	76·0	72·0	74·0	75·0	79·0	79·0	77·0	78·0	74·0	78·0	73·0
	19	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	20	55·0	75·0	73·0	75·0	71·0	76·0	76·0	75·0	78·0	74·0	69·0	57·0
	21	81·0	76·0	71·0	69·0	75·0	69·0	74·0	78·0	78·0	74·0	75·0	73·0
	22	14·0	73·5	98·0	94·3	66·0	63·3	75·0	73·2	77·2	64·8	64·3	83·7
	23												
	24												
	25												
	26												
	27												
	28												
	29												
	30												

(a) Unroofing Observatory.

Point Barrow, 1853; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.	Mean Point Barrow Time	
sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	D.	
.....	101·0	85·0	76·0	100·0	105·0	99·0	101·0	109·0	98·5	125·0	106·0	1	
104·0	72·0	62·0	86·0	106·0	120·0	128·0	146·0	(*b*)	(*b*)	105·0	101·0	2	
68·5	68·5	80·0	79·0	92·0	122·5	91·0	95·5	121·0	99·0	94·0	92·0	3	
78·0	82·0	116·0	95·0	98·0	102·0	90·6	89·5	99·5	89·0	92·0	92·0	4	
62·0	96·0	97·0	97·0	99·0	93·0	111·0	107·0	105·0	103·0	93·7	94·0	5	
89·0	104·0	131·0	122·0	113·5	98·0	104·0	125·0	95·5	104·0	81·0	93·0	6	
78·0	99·3	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	7	
.....	85·0	89·0	91·0	93·0	104·0	101·0	103·0	102·0	102·5	107·0	94·0	8	
88·5	86·0	89·0	91·5	98·0	95·0	94·0	99·0	96·5	94·3	92·0	92·0	9	
89·7	84·5	88·0	87·0	88·0	97·5	95·5	87·5	96·0	95·0	92·0	86·0	10	
88·0	84·0	81·0	90·0	89·0	89·0	95·0	94·0	95·0	91·2	89·0	96·0	11	
88·0	85·0	91·0	85·0	97·0	96·0	101·0	103·7	94·0	98·0	100·0	93·0	12	
89·0	87·0	91·0	92·0	96·0	94·0	95·0	100·0	96·0	89·0	90·0	97·0	13	
90·0	81·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	14	
.....	87·2	95·2	94·0	97·5	96·0	96·0	96·5	93·0	96·0	98·0	86·2	15	
87·0	87·0	93·0	92·0	90·0	94·0	91·0	105·0	103·0	89·0	94·5	104·0	16	May.
91·0	89·0	86·0	88·0	86·0	89·0	107·0	110·0	129·0	105·0	107·0	97·7	17	
85·0	83·0	102·0	96·0	27·0	72·0	116·0	112·0	126·0	96·0	80·0	73·0	18	
72·0	81·0	84·0	92·0	96·0	98·0	100·0	102·0	103·0	103·0	95·0	92·0	19	
88·5	82·0	87·0	86·0	89·0	87·0	110·0	114·0	113·0	98·0	91·0	88·0	20	
86·5	86·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	21	
.....	86·0	85·2	93·5	82·0	99·0	93·0	96·0	115·0	99·0	98·0	106·0	22	
83·0	87·0	86·0	85·0	83·0	76·0	79·5	89·0	98·0	92·0	130·0	87·0	23	
96·0	97·0	91·0	99·0	94·0	95·0	98·0	100·0	125·0	92·0	92·5	81·0	24	
85·2	81·0	84·0	87·0	92·0	96·0	98·0	102·0	101·0	104·0	96·0	91·0	25	
81·3	94·5	92·0	93·5	93·0	88·0	94·0	99·0	99·0	101·0	93·0	87·0	26	
79·8	92·5	86·8	97·5	69·0	103·0	100·0	123·3	118·0	145·0	65·7	104·0	27	
78·2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	28	
.....	94·0	102·0	93·0	94·0	91·0	104·0	145·0	114·0	103·0	114·0	100·3	29	
79·0	88·1	77·5	99·0	102·0	98·0	96·5	105·0	105·0	129·0	107·0	85·0	30	
78·0	81·0	86·5	82·3	82·0	88·0	101·3	(*b*)	103·5	157·8	(*a*)	.....	31	
46·0	55·0	83·0	72·0	85·3	94·7	37·3	46·0	45·5	103·6	104·6	101·2	1	
62·0	91·0	78·0	70·0	79·0	86·0	99·0	106·0	97·0	93·0	88·3	69·7	2	
84·0	79·0	76·2	82·0	94·0	86·0	98·0	94·0	100·1	85·0	87·0	83·0	3	
80·0	78·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	4	
.....	65·0	65·0	79·0	87·0	92·0	95·0	97·0	99·3	96·0	85·0	74·0	5	
75·0	78·0	82·0	86·0	80·0	84·0	89·0	105·0	106·0	118·0	100·0	98·0	6	
78·0	76·0	68·5	79·0	83·0	85·0	94·0	91·0	90·0	87·0	83·0	77·0	7	
75·0	69·0	82·0	80·0	84·0	83·0	86·0	94·0	87·2	89·0	84·0	83·0	8	
79·5	79·0	80·0	80·0	88·0	84·0	85·0	89·0	87·0	81·5	81·0	83·0	9	
75·6	69·0	74·0	64·0	78·0	89·0	92·0	95·0	98·0	99·0	91·0	73·0	10	
60·0	102·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	11	
.....	83·0	79·0	81·0	86·0	106·0	105·0	96·5	102·7	79·5	96·5	69·0	12	
24·3	45·0	62·0	64·0	78·0	67·0	69·0	85·0	110·0	80·5	83·0	86·0	13	
58·0	72·3	79·0	84·0	81·0	94·0	89·0	80·0	81·5	98·0	114·0	120·0	14	
74·0	78·0	67·0	81·0	82·0	86·3	85·0	106·0	94·3	91·0	81·0	94·0	15	
71·0	72·3	84·0	91·0	89·0	85·0	89·0	91·0	89·0	96·0	83·0	82·0	16	
78·0	78·0	79·0	83·0	85·0	93·0	89·0	89·5	91·0	88·0	89·0	83·0	17	
75·0	74·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	18	
.....	69·0	80·0	79·0	86·0	92·0	101·0	94·0	103·0	112·0	87·0	87·0	19	
72·0	71·0	70·0	78·0	84·0	110·0	116·0	98·0	80·5	82·0	82·0	80·0	20	
72·0	78·5	79·0	74·0	80·0	89·0	98·0	101·0	94·0	132·0	111·0	(*b*)	21	
76·8	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	22	
												23	
												24	
												25	
												26	
												27	
												28	
												29	
												30	June.

(*a*) Unroofing Observatory. (*b*) Off Scale East.

Point Barrow, 1853; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

	Mean Point Barrow Time.	0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.
	D.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.
October.	1												
	2												
	3												
	4												
	5												
	6												
	7												
	8												
	9												
	10												
	11	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	12	84·5	82·4	79·8	79·5	77·5	75·0	78·3	78·7	82·5	79·3	80·5	80·7
	13	78·5	79·7	80·0	84·3	80·3	81·0	81·5	76·9	80·7	80·5	78·0	70·0
	14	70·0	69·5	81·5	76·0	78·5	75·0	77·3	73·8	80·0	79·3	79·0	79·3
	15	76·2	75·0	72·5	75·0	79·7	77·0	71·0	79·3	85·7	78·5	71·8	81·0
	16	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	17	74·5	78·5	73·4	80·5	78·0	79·7	83·9	72·5	76·0	77·5	75·5	68·5
	18	75·3	73·0	74·0	71·3	77·5	79·3	79·7	79·7	82·0	83·3	79·5	76·7
	19	77·7	75·6	78·0	77·8	77·8	73·0	73·5	77·1	77·3	78·5	*73·0	73·4
	20	74·7	71·6	76·6	77·0	77·0	75·3	74·4	72·2	76·0	82·5	80·4	77·3
	21	76·7	78·6	78·3	78·9	78·9	76·5	78·5	80·0	78·7	76·5	*78·0	*76·7
	22	78·7	79·0	78·5	79·2	79·0	75·3	78·0	79·9	78·3	77·5	*79·3	*46·0
	23	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	24	81·5	78·0	80·0	80·0	75·0	71·7	*73·5	*73·0	*73·3	*74·0	*79·3	*38·5
	25	67·0	75·7	77·3	75·5	73·7	77·5	76·8	78·3	78·0	*63·0	*68·5	*72·5
	26	78·0	79·0	80·5	77·5	79·0	78·7	78·1	79·5	82·2	79·4	77·7	*157·7
	27	74·5	76·8	77·0	79·8	76·5	79·0	84·0	82·9	76·5	80·0	*83·3	*83·2
	28	79·8	79·5	79·7	79·5	81·0	81·3	79·3	82·0	81·5	81·3	81·3	*81·0
	29	81·0	79·5	80·3	79·7	78·5	77·5	80·5	79·0	82·5	82·0	84·0	83·5
	30	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	31	38·5	61·2	77·7	84·0	78·7	79·7	84·0	80·8	81·1	80·7	78·5	80·0
November.	1	73·4	77·8	77·5	82·2	81·5	80·3	82·1	82·5	83·0	81·7	82·5	*80·0
	2	72·7	80·6	81·8	79·0	81·1	79·3	75·7	82·3	*82·0	*80·3	*80·5	*90·7
	3	78·6	80·0	79·5	81·0	81·5	81·1	80·8	81·0	81·5	79·7	81·7	*79·3
	4	77·4	78·3	78·3	76·0	79·0	78·3	77·5	79·0	78·0	79·5	78·5	79·3
	5	77·3	82·9	72·0	78·7	78·1	79·3	80·3	79·5	78·0	*78·7	*78·5	78·7
	6	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	7	77·5	77·0	77·5	79·5	79·0	78·7	79·5	81·0	*79·0	*77·0	*63·3	*67·3
	8	93·0	75·0	74·0	79·3	75·9	80·5	80·5	79·7	79·6	79·5	*78·3	*67·5
	9	51·3	72·0	75·0	75·5	78·5	82·9	65·5	*83·5	*68·5	*68·6	*62·1	*90·0
	10	77·0	75·0	79·0	77·0	79·0	74·1	*77·2	*75·3	*85·0	*76·0	*80·0	*(a)
	11	79·3	80·0	77·5	79·7	72·5	82·5	*75·5	80·4	84·3	80·7	79·5	*78·4
	12	78·0	83·5	83·0	77·0	84·0	80·0	85·0	82·5	81·8	78·8	80·2	80·3
	13	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	14	80·0	80·5	79·7	80·5	80·0	81·5	80·9	79·1	80·4	76·5	88·0	78·0
	15	81·5	80·5	76·5	80·5	81·5	80·5	81·8	80·3	80·0	*80·4	*73·2	92·0
	16	76·5	74·5	75·3	82·0	82·7	80·5	82·0	79·0	*73·3	*83·7	*155·0	*61·0
	17	87·7	82·1	79·3	80·9	80·0	82·5	80·7	80·5	80·0	65·5	85·0	79·7
	18	79·0	79·0	78·8	77·6	79·5	80·7	79·0	*76·3	77·7	*77·5	*83·5	*98·5
	19	70·7	81·0	78·0	79·0	79·0	80·4	84·0	80·7	78·0	79·5	76·8	94·6
	20	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	21	80·3	80·1	79·2	84·0	81·7	77·9	81·7	77·5	*79·5	84·1	85·5	83·5
	22	79·0	82·7	72·5	78·8	81·5	81·5	78·5	90·7	83·7	82·0	82·0	80·0
	23	79·7	80·0	80·5	80·0	80·0	83·0	79·7	84·1	80·0	75·2	83·0	80·3
	24	81·5	78·1	78·5	75·3	77·5	*77·0	*50·0	*80·0	*76·5	*83·0	*81·1	*60·0
	25	81·3	83·0	80·5	80·9	78·0	76·3	80·3	80·0	*78·0	*119·0	*86·6	78·0
	26	80·5	81·6	81·6	81·7	82·5	82·8	83·5	83·8	82·9	77·5	80·0	77·5
	27	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	28	85·8	74·5	81·5	85·3	78·0	*82·1	*77·0	*85·0	81·5	*82·4	*79·5	*79·2
	29	76·7	77·1	80·0	82·3	77·6	82·5	78·7	79·0	81·5	*82·5	*82·0	*82·5
	30	79·5	80·9	78·0	82·0	79·0	*81·9	*80·4	*79·7	83·6	83·0	83·0	*81·5

Observations noted with a * were made when the Aurora was visible. (a) Off Scale East.

Point Barrow, 1853; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.	Mean Point Barrow Time.	
sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	D.	
												1	October.
												2	
												3	
												4	
												5	
												6	
												7	
												8	
												9	
												10	
.....	*78·5	*80·3	*82·3	*83·3	83·3	85·5	89·0	84·3	80·3	83·0	83·5	11	
*80·3	*81·5	*79·5	*83·5	*83·0	83·0	83·5	83·7	84·0	84·0	82·5	83·0	12	
76·5	72·7	*78·8	85·5	88·7	87·3	101·5	136·7	121·3	125·7	82·7	72·5	13	
79·7	79·1	75·5	82·0	83·5	94·3	94·7	97·3	102·3	93·7	78·5	73·8	14	
81·6	92·3	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	15	
.....	78·0	93·7	84·5	80·5	85·3	85·0	86·3	88·0	87·0	97·5	97·0	16	
79·5	84·7	75·5	63·7	93·3	81·5	95·0	77·5	83·5	83·5	82·0	79·7	17	
64·0	81·0	80·5	83·5	82·3	84·0	84·5	77·3	79·8	80·7	85·0	80·5	18	
77·0	76·3	*90·0	*76·0	*91·0	*104·0	92·5	92·5	87·7	85·0	85·0	80·0	19	
64·7	*67·5	*85·0	*123·0	99·5	97·7	91·3	91·3	85·3	73·0	79·0	75·0	20	
*76·5	94·5	88·0	84·5	87·3	86·3	81·0	82·0	82·3	81·5	85·5	81·3	21	
*74·3	*67·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	22	
.....	*117·3	86·3	80·7	84·7	86·0	88·0	87·3	87·5	70·0	71·0	79·7	23	
*70·5	*43·5	*68·5	*67·3	101·0	100·5	125·7	138·9	109·4	80·3	85·0	74·0	24	
*82·7	*81·5	*84·3	*83·3	82·3	81·3	80·5	82·7	83·7	82·5	85·9	79·3	25	
*49·7	*55·0	82·0	86·7	92·0	80·8	93·0	86·5	98·1	87·6	77·0	74·0	26	
*61·5	*79·3	90·3	86·5	87·0	89·5	82·0	86·7	86·5	85·1	80·3	81·3	27	
*81·0	*81·3	*81·3	81·3	88·0	85·7	84·4	91·9	90·1	84·3	82·3	83·5	28	
*79·5	83·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	29	
.....	*75·5	*73·0	*90·5	*82·0	*136·3	*122·2	(a)	59·1	113·5	100·0	59·0	30	
77·5	86·3	70·3	89·3	86·0	91·5	96·3	102·0	107·5	61·9	72·4	81·4	31	
*68·5	79·1	84·3	84·0	84·0	81·3	88·3	111·3	107·4	110·5	67·0	73·3	1	November.
*88·7	*80·9	*85·5	*84·1	84·5	84·1	83·4	85·0	99·5	84·0	81·5	82·2	2	
*82·7	*81·0	*84·7	*90·0	*93·0	*92·4	89·4	89·4	81·0	81·5	79·4	78·8	3	
80·0	78·3	78·7	78·5	79·7	79·5	80·5	82·7	84·5	84·0	78·9	77·7	4	
78·5	79·4	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	5	
.....	*81·4	*82·7	92·4	89·4	83·5	87·1	83·1	81·2	80·3	81·0	75·5	6	
*68·3	*77·5	*81·5	*79·3	*82·0	*83·5	*82·8	*90·3	120·1	119·0	118·0	63·5	7	
*62·5	76·5	126·5	75·6	103·0	83·7	108·0	121·5	100·0	62·4	76·7	82·1	8	
*76·6	*68·5	*3·7	*81·0	*110·5	*111·3	*99·4	106·5	97·5	66·5	70·0	72·3	9	
*100·0	*82·0	*98·5	*108·1	*83·8	*84·4	*88·0	89·5	73·5	80·0	78·7	74·7	10	
79·3	76·0	83·0	*85·0	*96·0	*91·0	*81·6	85·0	90·0	78·7	87·0	85·0	11	
78·0	78·2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	12	
.....	85·2	39·0	82·8	79·0	83·2	83·0	84·5	84·2	84·0	81·0	80·5	13	
79·5	82·0	82·5	87·5	79·8	98·5	89·0	86·2	81·0	83·6	79·3	80·5	14	
82·7	81·5	*85·5	87·5	*98·5	87·5	83·7	72·1	84·3	84·5	82·7	75·9	15	
*80·5	*82·3	76·5	116·7	108·5	92·0	99·0	84·5	83·7	87·8	79·0	79·5	16	
85·0	77·9	82·5	99·0	93·5	97·7	91·7	89·4	92·0	87·5	80·0	75·0	17	
*79·5	*89·3	*82·3	*94·5	84·6	83·3	81·0	90·0	89·0	95·4	82·7	69·7	18	
80·3	85·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	19	
.....	82·9	85·0	84·9	85·5	87·5	87·5	97·7	84·0	85·0	81·6	69·0	20	
80·0	*111·5	*99·3	82·5	85·9	82·3	98·3	84·0	87·3	95·0	94·0	53·5	21	
81·0	82·6	84·9	88·5	93·7	106·0	99·5	101·8	79·4	82·7	84·7	80·5	22	
81·0	80·5	87·0	84·7	86·0	83·0	88·6	85·5	82·5	83·5	89·0	84·1	23	
*44·0	82·0	88·3	85·5	87·7	86·0	82·0	83·0	*88·5	83·5	81·7	80·5	24	
*81·2	80·0	81·7	81·3	83·8	83·0	82·3	86·0	80·7	80·8	91·6	79·0	25	
80·0	*85·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	26	
.....	*52·0	*69·0	79·5	86·0	97·5	*104·7	*101·6	94·6	82·0	83·0	85·5	27	
*81·8	*81·0	*85·0	86·7	88·2	81·5	84·0	84·0	84·5	89·5	81·0	81·0	28	
*82·0	*82·2	*85·0	*88·5	*84·9	85·2	85·0	86·7	95·7	89·5	91·7	88·0	29	
*78·7	*55·5	*77·5	*87·1	*86·5	*85·6	*85·0	*86·1	88·3	87·2	81·0	81·8	30	

(a) Off Scale East.

Point Barrow, 1853–1854; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

Mean Point Barrow Time.	D.	0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.
		sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.
December.	1	80·8	81·5	81·3	82·5	82·0	82·0	*81·1	*79·3	*76·3	*77·0	*99·0	*82·0
	2	80·0	77·7	81·3	79·4	85·3	84·0	80·5	82·3	81·0	81·6	*95·0	*63·5
	3	79·8	77·0	73·5	79·7	83·3	82·1	81·6	79·0	*80·5	*76·3	89·3	*70·3
	4	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	5	75·3	77·5	73·5	74·0	76·5	81·3	*81·0	*80·3	*79·3	*76·0	*97·3	*77·4
	6	55·5	60·0	75·7	76·0	77·9	76·9	79·0	82·0	81·8	80·3	81·6	*75·5
	7	77·5	78·2	76·3	78·4	80·3	82·5	75·7	77·5	76·5	82·5	79·7	80·5
	8	82·5	82·5	77·5	80·0	82·5	80·8	77·0	81·4	80·0	84·0	77·7	*78·5
	9	78·8	76·5	79·0	76·0	81·0	81·5	80·3	81·6	79·7	82·5	*79·7	*84·5
	10	79·3	77·7	78·3	78·7	80·5	81·5	*79·0	81·3	80·7	*79·5	*82·6	*79·7
	11	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	12	76·5	77·0	80·5	79·0	80·0	80·8	*79·0	*71·5	*71·7	*65·5	*49·0	*63·5
	13	77·5	79·5	71·5	82·7	79·5	*85·0	*79·3	*78·5	*71·7	*97·0	85·3	84·7
	14	77·0	79·3	81·5	81·5	80·0	81·0	82·0	81·5	81·8	79·0	78·2	81·7
	15	74·6	77·5	78·4	77·6	79·8	85·0	83·1	82·0	78·5	84·6	81·0	86·0
	16	75·7	79·7	77·0	79·8	80·8	82·6	80·5	*73·5	82·4	82·3	73·0	*79·5
	17	78·7	77·5	76·7	76·5	78·0	85·0	76·3	72·0	*70·8	87·6	79·3	77·0
	18	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	19	74·0	79·7	86·8	81·3	81·5	73·7	81·3	82·0	82·0	82·5	*84·0	*82·8
	20	80·0	83·0	82·0	84·3	77·5	77·2	79·5	81·0	76·7	83·5	96·5	84·5
	21	74·0	72·0	76·5	76·7	78·1	*76·2	*74·5	*80·0	*79·2	*78·5	*77·8	*71·5
	22	74·7	76·3	78·3	68·5	74·0	76·3	76·0	81·1	66·0	87·0	*77·0	*(a)
	23	93·5	75·0	81·6	80·0	*70·5	*82·5	*81·0	*83·0	*82·0	82·6	*100·0	*88·0
	24	73·5	72·7	77·0	78·0	*86·3	*74·0	*84·5	*85·0	*77·5	*77·5	*88·0	*80·5
	25	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	26	82·0	84·7	78·5	80·0	80·5	83·0	80·5	84·3	*82·0	*82·9	*87·8	*83·4
	27	78·3	81·0	82·5	80·0	82·0	82·7	83·5	84·3	82·5	82·5	83·6	82·3
	28	79·0	80·3	81·0	75·0	77·6	78·7	*77·5	*85·0	*82·7	*78·0	*78·0	*72·5
	29	82·3	80·5	84·5	82·5	80·0	84·0	83·7	82·3	83·5	81·0	*84·3	*82·3
	30	83·4	79·8	78·3	80·5	81·1	*87·0	*77·5	*82·5	*78·0	*73·7	*84·9	*88·0
	31	73·5	77·5	80·0	82·7	83·5	*80·3	*81·7	*81·0	*85·5	*79·7	*80·1	*101·0
January.	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	2	51·7	82·5	77·5	90·0	*67·5	*69·0	*87·5	*79·0	*88·5	*76·5	*68·7	*109·0
	3	83·9	79·3	86·0	81·4	80·3	92·0	80·0	78·5	85·5	79·0	80·5	*85·2
	4	81·5	75·0	78·0	84·3	78·7	79·5	85·0	82·5	*84·0	*85·0	*82·0	*80·0
	5	81·5	81·0	81·5	81·7	78·1	79·5	81·7	79·4	80·0	82·0	84·0	83·5
	6	80·7	81·0	79·5	79·7	80·5	81·5	82·0	81·1	83·3	83·7	81·7	78·5
	7	70·9	73·7	78·3	81·0	81·5	*73·0	*74·0	*74·0	*74·0	*76·5	*137·0	*73·0
	8	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	9	80·0	81·3	85·0	84·0	84·3	84·0	84·0	86·7	87·7	89·5	84·5	81·0
	10	85·7	86·5	85·5	87·5	86·0	85·6	86·5	85·0	85·3	85·3	86·3	84·5
	11	82·7	86·3	83·7	81·5	82·9	83·0	83·5	84·0	87·3	88·7	75·5	78·5
	12	73·0	80·0	74·5	77·1	85·6	82·3	83·4	79·0	(b)	.....	.....	.....
	13	.....	.....	.....	.....	.....	.....	.....	.....	83·0	80·5	79·5	110·5
	14	76·5	80·0	74·5	77·5	82·0	76·5	74·0	80·0	*80·8	79·0	76·4	80·0
	15	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	16	76·0	82·5	86·0	84·5	74·9	91·7	90·5	80·5	78·0	69·5	95·7	79·0
	17	73·5	78·0	62·0	91·7	83·1	89·5	*82·0	84·4	86·3	85·3	87·0	82·0
	18	78·0	79·0	78·9	78·2	82·9	81·0	83·7	*81·8	*88·2	*82·0	84·9	81·2
	19	92·5	77·5	76·7	77·5	79·4	78·0	76·0	*82·5	91·0	80·5	87·3	80·5
	20	92·5	80·3	82·0	85·5	81·5	89·5	88·4	82·0	90·0	81·5	91·5	82·7
	21	87·0	73·5	81·5	86·1	87·0	88·8	88·1	87·5	89·7	85·5	*90·5	*92·5
	22	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	23	80·8	68·3	74·3	77·0	78·0	*83·5	*78·5	*81·3	*83·0	*78·5	*82·0	*80·0
	24	73·0	75·3	77·5	76·8	78·0	85·3	77·8	84·0	*80·5	*81·5	81·4	81·0
	25	77·5	74·0	76·6	78·5	82·0	70·0	79·1	74·0	72·3	72·5	79·5	72·0
	26	80·0	79·0	78·5	79·5	80·0	80·0	80·5	81·0	79·6	*80·0	80·0	81·0
	27	78·0	79·3	80·0	81·5	81·7	81·5	81·0	82·2	*82·0	*81·2	*81·5	*77·5
	28	85·0	82·0	66·5	72·0	80·0	77·0	80·0	84·5	82·0	67·0	68·0	*80·3
	29	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	30	75·0	74·0	76·0	73·0	76·0	80·0	85·0	*85·0	*82·0	*83·5	*83·0	*81·0
	31	84·6	76·0	80·0	81·3	84·0	83·0	82·0	*82·3	*83·5	*82·5	*128·0	*78·0

Observations noted with a * were made when the Aurora was visible. (a) Off Scale West.
(b) Observatory blocked up by the snow-drift raised by a heavy gale from the S.W.

Point Barrow, 1853–1854; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.	Mean Point Barrow Time.	
sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	D.	
78·6	*80·0	*80·0	*81·5	*83·5	*84·5	*86·7	*92·3	88·9	86·7	82·6	82·6	1	December.
63·0	81·5	92·5	86·4	86·5	87·5	*100·3	*90·0	85·5	98·6	83·5	83·5	2	
*65·5	80·2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	3	
.....	*84·5	*82·0	*90·0	*99·6	*89·8	*81·5	*78·6	80·8	83·5	81·9	72·5	4	
*74·4	61·5	*70·5	*84·0	*93·5	93·0	114·6	(a)	100·0	99·0	70·0	66·5	5	
*81·1	*73·3	*83·0	*80·5	*84·5	82·0	74·0	75·0	76·0	90·5	74·0	88·0	6	
81·0	79·7	*92·0	*91·1	*80·5	*89·7	*90·5	*102·0	*109·4	84·5	83·0	73·5	7	
*77·0	*77·0	*62·5	*86·8	*82·0	*84·5	*84·0	*82·5	83·5	82·3	79·5	84·4	8	
*85·0	*77·0	*84·5	*82·5	*82·0	87·9	80·0	79·3	82·0	83·0	81·5	80·7	9	
*78·5	*68·7	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	10	
.....	*79·5	*80·3	83·0	83·5	86·5	82·7	*92·3	*78·6	82·5	79·5	83·0	11	
*73·0	*104·5	*122·0	85·3	*136·0	*71·2	*80·1	*39·0	*75·6	127·0	*57·5	77·5	12	
80·0	75·5	75·5	80·6	79·0	85·0	77·5	83·0	81·5	82·5	80·5	74·5	13	
49·0	45·0	85·3	83·9	89·9	84·5	93·0	105·5	83·5	85·2	79·5	79·5	14	
66·3	*41·7	*66·5	*72·3	*100·7	*107·3	*102·7	75·3	92·0	84·5	83·5	81·0	15	
*76·0	*105·3	*105·0	*88·0	*85·0	80·8	89·0	83·0	83·2	92·3	88·5	80·3	16	
61·0	*61·9	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	17	
.....	*78·3	*85·8	84·5	*88·7	*88·5	*90·5	*84·0	*96·5	97·6	100·0	84·0	18	
*56·0	63·5	84·8	96·3	91·3	89·5	86·0	90·5	87·5	77·0	77·0	79·5	19	
76·0	*79·0	*116·5	*95·6	129·7	*78·0	*113·0	*84·4	90·6	*110·7	92·7	78·0	20	
84·2	*(b)	*85·5	80·0	84·0	93·0	93·0	98·0	100·0	98·0	84·7	76·0	21	
*94·0	83·0	*77·7	*89·7	*81·5	*84·5	*88·5	*88·5	*85·0	*114·1	*89·8	70·0	22	
*66·5	*118·2	*55·0	*80·3	*87·5	*93·3	*91·3	107·7	*129·7	*108·0	82·5	81·5	23	
*88·0	*83·2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	24	
.....	*88·0	*79·5	*85·0	*80·5	*87·7	*81·0	*85·0	*89·0	*84·0	87·0	87·5	25	
83·0	*79·9	*81·7	*81·7	*79·9	*81·0	*84·3	*85·0	*86·0	*84·3	85·0	85·0	26	
82·5	83·3	83·8	88·5	104·3	116·5	*113·3	*88·7	89·3	*95·7	81·0	80·5	27	
*87·0	*86·5	*86·5	*83·6	*93·5	*94·3	*130·3	*110·0	*77·5	76·5	81·0	77·4	28	
*77·5	*77·0	*71·5	*102·5	*87·5	*85·7	*86·0	*83·3	*85·0	84·0	85·0	82·5	29	
*87·0	*85·0	*75·7	*84·0	*77·4	*95·0	*87·5	*100·0	*94·5	80·5	77·5	85·0	30	
*54·6	*80·1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	31	
.....	*72·0	*108·0	*77·5	*99·1	*102·5	*149·0	*(a)	*90·8	80·3	97·0	86·0	1	January.
83·0	96·5	91·0	71·0	87·5	91·8	120·5	100·3	116·3	104·5	74·5	78·7	2	
*79·0	*77·3	*75·0	*62·5	*87·5	*85·5	*95·0	*86·5	*88·0	88·0	80·0	81·0	3	
*113·0	*75·6	*81·5	*88·0	*89·0	*88·7	*88·5	*102·0	*100·7	*74·5	80·0	77·5	4	
80·0	80·5	83·5	85·0	*82·5	*90·0	*87·5	91·5	78·7	81·5	82·5	81·5	5	
*79·7	*82·5	*90·5	*87·0	*90·0	*87·0	*93·0	*83·0	*95·5	91·1	103·5	79·0	6	
*78·3	*59·1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	7	
.....	*73·7	*89·0	*98·3	86·0	*90·7	*92·5	*89·5	*92·7	86·0	85·7	81·5	8	
79·0	76·0	79·5	97·3	88·0	89·0	91·1	90·5	90·8	88·5	86·0	83·5	9	
93·5	82·5	90·5	105·0	100·5	97·0	86·3	91·0	94·0	88·5	83·7	81·4	10	
51·0	89·5	82·5	102·0	108·4	89·0	95·0	105·5	112·5	85·0	110·0	62·0	11	
.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	12	
97·5	86·3	85·4	74·5	103·5	93·5	90·0	112·0	97·0	78·5	75·5	82·0	13	
76·3	86·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	14	
.....	82·5	85·0	87·5	*149·0	*106·5	*113·5	*75·5	*76·5	110·0	87·5	77·0	15	
84·0	85·3	131·0	86·1	92·2	85·3	85·7	97·0	90·5	84·7	121·5	112·5	16	
85·0	80·5	103·5	101·0	83·3	97·4	98·0	107·0	97·0	90·0	80·8	91·0	17	
*78·6	*142·0	*75·0	*77·2	81·8	87·0	142·0	127·8	83·0	75·6	97·0	104·0	18	
115·0	102·0	100·7	101·5	101·5	100·5	97·5	96·0	110·0	85·0	74·0	64·5	19	
70·0	73·0	89·5	95·3	101·0	90·5	95·0	107·7	81·0	89·0	85·0	86·0	20	
88·0	*89·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	21	
.....	76·5	*80·0	*77·0	*72·5	*92·0	*90·5	*95·5	*102·5	*105·5	128·0	70·0	22	
*76·5	*80·3	40·5	65·7	102·5	97·5	89·0	102·7	95·5	95·0	86·7	80·6	23	
80·6	80·0	81·0	81·7	81·5	82·3	78·3	86·3	81·0	83·5	85·0	83·0	24	
81·0	81·5	78·1	83·6	84·0	83·5	83·6	85·1	85·8	81·0	80·0	79·0	25	
80·0	*75·0	*80·5	*86·7	*84·0	*82·2	*86·3	*79·5	81·8	78·0	78·2	78·0	26	
*81·0	*80·0	*84·5	*93·0	*88·0	*79·0	*84·0	*85·5	83·0	87·5	84·5	98·3	27	
*55·9	*76·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	28	
.....	82·0	107·5	75·5	84·0	79·2	92·5	107·3	127·7	95·5	144·0	74·0	29	
*82·5	*67·5	*77·5	*75·0	*82·0	*88·5	*98·0	*87·0	94·5	100·0	85·3	76·0	30	
*77·0	*80·5	*82·5	*83·3	*88·0	*79·0	*91·5	*87·0	*89·5	94·0	81·5	81·3	31	

(a) Off Scale East. (b) Off Scale West.

Point Barrow, 1854; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

Mean Point Barrow Time.		0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.
	D.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.
February.	1	82·0	78·6	79·6	77·0	84·5	85·0	83·5	*81·5	*83·0	*85·0	*87·0	*83·3
	2	82·0	86·0	80·0	83·0	78·2	86·4	83·8	84·0	*84·5	*85·0	*84·0	*85·0
	3	81·5	82·0	82·0	78·0	80·0	81·3	*77·0	*85·0	*83·0	*79·7	*82·0	*82·0
	4	78·7	79·5	66·5	79·7	80·3	79·0	76·5	*80·0	85·0	76·0	75·2	135·0
	5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	6	81·0	76·4	86·5	84·1	84·1	83·3	84·0	*85·0	*83·5	*83·0	*83·0	*81·5
	7	81·0	80·5	81·0	83·0	82·0	79·3	80·0	*84·0	*130·0	*88·0	*76·0	*72·0
	8	81·0	85·0	86·5	88·0	83·5	84·5	84·5	85·2	84·8	83·2	84·5	80·0
	9	71·0	85·0	71·0	79·5	85·5	84·0	81·0	*80·0	*86·0	84·3	*75·0	*154·0
	10	54·5	75·5	85·0	82·0	84·5	82·2	84·0	83·0	77·0	*76·5	*32·5	*105·0
	11	74·0	85·0	81·0	80·0	81·0	80·0	91·0	85·0	84·0	81·5	78·0	84·0
	12	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	13	73·0	80·0	77·2	76·0	77·0	72·0	78·5	87·5	85·0	74·8	66·9	59·5
	14	71·5	76·3	80·0	74·5	71·2	74·6	*73·0	*82·0	87·0	*73·0	*72·5	*89·0
	15	82·0	72·0	76·5	72·0	83·0	80·0	80·0	*86·8	*76·0	*94·7	*75·0	*84·0
	16	77·5	84·7	83·5	67·0	89·0	66·0	75·0	72·5	81·0	91·5	85·0	62·0
	17	74·5	81·7	78·0	78·0	88·0	81·5	87·0	84·5	80·0	83·0	88·5	*81·0
	18	77·0	79·5	76·0	78·3	79·3	84·2	78·0	*81·0	85·0	*82·0	*81·0	*83·5
	19	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	20	84·0	82·0	84·0	84·5	84·5	82·5	83·0	*82·7	*82·0	*85·0	*83·0	*81·0
	21	85·0	81·0	79·0	80·0	84·5	81·5	81·7	78·0	80·5	84·3	84·0	83·0
	22	81·0	83·0	84·0	84·1	83·5	83·0	84·0	81·5	80·0	82·0	82·5	*81·5
	23	79·0	78·5	83·0	84·0	83·0	83·5	83·0	84·0	83·0	76·5	80·0	*78·0
	24	65·5	82·0	77·5	71·4	63·8	86·3	86·0	82·5	*83·7	*77·5	*79·3	*78·3
	25	75·5	76·0	81·0	80·7	83·5	56·0	94·5	*89·0	*71·0	*97·0	*87·5	*76·5
	26	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	27	75·5	82·5	60·5	68·5	70·5	90·0	76·0	82·0	85·0	92·0	81·0	71·0
	28	82·3	73·5	77·0	74·5	81·0	74·0	76·0	76·3	80·5	80·5	*84·0	*90·0
March.	1	80·5	79·0	89·6	84·5	84·5	81·5	79·0	78·5	*84·0	*82·5	*86·0	*82·2
	2	82·0	85·0	78·0	88·0	75·0	74·6	82·4	80·0	82·5	85·0	*84·0	79·0
	3	104·0	77·5	83·0	84·0	84·0	81·3	84·2	82·0	94·0	*87·0	*84·5	*74·0
	4	95·0	77·5	85·5	85·0	87·5	89·0	85·0	87·0	89·0	*87·3	*93·9	*83·5
	5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	6	87·5	92·1	81·1	88·0	84·0	87·0	82·0	89·0	81·0	85·0	*90·0	*73·1
	7	80·5	82·0	85·0	82·0	87·0	85·0	87·5	90·0	91·5	88·5	*89·0	81·5
	8	83·0	70·0	83·5	86·0	84·0	86·0	87·5	89·0	*86·3	*85·5	*77·5	*73·5
	9	87·3	86·0	86·0	88·0	87·0	88·0	87·5	88·0	88·0	*89·5	*88·0	*89·0
	10	91·0	89·5	89·0	87·5	88·0	88·0	86·7	85·0	85·8	*77·5	*101·5	*86·0
	11	92·5	90·0	88·0	87·5	86·0	88·2	89·0	89·4	88·5	87·0	88·5	90·0
	12	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	13	92·5	89·0	82·8	82·2	81·6	77·0	84·5	87·0	89·0	82·0	78·0	*27·0
	14	87·0	85·5	86·0	85·0	79·0	68·0	84·2	79·8	86·5	80·0	89·0	84·0
	15	63·0	107·5	133·5	85·0	76·0	49·0	92·0	74·5	*85·0	*98·7	*67·0	*56·0
	16	77·0	74·0	81·0	50·3	81·0	78·0	76·0	92·0	*63·5	*81·0	*78·0	*106·6
	17	86·0	81·0	78·0	75·3	85·2	81·0	80·0	86·0	84·0	*79·0	*79·0	*89·0
	18	84·0	82·0	81·3	85·0	85·3	85·0	84·6	87·5	85·5	86·0	85·0	*78·0
	19	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	20	80·0	81·5	85·5	81·5	83·0	84·0	82·5	83·0	85·2	*84·0	*82·0	*71·3
	21	81·0	76·0	83·2	78·0	79·0	80·0	78·5	84·0	81·0	*85·0	*84·0	*83·3
	22	74·0	80·0	82·5	81·0	81·5	82·0	81·0	81·0	82·3	83·0	*82·7	*83·0
	23	89·0	84·0	83·0	82·0	82·0	84·0	81·2	84·5	83·8	*80·0	*86·2	*79·0
	24	81·0	72·0	84·0	85·2	83·2	82·5	82·5	81·0	82·3	81·1	82·7	*78·6
	25	79·7	70·0	72·0	75·4	82·4	75·5	80·5	80·5	81·0	*78·0	*67·0	*106·5
	26	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	27	89·4	101·0	93·0	88·0	76·0	76·0	66·9	56·6	52·5	65·0	74·5	38·5
	28	81·7	65·3	75·0	67·5	65·5	66·0	71·0	76·6	72·4	*83·5	*80·0	*79·0
	29	76·0	79·0	79·2	81·2	75·3	83·5	77·2	80·0	79·0	78·5	*88·3	*77·0
	30	83·5	71·5	86·0	66·0	76·8	70·4	68·4	76·5	79·1	69·0	*65·0	*60·0
	31	80·5	81·5	81·0	79·0	80·0	81·5	81·0	82·2	77·0	80·5	*83·8	*73·8

Observations noted with a * were made when the Aurora was visible.

Point Barrow, 1854; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.	Mean Point Barrow Time.	
sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	D.	
*84·5	*81·2	*86·2	*85·0	*85·5	*90·0	*99·5	84·0	84·5	86·0	82·0	82·0	1	February.
*85·0	*85·7	*86·0	*89·0	*84·0	*86·6	*87·0	*87·4	87·0	87·0	87·0	87·0	2	
*78·0	*79·0	*75·0	*110·0	*100·0	*90·5	*85·0	*91·5	100·4	112·0	74·0	71·5	3	
70·0	91·7	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	4	
.....	*80·0	*72·0	*82·5	*112·0	*112·5	*112·0	*85·0	82·5	82·0	81·5	84·0	5	
*80·0	*83·5	*96·5	*96·8	*86·0	*87·0	*93·0	*90·5	117·0	91·0	78·0	78·0	6	
*79·0	*87·0	*95·0	*90·0	*87·5	*99·5	*88·5	*90·5	83·9	83·3	90·5	83·4	7	
*93·0	*81·5	*98·0	*91·5	90·0	*91·0	*90·0	*94·0	85·0	83·7	103·1	91·1	8	
*(a)	*48·0	*72·5	*105·0	*89·0	*116·0	*140·0	*112·0	88·0	(b)	123·5	129·5	9	
*50·0	*80·5	*143·5	*94·0	*89·0	*108·3	*112·5	*(b)	129·7	84·0	55·0	80·0	10	
70·5	90·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	11	
.....	81·5	81·0	90·0	72·0	88·0	90·0	95·0	112·5	136·0	82·3	116·0	12	
*80·5	87·0	82·0	*87·0	*122·4	*102·0	*95·0	79·5	117·3	125·0	107·5	101·5	13	
*93·5	*77·0	*110·0	*107·5	*81·0	*81·0	*130·0	(b)	105·2	81·5	38·0	86·0	14	
*80·5	*67·5	*46·0	*80·5	*72·0	*72·0	*102·0	*117·6	*117·6	71·0	68·0	75·5	15	
80·0	81·5	109·0	67·0	77·5	95·0	98·0	111·5	111·0	86·4	83·0	70·0	16	
*126·5	*87·5	*61·0	*112·4	95·0	*91·0	87·5	84·0	97·0	100·0	92·0	83·0	17	
*84·0	*79·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	18	
.....	*79·0	*85·0	*86·0	*85·5	*92·3	*88·0	*85·0	85·0	85·0	85·6	83·8	19	
*87·2	*85·5	*85·0	*84·0	*92·0	*94·0	*95·0	94·0	90·6	91·0	88·0	87·3	20	
*83·2	*83·4	*89·5	*86·0	*97·0	*87·5	*80·0	88·5	85·7	82·5	90·0	82·0	21	
*82·0	*83·5	*97·7	*81·5	*85·0	86·5	84·5	84·0	92·0	102·0	91·0	87·0	22	
*68·5	*76·5	*77·0	*83·0	*79·0	*110·7	*98·0	116·0	(b)	113·0	97·0	94·0	23	
*65·0	*69·4	*93·0	*96·6	*(b)	*(b)	97·5	87·8	94·0	94·5	83·0	74·0	24	
*68·5	*75·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	25	
.....	80·0	77·7	83·2	69·5	99·5	102·5	90·0	109·1	102·0	98·7	80·0	26	
95·3	89·0	96·5	93·0	87·0	92·0	99·5	102·7	86·2	85·0	77·8	85·0	27	
85·0	*73·5	*78·5	*84·5	85·2	*105·0	96·0	99·0	101·5	108·0	92·1	86·0	28	
*79·0	*73·0	*86·0	*86·2	*85·4	*84·0	86·8	95·0	96·5	106·0	87·0	89·0	1	March.
81·5	83·0	82·0	111·1	79·0	87·5	97·0	99·0	99·5	76·0	80·5	77·5	2	
*88·0	*77·5	*103·5	*101·0	94·0	99·0	95·9	100·0	83·5	86·0	93·0	77·0	3	
*88·2	*88·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	4	
.....	*89·0	*97·5	115·7	90·0	91·0	87·5	102·0	113·3	108·0	72·0	71·5	5	
*75·0	*85·3	*83·5	*98·5	*106·0	*82·0	*102·0	105·5	105·0	112·0	85·5	76·0	6	
85·5	*90·0	*86·0	*88·0	*92·0	*88·5	92·0	94·3	111·7	92·0	90·0	89·8	7	
*(b)	*81·5	87·0	90·5	*81·3	87·0	89·0	99·0	121·0	77·0	85·5	85·0	8	
*94·5	*89·7	*86·0	94·5	93·6	92·8	94·5	91·0	89·5	90·0	91·0	90·5	9	
*101·5	93·0	92·0	95·2	91·0	93·4	92·0	94·0	101·5	91·0	90·5	89·0	10	
91·8	84·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	11	
.....	89·0	93·5	91·5	94·8	92·0	98·0	103·5	108·0	112·0	116·6	81·0	12	
*81·0	89·2	85·5	85·5	91·9	103·5	94·3	98·5	91·5	89·0	87·0	88·0	13	
*86·0	*74·0	*80·0	*93·0	*86·0	89·6	91·6	91·0	87·2	124·0	138·3	83·3	14	
58·5	*89·0	*99·5	*88·0	*87·0	130·5	106·5	157·5	145·5	110·0	155·3	106·0	15	
*86·0	*67·5	*116·4	*88·0	*92·1	87·0	88·0	92·0	99·0	92·0	80·0	91·0	16	
*83·6	*32·5	*69·5	*87·5	*90·5	89·5	84·0	89·0	92·0	91·0	90·0	89·0	17	
*75·5	*88·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	18	
.....	*89·0	*85·0	*89·5	*93·5	108·0	88·0	93·5	96·0	100·0	103·0	88·0	19	
*76·9	*84·5	*75·5	*79·0	90·0	96·0	102·0	103·0	93·0	96·2	95·5	84·3	20	
*77·8	*85·0	*86·7	*87·5	86·0	79·0	80·7	118·0	111·5	99·0	87·5	86·0	21	
83·0	83·0	81·0	89·2	98·0	86·7	92·3	93·0	101·0	96·5	82·0	81·5	22	
*84·8	*81·0	*85·0	*88·0	86·0	87·2	90·8	87·0	105·5	106·7	94·0	77·0	23	
*77·6	*82·0	*90·5	*106·8	114·3	89·5	126·0	108·0	87·5	120·0	106·3	78·2	24	
*113·6	*86·6	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	25	
.....	*120·5	*85·2	*90·0	107·5	91·6	87·1	70·8	93·2	90·5	85·3	86·0	26	
118·0	102·0	98·0	68·5	123·5	56·7	116·9	101·3	108·9	90·6	55·8	99·6	27	
*78·0	*74·5	*81·7	*91·0	88·5	85·0	90·5	103·5	89·5	108·3	91·7	80·0	28	
*79·0	*64·1	*49·8	89·5	74·0	90·0	87·5	92·8	134·3	120·0	95·5	90·5	29	
*69·0	*96·0	*81·0	81·7	96·0	96·5	98·0	91·0	104·0	81·3	85·0	81·5	30	
*67·7	*†77·5	*†78·5	*†84·0	†87·5	84·0	90·0	83·7	†95·8	†110·0	79·5	81·0	31	

(a) Off Scale West. (b) Off Scale East. † Magnet vibrating.

Point Barrow, 1854; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

Mean Point Barrow Time.	D.	0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.
		sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.
April.	1	74·5	85·0	85·0	†91·9	†85·5	†82·5	†86·5	87·5	85·7	85·5	83·2	85·0
	2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	3	†79·5	81·5	†87·7	83·0	82·8	82·3	86·0	92·0	81·5	†89·5	†85·5	†84·7
	4	81·6	†85·0	86·5	80·5	82·5	82·3	87·5	85·2	87·0	86·0	84·8	85·0
	5	82·8	86·0	87·8	81·5	86·0	82·5	82·0	85·2	85·2	85·8	85·0	83·0
	6	86·0	84·0	82·8	83·0	82·6	86·0	84·8	84·0	80·5	79·0	88·0	86·0
	7	85·0	†84·3	85·0	*83·0	84·0	79·0	79·5	78·0	79·0	85·0	†*97·5	†*75·0
	8	85·5	85·5	†83·5	†81·5	81·0	†85·8	84·0	85·3	84·6	85·0	85·5	85·2
	9	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	10	86·3	99·0	†109·0	†58·0	90·2	75·2	55·8	60·3	93·5	71·2	40·0	†58·5
	11	†71·0	†91·1	86·0	†81·5	81·0	84·7	85·2	85·2	89·0	83·0	84·0	84·0
	12	†88·3	89·5	80·0	81·0	82·0	87·0	86·0	84·0	84·2	85·2	†79·8	†144·3
	13	84·6	80·5	†80·3	85·0	84·6	84·0	85·0	84·0	79·3	84·2	85·2	85·2
	14	†76·0	68·5	66·5	70·0	86·5	80·0	80·5	81·0	82·0	84·5	79·5	76·0
	15	87·0	79·5	†68·7	85·3	79·8	86·2	77·5	86·5	87·8	87·0	87·0	82·7
	16	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	17	89·0	86·5	86·0	87·5	85·1	84·0	85·5	85·5	88·0	81·5	83·3	83·4
	18	78·3	79·5	80·0	81·3	81·0	82·5	†74·7	†77·5	†64·5	†33·5	70·0	76·3
	19	†94·2	†84·0	†70·2	†84·0	†78·0	†69·5	87·3	†81·6	93·0	†81·3	†111·3	110·7
	20	76·5	78·0	84·0	†77·5	78·0	79·0	82·3	80·5	78·0	84·0	88·0	85·2
	21	76·0	80·5	81·0	78·8	78·0	85·0	90·0	89·7	86·0	85·5	86·0	†78·1
	22	91·0	74·6	†78·5	75·0	77·0	73·0	81·5	83·5	75·0	86·0	73·3	87·0
	23	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	24	†97·4	76·5	†62·0	†63·7	†80·0	†72·8	78·0	†64·5	†73·8	†77·3	85·8	64·0
	25	80·4	†77·1	78·8	†80·3	†75·7	79·3	80·0	80·0	80·3	81·0	†82·5	80·0
	26	†80·0	†77·7	†78·0	†78·5	†80·0	77·2	†76·7	†72·5	75·0	†71·0	†81·3	68·0
	27	†93·0	†68·3	†66·5	†58·5	80·0	84·5	†81·9	†79·3	83·5	96·7	85·0	78·0
	28	†70·0	†77·7	†80·0	†79·0	†80·5	†80·5	82·0	69·0	88·0	95·0	83·5	87·0
	29	†88·7	72·8	†75·0	†78·9	†75·3	81·0	78·5	90·8	67·5	82·5	84·0	90·0
	30	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
May.	1	82·5	79·7	82·0	82·3	78·0	65·0	80·0	83·0	81·5	82·5	82·0	†74·5
	2	83·0	81·0	81·0	80·5	81·0	80·0	81·0	73·0	87·0	†68·0	†68·4	†86·0
	3	85·0	82·9	†84·3	86·0	81·8	81·0	81·0	79·0	82·8	84·0	83·7	81·0
	4	81·0	92·3	93·0	†76·7	77·5	†77·0	†81·9	85·0	70·0	98·5	69·3	†44·5
	5	†75·5	79·5	77·0	81·0	83·0	84·0	83·0	†84·0	83·5	83·0	83·0	83·0
	6	80·5	81·0	81·0	81·0	82·0	82·7	81·5	85·3	84·0	82·5	84·8	82·8
	7	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	8	77·5	70·0	69·0	†77·5	75·2	73·0	†69·7	86·0	†76·0	66·1	53·8	76·2
	9	87·0	84·5	75·5	77·8	78·8	76·0	80·0	80·0	82·8	73·9	86·0	75·8
	10	74·0	81·5	67·3	76·5	80·5	79·0	81·3	†77·5	69·0	83·5	†66·1	†73·3
	11	88·0	80·2	81·0	†78·5	†72·5	77·5	83·0	82·0	87·5	77·3	76·6	80·0
	12	70·1	73·7	†75·5	77·4	79·0	79·0	79·0	81·5	85·3	81·0	75·9	72·3
	13	77·5	79·3	†79·5	79·0	81·1	81·0	81·0	80·0	†81·3	80·8	77·0	†80·0
	14	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	15	†75·0	†54·5	†103·9	†96·5	†81·9	†89·5	†86·0	86·0	86·0	86·0	84·3	81·0
	16	58·0	68·0	†79·8	†88·6	61·5	81·5	85·8	70·0	†102·0	87·0	72·5	42·8
	17	72·0	84·0	(a)	(a)	(a)	82·0	80·0	†84·5	†86·1	90·0	†75·0	†73·5
	18	†83·3	84·0	81·0	81·0	81·5	83·5	84·0	79·0	84·0	80·5	81·0	85·8
	19	75·6	†80·7	†79·6	74·5	80·0	79·2	72·0	79·0	†64·7	81·3	71·0	77·8
	20	†102·0	†72·7	†80·0	†75·4	†83·0	81·5	†80·8	78·0	81·0	76·8	77·0	84·5
	21	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	22	79·5	84·0	82·5	†80·5	†79·5	74·0	80·5	81·3	76·0	83·5	76·5	73·0
	23	80·0	82·0	82·2	†77·6	76·5	78·0	78·0	79·2	80·5	†73·6	80·7	73·0
	24	80·0	77·0	79 6	79·8	79·8	82·8	†83·6	82·0	84·0	78·5	79·3	78·3
	25	69·5	†77·1	†77·9	†80·3	77·0	74·0	77·4	88·5	†76·5	†102·5	†59·0	†76·0
	26	78·0	73·5	73·7	77·2	77·5	67·4	92·0	85·0	79·3	83·3	83·8	70·0
	27	83·0	75·5	82·0	87·5	†83·6	†80·0	†77·0	84·8	84·3	86·5	80·5	85·0
	28	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	29	†84·0	†78·7	87·0	79·7	77·5	79·0	86·0	74·5	84·5	81·0	83·0	78·0
	30	89·0	91·0	†85·5	81·0	81·0	82·8	86·0	75·0	85·2	79·5	84·7	80·0
	31	80·8	82·0	†92·0	83·0	71·0	88·5	77·3	82·5	78·7	†80·5	84·5	86·0

Observations noted with a * were made when the Aurora was visible.
† Magnet vibrating. (a) Unroofing Observatory.

Point Barrow, 1854; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.	Mean Point Barrow Time.	
sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	D.	
†*82·7	†*89·3	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	1	April.
.....	87·0	87·5	85·5	89·6	†101·5	†110·7	94·5	87·8	†90·7	†79·5	†90·5	2	
85·0	†87·5	81·3	91·5	91·5	88·0	98·7	†91·8	97·4	96·7	86·0	87·5	3	
85·0	82·0	82·0	†90·0	†97·0	100·0	112·2	†115·0	†87·4	†83·6	76·7	84·2	4	
84·5	†60·0	90·0	86·8	91·5	†101·7	†88·5	89·0	88·0	89·0	90·5	80·0	5	
83·5	86·0	87·5	89·0	90·3	89·0	88·0	91·8	89·0	89·0	88·0	85·0	6	
†66·0	†*81·3	†*90·5	88·0	84·5	88·0	90·7	92·0	89·7	89·8	91·4	84·5	7	
†*83·8	†*91·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	8	
.....	†83·1	87·5	88·0	89·8	89·0	98·5	†107·0	†136·0	†98·0	76·5	80·0	9	
*(a)	†70·7	87·5	89·2	86·0	100·8	96·5	†103·3	94·8	†91·0	84·5	†103·5	10	
†70·5	96·5	†98·6	88·0	88·0	91·7	93·7	95·4	†93·0	90·5	89·5	86·0	11	
87·5	85·0	88·5	88·5	89·5	80·0	89·3	†102·4	†134·0	94·3	†87·3	†86·5	12	
85·0	†86·3	83·5	82·2	84·0	†98·5	†142·0	(b)	†157·0	(b)	†114·5	†85·0	13	
†*119·7	70·7	†96·2	†91·1	84·6	83·0	85·5	†93·2	79·5	80·0	78·5	84·5	14	
91·5	†88·8	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	15	
.....	†92·5	87·0	†89·5	90·0	90·5	†96·0	97·0	96·0	93·6	88·0	89·0	16	
82·0	86·2	84·5	85·8	85·0	91·0	90·0	90·0	90·3	91·0	86·0	81·7	17	
*120·7	†88·1	75·0	†85·7	105·0	†114·5	†105·3	†141·3	†161·0	(b)	†136·0	†64·7	18	
†66·3	†79·1	†48·0	†90·0	†96·0	100·4	95·5	†97·5	†112·2	†87·5	†80·7	98·3	19	
†52·3	72·0	90·0	†90·5	87·0	88·5	†90·6	†102·5	†116·0	96·0	94·5	85·5	20	
†85·6	81·5	80·0	103·0	113·2	†121·5	102·0	†104·0	†97·5	126·0	†122·7	†85·8	21	
87·0	†89·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	22	
.....	†61·7	72·0	(b)	†79·8	†109·3	†49·5	†89·8	†75·9	†85·5	†75·9	†75·3	23	
†72·0	82·0	78·0	†80·0	†83·5	87·0	88·0	85·0	82·0	81·3	80·3	80·3	24	
†79·5	†70·7	†81·9	†83·0	88·0	88·0	88·5	87·5	†100·0	84·5	†81·5	81·5	25	
69·5	62·0	71·8	93·0	84·5	93·0	†93·0	97·5	†124·3	†81·7	†85·0	90·0	26	
73·5	†82·4	83·0	82·8	91·8	(b)	†80·8	†94·5	†88·5	†88·7	†95·3	†85·4	27	
†80·0	†67·0	†91·5	†89·5	†91·0	84·0	†86·3	†108·5	†103·6	†126·5	94·6	†91·7	28	
84·0	79·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	29	
.....	†44·6	†106·0	†98·7	83·0	†86·3	91·0	89·0	†88·3	81·5	†93·5	91·5	30	
†72·0	82·3	†80·3	88·3	99·5	†126·5	†94·5	†76·7	77·5	†87·7	†85·6	†83·0	1	May.
†67·3	†55·5	†88·5	†87·7	86·0	97·5	90·0	†92·0	89·8	87·8	88·0	85·2	2	
†82·3	71·2	†79·1	†97·5	94·0	98·0	98·0	†88·5	88·0	87·0	86·8	88·0	3	
74·0	78·5	†85·7	72·7	82·0	†95·3	†94·3	105·5	98·7	†98·7	†61·0	†103·0	4	
82·5	83·0	85·0	88·0	86·8	87·4	88·2	87·0	86·3	86·5	86·6	84·0	5	
83·0	83·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	6	
.....	†89·3	†85·0	†85·5	†87·8	†89·0	†97·0	†108·0	†105·7	93·7	86·1	97·8	7	
79·2	77·3	†97·1	†73·5	†92·3	†110·3	†118·9	†96·0	143·0	142·0	†65·7	91·2	8	
66·7	78·0	†86·8	82·5	92·2	94·0	93·5	97·3	†110·3	†114·8	87·8	85·4	9	
†91·6	85·6	82·8	97·7	95·0	89·8	109·5	100·2	137·5	†87·0	†81·6	93·5	10	
81·2	81·5	†83·7	†75·8	79·0	87·8	107·8	121·5	112·3	117·0	105·0	76·3	11	
†75·1	81·8	†92·6	90·0	86·0	92·0	87·5	†102·6	†101·3	†97·0	81·0	76·0	12	
85·6	68·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	13	
.....	†44·4	80·3	73·3	100·5	(b)	119·0	†84·7	120·0	†100·2	(b)	†161·7	14	
83·5	81·5	82·0	91·0	76·0	75·0	†141·0	(b)	†163·5	†119·7	†127·5	122·8	15	
66·5	81·0	†93·3	62·5	95·0	86·5	104·5	107·8	139·0	†149·0	†106·0	†107·0	16	
†82·0	†72·7	89·0	80·0	86·3	105·5	107·2	94·0	86·5	82·5	90·8	83·0	17	
†77·3	†75·0	75·0	†89·0	†90·5	95·0	100·0	108·0	†95·6	†103·0	98·5	78·0	18	
74·2	81·5	82·0	†76·5	95·0	95·0	†88·0	†89·5	†102·0	†121·9	†88·1	90·3	19	
81·8	84·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	20	
.....	78·5	80·0	81·5	78·3	90·6	†94·5	†103·5	†96·6	91·5	87·0	84·0	21	
73·0	75·0	†90·7	†80·0	82·5	86·8	111·7	87·5	87·1	86·0	†98·3	86·5	22	
72·0	89·6	†91·7	99·3	†86·3	†101·0	†102·0	†94·0	†91·5	88·0	87·0	84·9	23	
75·2	81·3	†70·0	90·0	64·5	†91·5	91·5	†64·0	(b)	†75·5	†76·5	†74·5	24	
85·0	72·0	56·2	73·0	89·5	118·0	89·5	107·0	120·0	110·0	85·0	82·0	25	
115·3	82·5	91·3	55·5	90·0	87·9	102·0	101·6	105·0	108·3	79·0	78·0	26	
78·0	78·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	27	
.....	†87·0	†84·0	79·5	†81·7	85·0	88·3	116·9	†85·7	†100·3	†95·0	†86·5	28	
77·0	83·7	82·0	82·9	85·0	89·0	†107·5	95·0	98·0	90·5	89·5	85·0	29	
78·8	85·0	82·0	84·0	86·0	90·0	91·6	†109·0	88·9	87·5	90·4	80·0	30	
†86·0	75·2	94·0	84·0	93·5	85·0	103·4	91·0	85·0	†77·7	91·0	89·5	31	

† Magnet vibrating. (a) Off Scale West. (b) Off Scale East.

Point Barrow, 1854; Lat. 71° 21′ N. Long. 156° 15′ W. Magnetic Declination.
One Scale-division = 2′·286. Increasing numbers denote increasing Easterly Declination.

Mean Point Barrow Time.		0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.
	D.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.
June.	1	84·7	78·2	81·0	83·0	81·5	80·3	79·5	83·7	85·5	91·5	84·7	78·3
	2	89·3	87·3	86·5	83·3	81·5	82·0	83·5	84·7	78·3	81·3	82·7	81·7
	3	87·0	82·0	85·3	†84·3	76·5	70·7	76·7	77·5	79·7	†79·3	82·0	83·5
	4	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	5	81·5	80·0	84·5	82·0	†80·7	†85·0	96·3	92·0	88·2	88·0	94·5	84·5
	6	78·7	82·5	†83·3	†84·7	†83·0	82·7	82·3	82·5	81·3	84·0	81·0	77·5
	7	†81·3	78·0	83·0	86·5	77·0	76·0	78·3	84·5	85·5	82·0	85·0	†81·0
	8	78·5	79·0	79·3	82·8	81·4	75·5	76·0	80·0	80·7	85·0	76·5	82·0
	9	86·0	79·5	79·5	81·2	81·5	82·2	79·2	81·5	81·0	85·0	83·7	81·0
	10	81·5	80·2	79·0	79·5	82·8	81·2	82·5	82·5	83·1	85·2	79·2	84·3
	11	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	12	100·4	66·2	78·1	†80·9	77·0	†57·0	71·0	88·5	56·0	77·5	†94·2	†91·7
	13	92·5	79·2	80·0	78·5	86·8	86·0	78·6	83·0	76·8	79·0	82·8	75·5
	14	87·8	96·0	88·8	86·5	†84·7	82·8	85·5	75·2	82·2	83·0	84·5	84·0
	15	†92·2	94·5	80·8	78·2	†79·9	81·0	76·5	83·0	79·5	†89·0	85·5	†70·3
	16	†75·5	100·5	†86·0	†77·0	†83·0	†54·6	95·2	†86·3	55·0	†57·0	†61·8	†71·9
	17	79·4	†71·7	84·0	80·5	78·8	83·0	81·0	83·2	82·5	†96·6	84·6	†86·4
	18	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	19	85·0	83·5	†79·0	82·5	78·5	78·0	82·4	83·2	82·8	76·0	80·0	80·0
	20	81·7	80·0	80·0	79·5	†82·5	79·0	†80·8	†80·5	85·4	84·8	†78·8	75·5
	21	79·5	88·5	†73·8	77·5	78·0	80·4	80·2	83·5	83·0	†73·2	82·5	†75·0
	22	90·0	86·5	76·0	79·0	84·0	86·5	82·0	79·0	84·0	†75·8	81·0	†82·6
	23	78·0	82·5	75·0	†75·1	75·5	79·0	82·0	81·2	79·5	84·0	85·0	78·8
	24	80·0	83·0	77·0	79·5	†77·6	80·0	80·8	81·0	83·0	84·7	81·2	80·0
	25	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	26	89·5	80·0	83·5	87·0	78·0	80·2	78·5	82·0	84·0	70·0	†72·3	†56·7
	27	83·0	82·0	81·5	80·5	†80·7	83·0	85·5	86·4	85·8	86·0	85·0	83·5
	28	82·0	71·0	78·3	†75·7	†80·9	†79·5	†81·5	83·2	82·5	81·8	83·4	80·8
	29	78·0	79·0	78·5	81·5	80·0	81·5	80·8	82·5	80·0	83·2	85·2	85·2
	30	83·5	85·0	80·0	83·5	82·5	83·0	81·8	82·8	83·5	81·5	†84·7	†81·5

12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.	Mean Point Barrow Time.
sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	sc. div.	D.
80·0	92·5	72·0	83·5	92·7	93·7	95·0	†95·3	90·0	86·3	88·0	86·2	1
84·0	78·0	73·5	†81·0	85·0	94·3	97·3	102·0	104·7	91·0	84·5	84·3	2
82·5	81·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	3
.....	79·0	80·5	83·5	84·0	83·7	†82·5	85·0	83·5	84·6	80·6	81·3	4
81·5	81·5	89·0	92·0	99·0	†101·0	97·0	99·5	97·5	90·3	89·3	83·5	5
76·0	79·0	81·5	82·0	†84·0	92·8	92·5	114·0	104·7	93·5	82·0	†80·3	6
81·0	79·3	79·5	101·7	88·7	94·0	100·0	91·5	†93·0	89·3	84·0	81·0	7
83·2	†82·5	84·0	90·5	87·5	94·2	102·3	98·5	95·0	85·6	86·7	85·8	8
82·8	†79·6	82·5	85·5	†89·5	94·6	95·0	92·0	91·8	91·8	89·8	81·6	9
78·2	82·0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	10
.....	107·5	†84·8	85·2	†110·5	101·6	†100·4	†115·7	103·2	(a)	†78·5	78·5	11
†60·4	108·5	92·2	91·2	100·0	100·2	98·0	95·5	107·0	†99·5	†86·2	91·0	12
79·8	68·4	†81·6	86·6	96·5	87·5	87·2	†116·2	†100·7	88·5	86·5	82·2	13
86·5	80·0	†80·4	82·5	†83·3	116·8	†114·9	91·5	†114·3	†124·0	†100·5	74·5	14
95·5	77·8	82·3	†85·6	90·3	†102·0	†105·6	†132·6	†124·5	†101·4	†83·6	†73·1	15
75·5	87·0	†82·5	†83·5	†68·0	†120·4	†122·0	141·8	†148·1	†132·9	87·9	†80·8	16
81·3	†78·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	17
.....	81·8	78·5	84·0	91·5	†91·6	87·0	†91·7	†91·6	†90·7	†90·0	82·5	18
71·0	†86·9	†82·3	89·8	90·0	94·5	98·0	96·7	96·0	†91·3	86·0	81·5	19
79·2	79·5	79·5	83·0	89·0	91·2	96·5	93·7	90·2	93·2	91·8	80·6	20
76·0	84·7	85·2	89·8	86·2	†93·3	93·0	93·3	93·5	95·8	98·0	99·0	21
†69·3	80·0	78·5	79·0	78·0	83·5	127·4	120·0	110·5	96·5	†85·0	84·6	22
†84·0	72·0	72·5	92·1	88·8	84·0	86·0	90·5	†97·3	91·6	84·3	82·2	23
82·5	78·2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	24
.....	82·3	72·5	82·5	85·3	90·0	90·5	92·0	†95·8	†91·5	†89·4	†89·7	25
85·4	79·0	†81·5	82·7	91·5	93·2	94·0	91·4	89·8	91·0	86·0	84·0	26
77·8	86·5	†82·4	79·2	†91·4	†92·5	91·5	†94·5	90·0	†93·0	89·0	87·5	27
65·7	58·1	60·6	82·5	84·2	90·0	†95·5	†101·5	†106·7	97·2	†86·6	83·2	28
81·5	72·0	94·0	87·2	†88·2	90·0	†90·0	†106·8	92·2	91·5	89·0	84·0	29
†79·5	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	30

June.

† Magnet vibrating.

(a) Off Scale East.

XXV. *Researches on Silica.* *By Colonel* PHILIP YORKE, *F.R.S.*

Received March 25,—Read April 2, 1857.

SINCE chemistry has been studied as a science, it has been an object with its cultivators to arrange the bodies which have been the subjects of their attention, into groups in which the individuals should have a natural relation to each other. Probably at no time in the history of the science has the importance of such a classification been more strongly felt than at the present day, not only on account of the number of known elements, but also from the number of compound bodies appearing to act as elements, which organic chemistry has made known. Although great advances have been made in this direction, the place of the element silicon in such a series as above alluded to, is very doubtful. Yet the binary compound of this element, silicon with oxygen, is familiar to every one; it constitutes by itself a considerable portion of the crust of the earth, and enters into a long series of definite crystallized compounds.

It has been satisfactorily determined that this substance, silica, belongs to the class of bodies designated as acids, but one essential point is wanting to enable chemists to give it, or its peculiar element, its proper position, and that is, the formula of this silica or silicic acid.

Three different formulas have been proposed and made use of by different chemists,—

$$SiO_3, \quad SiO_2, \quad SiO.$$

The first was originally adopted by BERZELIUS, from a consideration of the composition of felspar; and subsequently it was supported by arguments deduced from a law relating to the boiling-points of the volatile compounds of chlorine and bromine, suggested by HERMANN KOPP. The composition of a compound of chlorine, sulphur and silicon, discovered by Is. PIERRE*, is also favourable to this formula. The second formula is adopted by GMELIN, and is supported by some experiments of H. ROSE†, who found that when silica is fused with excess of carbonate of potash, the quantity of carbonic acid driven off is such that the oxygen contained in the carbonic acid expelled is equal to that of the silicic acid used. This formula, SiO_2, agrees with arguments derived from the specific gravity of the vapours of chloride and fluoride of silicon. The last formula, SiO, originally suggested by Dr. THOMSON‡, has been supported by M. DUMAS§, and also by EBELMEN‖, from arguments suggested by the constitution of the silicic ethers.

* Ann. de Chim. &c. 3 sér. xxiv. 286.

† On Titanium and its Compounds, GILBERT'S 'Annalen,' lxxiii. 83.

‡ System of Chemistry, 5th ed. 1817.

§ Ann. de Chim. &c. 2 sér. xxxiii. 368.

‖ Ann. de Chim. &c. 3 sér. xvi. 163.

With regard to the arguments in favour of the first-mentioned formula, derived from a consideration of the boiling-point of chlorides and bromides, I should mention that M. Kopp, since the publication of a notice by Mr. Duppa communicated to the Royal Society by Dr. Hofmann, has appended to a translation of this note some remarks of his own*, in which he states that the law he had suggested cannot now, in its primitive generality, be regarded as true.

From what has preceded it may be observed, that the only instance in which the formula of silicic acid has been sought for by a direct method, as for example, in which its equivalent has been determined by the quantity of some other acid displaced, is in the experiments of Rose. These experiments, however, which are introduced incidentally in a memoir on titanic acid, have not generally attracted the attention of authors of chemical systems, but as it appeared to me that the method deserved preference over the others which have been used, or any that I could devise, I resolved to repeat the experiments, entertaining the notion that if the results were confirmed, this ought to establish the formula which they had suggested to Rose.

I imagined that with the means I meant to apply, what I had in view would prove an easy matter; in this I was deceived; but though I am still unable to give a definite answer to the question which I had proposed to myself, I have obtained some results which I venture to think have interest, and which, under the uncertainty I am in of the time it may take before I can complete these researches, I desire to offer to the Society in their present state.

Experiments on the quantity of Carbonic Acid expelled from excess of Alkaline Carbonates by fusion with Silica.

The silicic acid used was obtained by the ignition of perfectly transparent and colourless rock-crystal, and subsequent reduction of this to powder. The carbonate of potash was produced by the ignition and subsequent fusion of crystallized bicarbonate of potash. The carbonate of soda was mostly obtained by a similar treatment of the purest bicarbonate I could procure, though a few experiments were made with some carbonate obtained by the purification of the crystallized carbonate, by Gay-Lussac's method.

In my earlier experiments I adopted methods varying a little from one another, of adding the silicic acid to the alkaline carbonate, whilst in a state of fusion, in a platinum crucible; after some experience I found that it was in all respects a better plan to add the alkali to the silica. The latter was placed in a counterpoised platinum crucible, and weighed; the fused alkaline carbonate in fragments was then weighed in a counterpoised closed glass tube, then gradually added to the silica heated in the crucible, and the tube again weighed, by which the quantity of alkaline carbonate was determined. The heating was performed by a gas-flame, through which a stream of air was forced by means of a table bellows. The crucible was heated for some time after all effervescence had ceased, and the materials used appeared in a quiet state of fusion. The crucible

* Ann. de Chim. &c. 3 sér. xlvii. 66.

was weighed when cold, and again heated for five minutes until the last two weighings agreed within a few hundredths of a grain. The loss was then estimated as carbonic acid. The equivalent of the silicic acid is then readily obtained by the following proportions, 22 being the equivalent of carbonic acid on the hydrogen scale:—

$$x = \frac{22 \times \text{weight of silica used}}{\text{weight of carbonic acid expelled}}.$$

Experiments with Carbonate of Potash.

No.	Quantities used. Carbonate of Potash. grs.	Silica. grs.	Loss = Carbonic Acid.	Equivalents of Silica deduced.
I.	52	10·67	7·5	31·3
II.	61·3	13·45	9·62	30·06
III.	30·7	10·07	7·04	31·4
IV.	47·4	10·10	7·26	30·06
			Mean	30·07

This result agrees very nearly with that of Rose, and also with the formula SiO_2, which, if calculated from the percentage composition of silicic acid as stated by Berzelius, would give 30·75. If from that assigned to it by Pelouze, it would be 30·22.

Experiments with Carbonate of Soda.

No.	Carbonate of Soda. grs.	Silica. grs.	Loss = Carbonic Acid.	Equivalent of Silica deduced.
I.	58·16	13·08	13·7	21
II.	59·12	12·17	12·15	22
III.	60·68	13·86	13·94	21·8
IV.	53·04	16·45	15·2	23·7
V.	34·84	9·06	11·3	17·6
VI.	55·78	15·85	15·89	21·2
VII.	59·12	16·1	16·34	21·7
			Mean	21·3

Here then I obtained a very different result from that with carbonate of potash, agreeing more nearly with half the number represented by the formula

$$SiO_2, \text{ or } \frac{45·2}{2}.$$

As M. Jacquelain had stated that under certain circumstances carbonate of soda lost part of its carbonic acid by fusion*, I made the following experiments to ascertain whether, in my method of experimenting, the loss I had observed could be attributed to the action of heat alone. A portion of the carbonate of soda used in the last-mentioned experiments was fused for three or four minutes, it weighed = 78·5 grs.

* Ann. de Chim. &c. 3 sér. xxxii. 195.

	grs.	diff.
I. Fused a second time by gas-flame, urged by bellows for fifteen minutes, crucible open	=78·29	0·21
II. Fused a third time by gas-flame, urged by bellows for fifteen minutes, crucible open	=78·26	0·03

On the following day,—

III. Fused a fourth time by gas-flame, urged by bellows for fifteen minutes, crucible open	=78·20	0·06
IV. Fused a fifth time by gas-flame, urged by bellows for fifteen minutes, crucible open	=78·14	0·06

The carbonate of soda had acquired a slight brown tint; on examination I found that this was owing to its containing a very minute portion of oxide of platinum derived from the crucible. From these experiments, it will be evident that the results I had obtained by the fusion of carbonate of soda with silica could not be accounted for by the action of heat alone.

The carbonates of potash and soda having afforded me such different results, I now had recourse to carbonate of lithia.

Experiments with Silica and Carbonate of Lithia.

The carbonate of lithia, as purchased, was purified by dissolving it in 100 parts of boiling distilled water, filtering the liquid from small portions of lime and other matters which remained undissolved, and evaporating the filtrate in a porcelain basin until a small portion of liquid only remained; the carbonate of lithia was obtained as a crystalline crust deposited on the basin. The mother-liquor contained potash.

The carbonate of lithia thus obtained was fused in a covered crucible. It lost very little weight. I found that although carbonate of lithia by itself fused at a temperature below a red heat, visible by daylight, yet that when any considerable proportion of silica was added, the fusing-point rose so much that I was obliged to conduct the experiments in a platinum crucible.

Experiments with Carbonate of Lithia.

No.	Carbonate of Lithia. grs.	Silica. grs.	Loss = Carbonic Acid.	Equivalent of Silica deduced.
I.	34·46	10·33	14·9	15·2
II.	26·87	8·24	12·09	14·99
III.	42·74	15·84	23·24	14·99
IV.	31·28	9·14	13·61	14·77
			Mean	14·99

Experiments III. and IV. were made with a different specimen of carbonate of lithia from Nos. I. and II.

Hence it appears that the equivalent for silica, deduced from experiments with car-

bonate of lithia, differs from that obtained with the carbonates of the other two fused alkalies, and agrees with the formula SiO.

These unexpected results suggested to me two lines of research, by following which I hoped that some light might be thrown on the apparent anomaly involved in them. The first was to determine whether any other acid than the silicic yielded results similar to those already obtained with carbonates of potash and soda; and the second was to ascertain whether the equivalent of silicic acid could be obtained by means of any other volatile acid than the carbonic.

I desired to ascertain, in the first place, whether at the temperature used, any quantity of carbonic acid, different from its received equivalent, could be expelled by a well-known acid of the class of hydrogen acids, as *e. g.* the sulphuric acid. But as it was obviously impossible to experiment with this substance in its usual form, and considering that carbonate of magnesia could not exist at the temperature of the experiments, I made use of anhydrous sulphate of magnesia for my purpose.

With this substance I made the following experiments:—

With Carbonate of Potash and Sulphate of Magnesia.

No.	Carbonate of Potash. grs.	Sulphate of Magnesia. grs.	Loss = Carbonic Acid.	Equivalent of Sulphuric Acid deduced.
I.	58·14	27·81	10·51	39·4
II.	63·99	32·11	11·44	41·0
			Mean	40·2

With Carbonate of Soda.

	Carbonate of Soda.			
I.	43	30·27	11·3	39·4
II.	60·9	34·46	12·22	41
III.	67·98	35·28	12·63	40·1
			Mean	40·1

The number 40 is the received equivalent of sulphuric acid. These experiments are sufficient to show that with sulphuric acid, at the temperature used, no such effect as that with silicic acid is obtained.

I made several experiments of the same kind with tungstic acid, with boracic acid (using, however, biborate of soda for the purpose), alumina, and red oxide of iron. Boracic acid only, of the bodies named, afforded results analogous to those given by silicic acid, but I did not succeed in obtaining numbers which accorded well together. The other substances mentioned gave like results, whether carbonate of potash or soda was employed. As alumina appears in some instances to take the place of silica, I will insert here the results I obtained with that body.

The alumina was obtained by GREGORY'S process.

Experiments with Carbonate of Potash.

No.	Carbonate of Potash. grs.	Alumina. grs.	Loss = Carbonic Acid.	Equivalent of Alumina deduced.
I.	46·98	21·2	9·15	50·9
II.	48·28	21·16	9·05	51·4
			Mean	51·1

With Carbonate of Soda.

	Carbonate of Soda.			
I.	46·12	20·03	8·83	51
II.	52·21	27·18	11·93	50
			Mean	50·5

The equivalent usually assigned to alumina is 51·4.

Turning now to experiments made with the view of obtaining the equivalent of silicic acid in the weight of some other acid than the carbonic, I made some attempts with chloride of sodium and sulphate of soda, but with these did not succeed; I then had recourse to the hydrates, and will give the results I obtained. But I must premise that the difficulties of obtaining accordant quantities are much greater with these bodies than with the carbonates, difficulties which arise from the volatility of the alkalies on the one hand, and the tendency to absorb carbonic acid during the experiments on the other.

Of the various methods which I tried, I think that the best and simplest was to fuse the alkaline hydrate and weigh it in a deep narrow silver crucible provided with a cover, and of large capacity compared with the bulk of materials employed, then to add the silica and weigh the two bodies together; to heat this crucible while much inclined very gradually, and finally to repeat the weighings and heatings until the weight appeared constant. The last two experiments with hydrate of soda were made in this manner.

Experiments with Silica and Hydrate of Potash. Water = 9.

No.	Hydrate of Potash. grs.	Silica. grs.	Loss = Water.	Equivalent of Silica deduced.
I.	73·27	18·39	5·39	30
II.	105·39	19·22	4·7	36·7
III.	96	16·08	4·95	29·2
IV.	87·44	6·55	1·8	32·7
V.	70·38	7·37	2·39	27·7
VI.	74·7	7·65	2·59	26·1
			Mean	30·8

With Silica and Hydrate of Soda.

No.	Hydrate of Soda. grs.	Silica. grs.	Loss = Water.	Equivalent of Silica deduced.
I.	100·04	18·61	9·18	18·2
II.	81·29	10·07	5·48	16·4
III.	128·24	15·86	8·37	17·0
			Mean	17·2

In these experiments with the hydrates, it will be observed that the equivalent obtained with hydrate of potash is the same as that obtained with the carbonate, but that obtained with hydrate of soda approaches more nearly to that afforded by carbonate of lithia.

Reverting now to the experiments with the carbonates, I found that I had, in order to secure the requisite fusibility in the case of the carbonate of lithia, used a greater proportion of that salt to the silicic acid than with the other two carbonates, so that considering the ratio of the oxygen in the base to that in the silicic acid in the case of the lithia experiments, it was nearly as 2 : 1, whilst in the other two carbonates the oxygen was equal in the base and in the silicic acid. To ascertain if this would in any way explain the results I had obtained, I made two experiments with carbonate of potash, in which the oxygen, considered as belonging to the oxide of potassium, was double that contained in the silica, and obtained for the equivalent the number 29 very nearly. Operating in a similar manner with carbonate of soda, the mean of two experiments gave 19·5.

Both the numbers thus obtained are slightly smaller than those given in the former experiments, but I do not think they should materially affect any inference to be drawn from them; and I may remark here, that, though I am of opinion that generally the quantity of carbonic acid expelled by an anhydrous acid from the alkaline carbonates in fusion may be very valuable for determining between two different formulas for such acid, I do not think that the method is capable of determining with precision the equivalent weight of such an acid, as I have satisfied myself in the course of these experiments, that when the heat is long-continued, minute, but appreciable quantities of the alkali are volatilized.

On Crystallized Hydrate of Silicate of Soda.

A silicate of soda was formed by taking one equivalent of carbonate of soda, or 54 parts, and one equivalent of silica as given by the results of the experiments before detailed, or 23 parts, and fusing them together, the quantity of carbonic acid present would be 22; the quantity actually expelled was 20, so that 2 parts = 5 of carbonate of soda had remained in the fused mass. This substance appeared crystallized throughout, and to be made up of spheroidal masses of prismatic crystals radiating from a centre. This kind of crystallization invariably followed on fusing together the two substances in

the proportions I have mentioned. The fused substance readily dissolved in water, affording a perfectly colourless transparent solution. The solution was evaporated at atmospheric temperatures *in vacuo* with sulphuric acid; after some days it began to crystallize, and eventually formed a mass of crystals with very little liquid remaining.

Analysis.

I. 17·62 grs. of the salt, dried by pressure in filtering paper, was treated with hydrochloric acid, effervescence was produced. It yielded 3·72 grs. of ignited silica. The filtrate evaporated and residue ignited, gave 8·33 grs. chloride of sodium =4·41 soda.

II. 16·24 grs., dried as before, heated in a sand-bath, melted into a clear liquid, which soon became viscid and opaque; ignited, it weighed 7·78 grs.; loss =8·46.

III. 22·17 grs. of the salt, dried as before, heated red in a platinum crucible =10·66; loss =11·51.

To expel the carbonic acid, a little silica was added to the remaining substance, the whole =13·78.

This was heated by the gas-flame, urged by bellows for ten minutes, and then weighed . . . . . } =13·28

Loss = carbonic acid = 0·50

These experiments would give as the constituents of 100 parts of the salt (*a*),—

		I.	II.	III.
Silicic acid		21·11	——	——
Soda		25·02	——	——
Carbonic acid and water by loss	53·87		——	2·25
			52·1	51·90

and as 2·25 carbonic acid would combine with 3·15 soda, 100 parts would contain—

Silicic acid	21·11
Soda	21·87
Carbonate of soda	5·40
Water	51·90
	100·28

If the carbonate of soda be the ordinary ten-hydrated salt, and this be considered as mixture and subtracted, we obtain—

		By Theory, in 100 pts.	By Experiment, in 100 pts.
SiO_3	30·2	24·19	24·61
Soda	31·0	25·01	25·49
7HO	63·0	50·80	49·90
	124·2		

Two crystallized hydrates of silicate of soda have been described by FRITZSCHE*, one of which crystallizes in rectangular prisms, and contains 56·6 per cent. of water, the other 47 per cent. HERMANN† has also described a salt of this kind crystallizing in rhombohedrons and containing 54·4 per cent. of water. In all these salts the proportion of acid to base is the same as in the one I have described above, and the number of equivalents of water contained in them respectively is 9, 6 and 8. Lately M. FREMY‡, in a communication on the Silicates, has described a silicate which he states he obtained in large crystals, to which he assigns the formula $(SiO^2)^3\ 4NaO, 26HO$, which would give 48·6 per cent. of water. I regret that I have not been able to determine the crystalline form of the salts I have obtained; but the point to which I wish to direct attention is, that in the process of obtaining a crystallized hydrate from the fused anhydrous salt a portion of soda is extruded. I found that this extrusion also took place when the salt was formed by dissolving hydrated silica in such a quantity of a solution of caustic soda, as that the soda should be in the same proportion to the silica as in the fused salt before described, viz. as 31 : 23.

A specimen of a crystallized salt, formed in this manner, of which the solution had been exposed to the atmosphere during crystallization, yielded, on analysis,—

Silica	22·38
Soda	24·44
Carbonate of soda .	2·41
Water	50·55

I also obtained a crystallized salt by dissolving silica in a solution of caustic soda in such proportion that the oxygen in the silica should be double that in the soda. The crystallized salt was redissolved and recrystallized, the crystallization in both instances taking place out of contact with the atmosphere.

Analysis gave the following composition of this specimen (*c*):—

				in 100.
Silica . . .	24·45	2SiO	30·2 . .	24·193
Soda . . .	25·276	NaO give	31 . .	25·00
Water . . .	50·374	7HO	63 . .	50·80

As it appeared to me desirable to ascertain if any portion of the water in these salts should be considered as basic, I made some experiments on specimens (*a*) and (*c*) with this view. I found however no decided difference in the comportment of the two specimens; and further, that all the water, with the exception of less than 1 per cent., was expelled at a temperature of 300° FAHR.

With regard to the silicate of lithia (LiO, SiO), I have not yet been able to examine it particularly; but when treated with water it appears, like the soda compound, to split into different combinations, as a substance which remained when more than one-half of

* POGGENDORFF'S Annalen, xliii. 135.

† ERDMANN'S Journal für prakt. Chemie, xiv. 275.

‡ Comptes Rendus, December 1856.

the salt had been washed away, gave a result on analysis showing that its silicic acid contained at least double the quantity of oxygen contained in the lithia.

As to the formula SiO, deduced from the lithia compound, I would add, that some observations have lately been published by Troost*, which go to show that lithia in its comportment has more analogy with the magnesian class of bases than with the potassium class. Now, of the known crystallized silicates of bases belonging to the former class, whether found in nature or obtained in the processes of the arts, a large proportion have a composition in which the oxygen of the base is equal to that of the acid.

In concluding this paper, I desire to guard myself against drawing any decided inference from the experiments I have related, particularly as I hope to be able to follow up this communication by further researches, which may afford more definite grounds for theory. I venture only to remark at present, that as I cannot but think that the quantity of carbonic acid displaced must afford a true measure of the equivalent of silicic acid; so, looking to the results of my experiments, I see at present no alternative but to admit of at least two equivalents (that is to say, of two acids), the value of which is determined by circumstances, such as the presence of water, and the nature of the base to which the silicic acid is presented.

The existence of different silicic acids has been already suggested by chemists, particularly by Ebelmen† and by Laurent‡. Fremy also considers that his late experiments, before quoted, support the same view.

* Comptes Rendus, November 10, 1856. Chemical Gazette, December 1856.

† Ann. de Chim. &c. 3 sér. xvi. 162.

‡ Chemical Method, Engl. Trans., p. 127–8.

XXVI. *On the Fructification of certain Sphæriaceous Fungi.*

By Frederick Currey, *M.A., F.L.S. Communicated by* W. J. Hooker, *M.D., F.R.S.*

Received May 14,—Read June 18, 1857.

The inquiries which have been carried on of late years with regard to the reproductive organs of Fungi, although they have caused but little progress towards the solution of the problem of sexuality, have nevertheless been by no means barren or unproductive. They have revealed the existence of a diversity in those organs, not only amongst genera and species, but in the same individual plant, to an extent which would excite surprise, if not incredulity, in the mind of an observer unacquainted with the subject or approaching it for the first time.

It has been proved, that in a vast number of cases the same Fungus, or to speak more correctly, the same mycelium or vegetative system, has the power of producing fruits differing so much in form and appearance, that it would never occur to a casual observer to suppose that they were the produce of the same plant. Many Fungi placed by former observers in different genera and even in different families, are now ascertained to be different forms of fruit of one and the same plant.

Without going into details, one striking instance may be mentioned. Dr. De Bary has shown that the common cheese-mould, *Aspergillus glaucus*, and *Eurotium herbariorum*, an ascigerous fungus too well known to botanists as the pest of herbaria, are two forms of fruit produced from the same mycelium, and numbers of other instances of a similar nature have also been published.

This polymorphism of fructification, which is of the greatest interest in a physiological point of view, is of hardly less importance when considered with reference to the effect which it must produce upon the systematic arrangement of Fungi. Its obvious effect (if it should be proved to exist as extensively as there seems reason to suspect) must be the striking out of a number of ill-founded genera, thus affording relief to the overloaded terminology of the science of Mycology.

In the present state of the question accumulation of observations is the great desideratum, the means by which it may hereafter be possible to arrive as it were by a process of induction, at more general conclusions than have hitherto been reached.

In the present paper my object is to lay before the Society the result of some observations upon certain Fungi of the division Sphæriacei, a division which seems likely to be productive of a greater number of instances of the diversity of fructification above alluded to than any other tribe. The observations in question have been conducted certainly with care, and I believe with accuracy, and I venture to hope that they may not be considered without importance in their bearing upon the general subject.

1. *Sphæria verrucæformis*, Ehr.—The first plant which I have to mention is *Sphæria verrucæformis*, one of the most common of the genus, at least in this country. It occurs at all seasons of the year, principally on fallen branches of Oak, but also on Spanish Chestnut, Beech, Alder, and other trees, in the form of prominent black warts protruding through, and surrounded by the laciniæ of, the ruptured epidermis. The ordinary fruit of this Sphæria consists of minute curved sporidia of a yellowish tinge. In most Sphæriæ the number of sporidia in each ascus is limited to eight, but *Sphæria verrucæformis* is one of the comparatively few species in which the number is quite unlimited; the sporidia indeed are so densely crowded that it is difficult to arrive at even an approximate calculation of the number contained in any one ascus. Plate XXIV. fig. 1 represents an ascus of the ordinary form filled with sporidia, and fig. 2 a few of the latter detached from the ascus, all magnified 315 diameters. The colour of the sporidia, when in a mass, is a dull yellow, but they are almost colourless when viewed singly. Their mean size is $\frac{1}{1800}$th of an inch, and there is usually a globular nucleus at each end. I am not aware that any second form of fruit has hitherto been observed in this Fungus, but in the course of last autumn (1856) I found some very young specimens of the Sphæria growing in close proximity to a species of *Cytispora*, and subsequent investigation convinced me that the latter was merely a cytisporous condition of the former. The appearance of the perithecia of the young Sphæriæ was different from that of their mature condition; they were of a light ochrey-yellow colour, instead of being black, or nearly so. The perithecia of the Cytispora were so exactly like those of the Sphæria, that without a microscopical examination of the contents, it was not possible to distinguish the one from the other. The association in growth, the exact similarity of the perithecia, and the fact, now well established, that other species of Sphæriæ possess a cytisporous as well as an ascigerous fructification, would seem almost sufficient grounds for assuming the identity of these two plants, but a further argument in favour of such identity was afforded by the occurrence of the perithecia of the Cytispora and of the Sphæria within the same circumscribing line. *Sphæria verrucæformis* belongs, as is well known, to the division Lignosæ, one of the characteristics of which is the existence of a black line surrounding the base of the stroma, in which the perithecia are imbedded. If a thin section of the substance of this black line be examined under the microscope when the Sphæria is ripe, it appears to be composed of dry carbonaceous matter in which it is hardly possible to trace any structure; but if examined at the period of growth of the above-mentioned specimens, it is seen to consist of strings of oval cells of a clear dark brown colour. Precisely similar cells are frequently to be seen near the base of the perithecia of other species of *Sphæria*, which doubtless belong to the mycelium, and I suspect therefore that the black line above alluded to, and which has been supposed to be the outer coat of the stroma, is in fact formed by the cells of the mycelium. If this be so, the occurrence of the Sphæria and the Cytispora within the same circle would seem to make it reasonable to assume that the two are the produce of the same mycelium.

Fig. 3 represents the oval cells above-mentioned, and fig. 4 the sporidia of the

Cytispora, which are fusiform and strongly curved. They are all magnified 315 diameters.

2. *Sphæria favacea*, Fr.—This Sphæria is closely allied to *S. verrucæformis*, and extremely liable to be confounded with it. Differences, however, exist which on careful examination suffice to distinguish the one from the other. The asci are usually clavate, instead of fusiform as in *S. verrucæformis*; the sporidia have not ordinarily the globular nuclei, and their size is somewhat less, being about $\frac{1}{850}$th of an inch; the plant is moreover larger and of a coarser habit. No reliance is to be placed upon the arrangement of the perithecia in two tiers, for this occurs indifferently both in *Sphæria verrucæformis* and *S. favacea*. I notice the latter plant here in order to call attention to some strange modifications in the shape of the asci which have come under my observation. The ordinary fruit is shown in fig. 5, while fig. 6 (*a*—*m*) represents aberrant forms in which the asci have become most fancifully distorted. But perhaps the most curious instance of change is that shown in fig. 7, where the outer membrane of the ascus assumed a perfectly globular shape. In this instance the inner membrane remained unaffected, and lay bent almost in its natural shape along one side of the globose external membrane, the sporidia being entirely confined within it; but in other instances, as in fig. 8 (a much smaller ascus), the inner membrane shared the change of form of the outer one, and the sporidia filled the whole of the globe. The specimens in which this abnormal fructification occurred had been subjected to an unusual amount of moisture, by which the change in the form of the asci might possibly have been produced. The figures in this instance are all magnified 220 diameters.

A similar change of form occurred in another Sphæria, the only difference being, that in the latter case the irregular ascus was not exactly globular, but slightly ellipsoidal. Figs. 9 and 10 represent the common and the abnormal forms of fructification in this latter Sphæria, which belongs to the division Concrescentes, the figures being magnified 415 diameters. I may add, that the sporidia are of a clear brown colour, and sometimes, though not usually, nucleated at one or both extremities. I cannot ascertain that it has been previously described, and propose therefore to call it *Sphæria stipata*, the perithecia being very closely packed. The mean size of its sporidia is $\frac{1}{1150}$th of an inch.

3. *Sphæria Tiliaginea*, Currey.—This species is new to this country, and I have not been able to find any description of it elsewhere. I have met with it at different seasons of the year in my own neighbourhood (Blackheath), always on small dead branches of Lime. It belongs to the division Circinatæ, but the perithecia are more deeply immersed in the inner bark than is usual in that division. The ostiola sometimes penetrate through the epidermis, but sometimes remain concealed by it; in the latter case raising the epidermis into a number of small pustules. When the epidermis is stripped off, the inner bark is seen to be covered with small, green, circular umbonate disks, the umbonate appearance being caused by the ostiola, which, as in other Circinatæ, are in contact at their apices, and protrude through the exact centre of each of the

green disks. The fact that each one, without exception, of the circular masses of perithecia is always crowned by one of these green disks, would seem to point to something more than an accidental connexion between them; and although it is possible that the disks might be only a fungoid growth parasitic upon the ostiola of the perithecia, it seems far more probable that they really belong to the latter, forming an integral portion of their structure. If a very thin vertical section be made passing through the centre of the mass of ostiola, and consequently through the centre of the disks, it is seen that the latter are composed of densely packed filamentous tissue, hardly differing, if at all, from the tissue of which the ostiola are composed. The tissues moreover are in actual contact, so close that it is hardly possible to distinguish any line of demarcation between the two. Such a section is shown in fig. 11, magnified 45 diameters; and it must be taken whilst the plants are quite young, for in a later stage, when the ostiola have become black, hardened and brittle, it could not be made.

The perithecium in fig. 11 was just beginning to fructify, and fig. 12, Plate XXV. shows one of the asci with its sporidia, which, although of a greenish tinge and therefore, I should think, not perfectly ripe, were nevertheless fully formed, constituting the normal fruit of the Sphæria. Their average length is $\frac{1}{1315}$th of an inch. The filaments of which the disks are composed produce two different sorts of fruit; the one being minute, colourless cylindrical bodies, answering exactly to the spermatia which have been observed in the Lichens generally, and in many other Fungi; the other being of the same nature as the bodies which M. Tulasne has called stylospores. I have not been able to ascertain the mode of attachment of the spermatia, but the stylospores are borne upon the apices of the filaments, and are formed by the swelling out of their tips, which afterwards become divided from the main filament by a septum. These stylospores, when viewed in a mass, are of a peculiarly rich, bluish-green colour, and when sufficiently magnified, are seen to be very narrowly elliptical bodies, sometimes simple, sometimes with one, two, or even three septa, with an endochrome which is sometimes smooth and refractive, and sometimes granular. I found them vary in length from $\frac{1}{1315}$th to $\frac{1}{700}$th of an inch. These bodies are shown in figs. 13 *a* and 13 *b*, and the spermatia in fig. 14. Figs. 12, 13 *a*, 13 *b* and 14, are magnified 315 diameters.

If the plants are kept in a sufficiently moist atmosphere, these stylospores are sometimes ejected through the bark in tendrils, after the fashion of a *Melanconium*; they differ from the spores of the latter genus in being septate, and from *Stilbospora* by the formation of tendrils. If the green disks were autonomous Fungi, these differences would require consideration, as it would in strictness prevent their classification under either of the two genera just mentioned; but there being, as it seems to me, sufficient grounds for considering the disks only as forming part of the accompanying Sphæria, and not as independent vegetable productions, the differences in question cease to be of much practical importance.

4. *Sphæria vestita*, Fr.—This Sphæria belongs, like the last, to the division Circinatæ,

and has not hitherto been recorded amongst the British Sphæriæ; but, not long since, I found a considerable number of specimens growing upon a fallen branch of Beech in company with *Sphæria quaternata*, *Sphæria turgida*, and a third Sphæria which I am unable to name, and upon these specimens the following observations were made. One peculiarity of *Sphæria vestita*, and that from which it derives its specific name, is a dense woolly covering of a dirty white or yellowish colour, which in their early state clothes the young perithecia, and which covering has always been considered as properly belonging to the Sphæria. It consists of a dense filamentous tissue, not altogether unlike that which forms the green disks in the Sphæria above described, and like the latter tissue it produces a secondary kind of fruit, which I will presently describe. Before doing so, however, I must mention a striking peculiarity in the mode of growth of some of the specimens which came under my observation. Ordinarily the perithecia of *Sphæria vestita* are seated beneath the epidermis upon the surface of the inner bark, but in the above specimens the perithecia were imbedded in the substance of the bark, or rather they were enclosed in a sort of conceptacle formed from the bark and having a perforation at its apex.

The perithecia themselves were surrounded on all sides by a dark dense mass, which the microscope proved to be composed entirely of spores having all the appearance of the spores figured by Corda under the name of *Steganosporium pyriforme*. The perforation at the top of the conceptacle served as a common orifice for the escape of the sporidia of the Sphæria, and of what I may call the Steganosporium spores. The perithecia, which were enclosed in the conceptacles, had for the most part lost their woolly covering, and the conceptacles themselves were surrounded on all sides by vast numbers of the Steganosporium spores which had been ejected through the orifices. Fig. 15 represents one of these latter spores, which are of a dark brown colour with a granular endochrome, and, when ripe, have two and sometimes three transverse septa. Now if the woolly covering of the perithecia be examined before it loses its pale colour, it will be found that the threads of which it is composed produce a number of pale-coloured globular bodies, bearing a strong resemblance to the immature spores of many species of *Uredines*; and it will also be seen that these bodies by degrees lose their globular form and assume a pyriform or turbinate shape, shortly after which a septum is formed across the lower extremity. When the spore has reached this latter condition, it is impossible not to see that it is the young state of the dark-coloured Steganosporium spores, which subsequently become developed in enormous quantities, and which are afterwards found surrounding the perithecia, both when the latter are naked and when they are enclosed in conceptacles.

A case precisely analogous to the above was noticed some time since by Messrs. Berkeley and Broome, and was published in Hooker's 'Journal of Botany,' in which a well-known Stilbospora (*Stilbospora macrosperma*) was found growing on the outer surface of the perithecia of *Sphæria inquinans*, the Stilbospora and the Sphæria being both situated beneath the bark, and having a common orifice for the escape of both the

kinds of spores. This latter has been admitted to be a case of double fructification, and there seems no reason to doubt that the instance above recorded must be classed in the same category, and that the Steganosporium spores must henceforth be considered as a secondary fruit of *Sphæria vestita*.

The perfect fruit of the Sphæria, *i. e.* the sporidia which are produced in the asci in the interior of the perithecia, are of an oval shape, with transverse and longitudinal or oblique septa, somewhat irregular in number and position. Fig. 16 Plate XXV. represents an ascus with sporidia in its interior, and fig. 17 some free sporidia. In this, as I believe to be the case in many if not most Sphæriæ, the sporidia appear to increase in size after their escape from the ascus. The smaller ones (fig. 16) were drawn from fresh specimens in which the asci were only just matured, and measured $\frac{1}{1450}$th of an inch in length; the larger ones (fig. 17) from a dried specimen in the Hookerian herbarium, in which the asci were completely absorbed. These latter measured $\frac{1}{1333}$th of an inch in length.

5. *Sphæria fragiformis*, Pers.—There is a peculiar state of this Sphæria of not unfrequent occurrence, in which the base of the pulvinate stroma is surrounded on all sides by a small pale brown arborescent growth, having the appearance of being parasitical upon the Sphæria.

This state is figured by Sowerby, who, mistaking the unripe stroma of the Sphæria for a species of *Lycoperdon*, gave it the name of *Lycoperdon acariforme*. It is mentioned also in the 'English Flora,' where it is suggested that the growth in question is a species of *Isaria* parasitical upon the stroma of the Sphæria. I have long suspected it to be an essential part of the Sphæria itself, but it was some time before I was able to find specimens in a state favourable for examination. During the last autumn (1856) some such occurred in the neighbourhood of Tunbridge Wells, which enabled me to verify my previous suspicions. The stroma of *Sphæria fragiformis* is covered in its early stage with a veil of dense cellular tissue of a rusty brick-red colour, which disappears as the Sphæria ripens. If a specimen be taken covered with its veil and accompanied by the supposed Isaria, and a very thin vertical section be made passing through the stroma, the veil, and one of the Isariæ, it will be seen, on examining it with a sufficient magnifying power (about 200 diameters), that the threads which form the Isaria are in fact the outlying portions or ragged edge of the mass of cellular tissue which constitutes the veil. In fig. 18 (*a*, *b*), some of these threads are drawn as they appeared under a magnifying power of 415 diameters; these are barren, and have nothing remarkable in their structure. Fig. 18 (*c*) shows some fertile threads similarly magnified, which it will be seen produce at their summit small heads of minute cellules arranged something in the manner of the spores of a Botrytis, and which must be considered as a conidioid form of fruit of *Sphæria fragiformis*. These cellules are elliptical, colourless, and very minute.

6. *Sphæria salicina*, Pers., and *Coniothecium Amentacearum*, Corda.—The latter Fungus is exceedingly common upon the branches of Willows. It appears to the naked eye in the form of small, round or irregular patches of black powder, which leave a sooty mark upon the fingers when touched. The microscope shows that these patches are

composed of multitudes of spores of all sizes and shapes, sometimes strung together in moniliform rows, sometimes united together in a tabular form, presenting altogether a most irregular appearance. It is probable, indeed there can be little doubt, that the moniliform rows of spores are the extremities of fructifying threads proceeding from the mycelium, and upon the point of breaking up into spores. Until lately I had always found the black patches of Coniothecium unconnected with any other sort of fungoid growth; but in some specimens which I met with during the last summer (1856), I found upon stripping off the epidermis of the Willow branch, that the Coniothecium was in immediate connexion with a perithecium which lay buried beneath the outer bark of the Willow. Upon extracting the perithecium the Coniothecium continued adherent to it, the threads of the latter projecting from the apex of the perithecium, and giving it the appearance of being crowned with a little brush. The question then arose, whether the Coniothecium really belonged to the perithecium, or whether it was merely parasitical upon it, and I satisfied myself that the former was the case; for upon examining the perithecium with the microscope, there was no trace to be seen of any threads traversing its outer surface, and that being the case, the only other conclusion in accordance with the facts was, that the threads of the Coniothecium were in connexion with and formed a continuation of the threads constituting the lining of the perithecium.

In this instance most of the perithecia had nothing in their interior but a tangled mass of colourless filamentous tissue, but one or two of them contained a number of spores not enclosed in asci, but intermixed irregularly with the filamentous tissue. The spores were colourless, subelliptical, slightly acuminate at each end, and constricted in the middle with a single central septum. It could hardly be doubted that the perithecia belonged to some species of *Sphæria* of which the free spores might either be an imperfect or stylosporous state of fruit, or normal sporidia set free by the absorption of the asci. Since then, I have had the satisfaction of finding the Coniothecium and the Sphæria connected in the same manner, the latter being in perfect fruit, and proving to be, as might almost have been anticipated, *Sphæria salicina*, Pers. The asci and sporidia of this Sphæria are shown in fig. 19, Plate XXV. The mean length of the sporidia is $\frac{1}{1100}$th of an inch.

There is another species of *Coniothecium* almost as common as the last, known by the name of *Coniothecium betulinum*, and which is to be found upon twigs and small branches of Birch. It bears a strong resemblance to *Coniothecium Amentacearum*, and I have some grounds for supposing that, like that species, it is a conidioid form of fruit of a Sphæria (*Sphæria lanciformis*) which is not uncommon in this country upon Birch. I have not yet been able to prove satisfactorily the connexion between the two, but further observation may possibly establish it.

In the plants above described, the two forms of fruit, assuming them to arise from the same mycelium, must be considered as essentially distinct, that is, the secondary form has no further connexion with the normal fructification, than the fact of being

produced upon the same mycelium; but other cases occur in which the normal form of fruit is so far modified and altered by external circumstances, as to appear at first sight as if belonging to a different species or even genus.

It is an opinion rapidly gaining ground, that by far the greater number of the plants composing the genera *Diplodia*, *Sporocadus*, *Sphæropsis*, *Hendersonia*, *Phoma* and others, are only imperfect forms of different species of *Sphæria*, and a wide field of observation is thus opened in order to allot to each individual member of these pseudo-genera its proper place in the mycological system.

The following instances of this change of form in the fruit have come under my own observation:—

1. *Sphæria angulata*, Fr.—This Sphæria is very common, being found upon almost every small dead fallen branch of Oak. It belongs to the division Incusæ, the perithecia being imbedded in a stroma formed from the bark in small circular patches. The perithecia, with the exception of the ostiola, are usually completely hidden by the bark; but the space which they occupy beneath it is generally well defined by a circular mark surrounding the base of the broad-angled cone, into which the bark is raised by the growth of the ostiola. The sporidia are oblong, but constricted in the middle, varying in length from $\frac{1}{1333}$th to $\frac{1}{815}$th of an inch, and having a septum at the constriction. They are colourless, and their appearance varies considerably according to the state in which the endochrome happens to be at the time of observation. This difference in the endochrome will be seen by referring to figs. 20, 21 and 22, Plate XXV.; in the former the endochrome has a clear oleaginous nucleate appearance, and fills each cell of the sporidium, leaving however a manifest interval between it and the outer membrane; in figs. 21 and 22, on the other hand (the latter representing free sporidia), the endochrome is broken up into a granular mass, which fills each cell of the sporidium so completely as to render the outline of the outer membrane no longer distinguishable. I may observe that this change of state in the endochrome, although striking at first sight, is not uncommon in the sporidia of Sphæriæ, and it is a circumstance which must always be carefully attended to when the form and structure of the sporidia are resorted to for distinguishing species. In this instance the difference between what may be called the nucleate and the granular sporidia is so great, that they might at first sight be supposed to be different forms. In fig. 20 (*a*) is magnified 315 and (*b*) 415 diameters. Figs. 21 and 22 are magnified 315 diameters.

There is another peculiarity connected with this Sphæria, which is, the existence in all the *free* sporidia of four delicate cilia, one proceeding from each pole, and one from each side, which are not perceptible as long as the sporidia are enclosed in the ascus. The fact of these cilia being invisible in the ascus might seem to point to germination as their origin, but the constancy and regularity in their appearance, size and position almost preclude this supposition.

The above varieties in the sporidia are not, however, the main peculiarity in the fructification of this Sphæria, for I have found its perithecia producing in lieu of the

regular asci, immense numbers of the bodies drawn in fig. 23, Plate XXVI. These bodies are shaped like the spores in the genus *Cryptosporium*, and if the perithecia producing them had existed singly, or if a number of perithecia in one group had produced nothing but similar bodies, the plant would have had to be classed with the Cryptosporia and not with the Sphæriæ. But in the specimens upon which my observations were made, the same stroma produced contiguous perithecia, of which some produced the regular asci and sporidia, others the Cryptosporium fructification. The origin of the latter is, I think, sufficiently obvious: it seems to arise from an arrest of growth taking place in the asci, by which not only do the latter not acquire their usual size, but the endochrome, instead of breaking up into separate portions and forming sporidia, remains united in one mass which is uniformly distributed through the ascus. The real nature of these bodies must depend upon whether they possess the power of germination, a fact which I have not yet ascertained. If they have such power, they would rank as real fruit; if not, then they must be regarded simply as abortive productions, worthy of notice only from their resemblance to the spores of a distinct genus.

2. *Sphæria lanciformis*, Fr., and *Hendersonia polycystis*, B. & Br.—There is, as I have stated above, a prevalent opinion amongst mycologists, to the effect that certain Fungi, including those of the genus *Hendersonia*, are only Sphæriæ in an imperfect condition. The species just mentioned (*Hendersonia polycystis*) was first described by Messrs. Berkeley and Broome in their notices of British Fungi in the 'Annals of Natural History,' and it may be found figured, though under a different name (*Myxocyclus confluens*), by Fresenius in his 'Beiträge zur Mykologie.' It is remarkable for the beauty of its spores, which are club-shaped, multicellular, and of a deep olive-brown colour, with the exception of the pedicel-cell, which is hyaline and colourless: the spores are surrounded by a broad gelatinous margin and usually grow singly, but occasionally (as is shown in fig. 24) one pedicel serves as a support to two spores. The double spore in fig. 24 is magnified 415 diameters.

The plant grows upon Birch and is in perfection in very moist weather, when it may be easily recognized by the large, black, soft, gelatinous protuberances on the bark, formed by spores escaping and depositing themselves upon and about the apex of the perithecium. This Hendersonia I suspect to be an abnormal state of a well-known Sphæria (*Sphæria lanciformis*), which also grows upon Birch, and upon Birch only.

The regular fruit of this Sphæria (see fig. 25) consists of asci having eight large, elliptical, dark olive-brown sporidia, of an average length of about $\frac{1}{500}$th of an inch; the sporidia are furnished with several septa, and when seen under the microscope are very beautiful objects. Fig. 25 is magnified 315 diameters.

In the first place, the sporidia of *Sphæria lanciformis* are apt to assume irregular forms. Some of the irregular forms are shown in fig. 26 (*a–e*) and fig. 27, amongst which it will be observed that the spore (*e*), fig. 26, which occurred singly in an ascus, bears considerable resemblance to the fruit of *Hendersonia polycystis*. It is true that it is not divided by longitudinal as well as transverse septa; but that this is not a safe mark

of distinction, may be seen by referring to the sixth spore from the top in the ascus, Plate XXVI. fig. 25, where the cell is divided by a longitudinal septum. Besides this variety in the sporidia, which establishes at least a capacity if not a tendency to change of form in the fructification, I have found perithecia in the same stroma producing the very dissimilar bodies shown in fig. 28, *a*, *b*, *c*, *d*, of which *a* and *b* do not differ from the ordinary form of sporidium of *Sphæria lanciformis*, and *c* and *d* are exactly the fruit of *Hendersonia polycystis* minus the gelatinous border. The absence of the latter is easily accounted for, since all sporidia which have gelatinous envelopes lose them in process of time, and the specimens in which these different forms were found were unfortunately so far past their prime that the perithecia were almost empty. This latter circumstance may perhaps be considered as throwing some doubt upon the observation, and it certainly would have been more satisfactory to have found the Sphæria and Hendersonia in full fruit in the same stroma; but if the facts I have stated be not considered sufficient to establish the identity of the two plants, they must, I think, be admitted to afford reasonable grounds for a suspicion of relationship between them.

If the Hendersonia be an imperfect form of the Sphæria, then the so-called perithecia and fruit of the former will be (to adopt M. Tulasne's terminology) the pycnidia and stylospores of the Sphæria. In the last vol. of the 'Annales des Sciences,' M. Tulasne has described what he considers to be the conidioid form of fruit of *Sphæria lanciformis*, and which from his description (there being no figures) I take to be a species of *Coryneum*, which occurs in the neighbourhood of London upon dead branches of Birch.

3. *Sphæria siparia*, B. & Br., and *Prosthemium betulinum*, Kunze.—This Sphæria, perhaps the most beautiful of all the British species in regard to fructification, was first described in the 'Annals of Natural History' by Messrs. Berkeley and Broome. It belongs to the division Obtectæ, and occurs upon twigs of Birch. The perithecia are large and depressed, and furnished with a short central ostiolum, and are stated in the 'Annals' to be clothed with a more or less dense ferruginous wool. This description quite accords with specimens in my herbarium, with the exception of the colour of the wool, which I find quite white, a difference which may depend upon the period of growth at which the plants are examined. Fig. 29, Plate XXVI. represents an Ascus with sporidia and two free sporidia. The sporidia are oblong-elliptic, but sometimes slightly curved, and divided into a number of cells by transverse and longitudinal septa. There is a slight constriction at each transverse septum, and each sporidium is surrounded by a gelatinous border, which, after the sporidia have escaped from the ascus, is of some width. The colour of the sporidia, when in their prime, or rather, I think, just before maturity, is a bright golden yellow, which subsequently changes to a clear brown tint, after which the gelatinous coat disappears. Their mean length is $\frac{1}{443}$th of an inch. The brown colour is precisely that of the spores of *Prosthemium betulinum*, which latter plant I suspect to be an altered form of the Sphæria.

In the first place, *Sphæria siparia* and *Prosthemium betulinum* are almost always associated in growth. Whenever the one is found, a very slight search will usually

enable the observer to detect the other; and although the mere fact of their growing together would hardly afford any ground for suspecting a relationship between the two, it is a circumstance which, taken in connexion with others, must be allowed to be of considerable weight. But besides their companionship, there is a very strong external resemblance between the Sphæria and the Prosthemium, so much so that it is frequently impossible, without a microscopical examination of the fruit, to decide whether a particular specimen belongs to the one or the other. The perithecia of the Prosthemium are large and depressed, with a central ostiolum, and, although not invariably, are sometimes clothed with the dense wool which is one of the characteristics of the Sphæria. There is, no doubt, a total dissimilarity at first sight between the eight-spored ascus of the Sphæria and the fascicle of spores of the Prosthemium; but there is no great difference, on the contrary, there is not unfrequently a strong resemblance, between the individuals constituting the fascicle and the free sporidia of the Sphæria after the latter have lost their gelatinous coat, which takes place soon after their escape from the ascus. Upon tracing the asci and the fascicles from their earliest stages, it will be found that they originate in precisely the same manner, viz. in the form of a clavate cell proceeding from the wall of the perithecium. This shape is retained throughout its existence in the case of the ascus; but in the fascicle, very shortly after the endochrome has begun to form distinct masses, *i. e.* shortly after the state shown in fig. 30, *a*, a bulging-out of the membrane takes place as shown in fig. 30, *b*, *c*, and the contents of the cell become distributed between the branch-cell and the original one (see fig. 30, *d*. The contents of the branch-cell become separated into distinct masses, precisely as in the primary cell, septa or pseudo-septa are formed at the point of bifurcation as well as in the body of the cell, and after the formation of a similar branch-cell on the other side of the original cell the fascicles eventually assume the form shown in fig. 30, *e* and *f*, and in fig. 31.

The small processes which project from the central cell with the principal spores I take to be only abortive spores, *i. e.* bodies originating in the same manner as the spores themselves, and failing to become perfect from a deficiency in quantity or vitality of the formative matter in the interior of the primary cell.

Figs. 29 and 30 are magnified 220, and fig. 31, 315 diameters.

XXVII. *Researches on a New Class of Alcohols.*

By AUGUSTUS WILLIAM HOFMANN, *LL.D., F.R.S., and* AUGUSTUS CAHOURS, *F.C.S.*

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ON submitting glycerin to destructive distillation, either alone or in the presence of bisulphate of potassium or anhydrous phosphoric acid, a very remarkable product is obtained, which has been elaborately studied by Professor REDTENBACHER.

According to the researches of this distinguished chemist, this substance, to which the name of acrolein has been given, presents all the characters of an aldehyde, and bears, in its reactions, a close resemblance to the vinic aldehyde. Under the influence of oxidizing agents, and especially of oxide of silver, this body changes into an acid which REDTENBACHER designates by the name "Acrylic Acid," and which stands to acrolein in the same relation that acetic acid bears to common aldehyde.

The researches of MM. WILL and WERTHEIM upon the volatile oils of garlic and mustard established a relationship between these compounds and acrolein which has been rendered still more evident by the recent investigations of MM. BERTHELOT and DE LUCA.

In studying the action of iodide of phosphorus upon glycerin, these chemists obtained a volatile liquid containing iodine, the composition of which is expressed by the formula C_6H_5I, and to which they applied the name Iodide of propylene.

This substance is evidently allied to the chloride of propylene, C_6H_5Cl, and bromide of propylene, C_6H_5Br, discovered previously by CAHOURS, REYNOLDS and HOFMANN, in studying the action of chlorine and bromine on the gases which are obtained by submitting either amylic alcohol or valeric acid, as well as their higher homologues, to the action of heat.

A glance at the formulæ of the propylene-compounds, and the volatile oils of garlic and mustard, viz.

Chloride of propylene . . .	C_6H_5Cl,
Bromide of propylene . . .	C_6H_5Br,
Iodide of propylene	C_6H_5I,
Garlic oil	C_6H_5S,
Mustard oil	$C_6H_5C_2NS_2$,

obviously shows that the same radical Allyl, C_6H_5, may be assumed to exist in all these compounds.

This relation did not escape REYNOLDS, who, in his memoir published about six years

ago*, distinctly sets forth the probability of the transformation of the chloride and bromide of propylene into the oils of garlic and mustard by the action respectively of the sulphide and sulphocyanide of potassium.

The experimental proof of the correctness of this idea was however reserved for MM. Berthelot and De Luca, and for M. Zinin. In fact these chemists demonstrated that the action of iodide of propylene upon the sulphocyanide of potassium or silver, gives rise to an oil which is absolutely identical with that furnished by distilling the seeds of the common mustard (*Sinapis nigra*) with water.

The admission of a radical allyl (C_6H_5) analogous to ethyl (C_4H_5), establishes a perfect analogy between the compounds derived from propylene and glycerin, and the several terms of the ethyl-series.

Ethyl-series.		Allyl-series.
C_4H_5Cl	Chloride	C_6H_5Cl.
C_4H_5Br	Bromide	C_6H_5Br.
C_4H_5I	Iodide	C_6H_5I.
C_4H_5S	Sulphide	C_6H_5S.
C_4H_5,C_2NS_2	Sulphocyanide	$C_6H_5C_2NS_2$.
$C_4H_6O_2$	Alcohol	
$C_4H_4O_2$	Aldehyde	$C_6H_4O_2$.
$C_4H_4O_4$	Acid	$C_6H_4O_4$.

There yet remained the key-stone to be inserted into this edifice, in other words, the alcohol had still to be discovered, to which not only all the preceding compounds could be referred, but which might serve in producing a long series of substances analogous to the host of compound ethers derived from common alcohol.

After many attempts, which remained long unsuccessful on account of the facility with which this class of bodies is altered, we at last succeeded in obtaining, not only the alcohol and ether of this series (for which we have adopted the name of the "Allyl-series†"), but also a very considerable number of compound ethers, and some of their derivatives.

In order to arrive at this result, we have submitted many silver-salts to the action of iodide of allyl, expecting to obtain by double decomposition iodide of silver and a compound ether, from which it would be possible afterwards to extract the alcohol. The silver-salts are in general powerfully attacked by iodide of allyl, but comparatively

* Quarterly Journal of the Chemical Society, vol. iii. p. 111, 1851.

† In a preliminary note communicated some time ago to the Royal Society (Proceedings, vol. viii. No. 19), regarding the principal facts detailed in the present memoir, we adopted the term "Acryl-series;" but as the name acryl belongs more properly to the derived radical which may be assumed to exist in acrolein, we now prefer the appellation "Allyl-series," in accordance with the nomenclature originally adopted by MM. Will and Wertheim. This name has also been adopted in several papers which appeared after the above-mentioned notice had been published.

few of the products obtained in this reaction are themselves appropriate for the separation of allylic alcohol. The most satisfactory results we have obtained with oxalate of allyl. Iodide of allyl acts very energetically upon oxalate of silver. After several hours' digestion at 100°, which is most conveniently accomplished in the presence of anhydrous ether, the reaction is completed. The oxalate of allyl furnished by this process can be easily purified; on filtering the ethereal liquor from the iodide of silver and distilling, the ether passes over first, the temperature then rises rapidly, and between 206° and 207°, a colourless transparent liquid distils over, which is the oxalate of allyl. According to an analysis, the details of which will be given further on, this ether contains $C_{16}H_{10}O_8=2C_6H_5,C_4O_8$.

When once the oxalate of allyl has been procured, nothing is easier than to separate from it the allylic alcohol. It is only necessary to treat this product with an excess of dry ammonia, by which oxamide and the allylic alcohol are obtained, according to the following equation:—

$$\underbrace{2C_6H_5,C_4O_8}_{\text{Oxalate of allyl.}}+2H_3N=\underbrace{C_4O_4,H_4N_2}_{\text{Oxamide.}},+\underbrace{2C_6H_5,O,HO}_{\text{Allylic alcohol.}},$$

a result in perfect accordance with the reciprocal action of ammonia and oxalic ether.

$$\underbrace{2C_4H_5,C_4O_8}_{\text{Oxalic ether.}}+2H_3N=\underbrace{C_4O_4,H_4N_2}_{\text{Oxamide.}}+\underbrace{2C_4H_5,O,HO}_{\text{Ethylic alcohol.}}.$$

The alcohol separated by distillation from the oxamide is not absolutely pure—it may retain water and ammonia; but a simple rectification over anhydrous sulphate of copper is sufficient to render it perfectly pure.

Allylic alcohol is a colourless, very mobile liquid, possessing a pungent odour, recalling at the same time that of common alcohol and of mustard oil. It boils at 103° C., and is combustible, but burns with a far more luminous flame than ordinary alcohol. Its flavour is spirituous and pungent. Water, alcohol and wood-spirit dissolve it in all proportions.

The analysis of allylic alcohol has furnished the following results:

0·182 grm. of substance gave 0·414 grm. of carbonic acid and 0·172 grm. of water.

The percentage composition corresponding to these numbers leads to the formula

$$C_6H_6O_2,$$

which requires—

	Theory.		Experiment.
6 equivs. of Carbon . .	36	62·07	62·08
6 equivs. of Hydrogen .	6	10·34	10·43
2 equivs. of Oxygen . .	16	27·59	——
1 equiv. of Allylic alcohol	58	100·00	

Accordingly, allylic alcohol has the same composition as acetone and propylic aldehyde, from both of which substances it nevertheless essentially differs.

Potassium and sodium attack allylic alcohol energetically, especially if the temperature be slightly raised; hydrogen is copiously evolved, and a gelatinous substance is formed obviously analogous to potassium-alcohol, the composition of which must accordingly be represented by the formula

$$C_6H_5KO_2.$$

On treating this last product with iodide of allyl, much heat is evolved, iodide of potassium is deposited in abundance, and at the same time there is formed a colourless volatile liquid, very mobile, lighter than water, and entirely insoluble in that menstruum, which corresponds both in composition and reactions to ordinary ether. This compound, which we will designate by the term allylic ether, and which boils at 82° C., also results from the reaction of iodide of allyl on the oxides of silver and mercury. The following equations represent these reactions:—

$$\underbrace{C_6H_5KO_2}_{\text{Allyl-potassium-alcohol.}}+\underbrace{C_6H_5I}_{\text{Iodide of allyl.}}=KI+\underbrace{C_{12}H_{10}O_2}_{\text{Allylic ether.}}.$$

$$\underbrace{2(C_6H_5I)}_{\text{Iodide of allyl.}}+2AgO=2AgI+\underbrace{C_{12}H_{10}O_2}_{\text{Allylic ether.}}.$$

On treating ethylic potassium-alcohol with iodide of allyl, or allylic potassium-alcohol with iodide of ethyl, iodide of potassium is deposited, a colourless, aromatic, very volatile liquid being formed, which boils at about 64° C., and is a mixed ether containing both the radicals ethyl and allyl.

The analysis of this substance has furnished the following results:—

0·338 grm. of substance gave 0·864 grm. of carbonic acid and 0·352 grm. of water.

The formula

$$C_{10}H_{10}O_2=\left.\begin{matrix}C_4H_5\\C_6H_5\end{matrix}\right\}O_2$$

requires the following values:—

	Theory.		Experiment.
10 equivs. of Carbon . . .	60	69·76	69·69
10 equivs. of Hydrogen . . .	10	11·62	11·56
2 equivs. of Oxygen . . .	16	18·62	——
	86	100·00	

Methyl-, amyl- and phenyl-potassium-alcohols, when brought in contact with iodide of allyl, furnish analogous results.

By distilling allylic alcohol with the chloride, bromide or iodide of phosphorus, the hydrochloric, hydrobromic and hydriodic ethers of this series are obtained with the

utmost facility. Iodide of allyl obtained by this process, possesses all the properties of the iodide of propylene which is produced by the action of iodide of phosphorus on glycerin; and in a similar manner the chloride and bromide of allyl, and the chloride and bromide of propylene, are identical in every respect.

Allylic alcohol dissolves without colour in concentrated sulphuric acid, and gives rise to a conjugate acid, which forms soluble and crystalline salts with baryta, strontia, and oxide of lead.

Anhydrous phosphoric acid powerfully attacks allylic alcohol at a moderate temperature, a colourless gas being disengaged which burns with a very luminous flame. Its analysis has not been made, but most probably it is a hydrocarbon represented by the formula $C_6 H_4$.

Allylic alcohol is readily acted upon by oxidizing agents. A mixture of bichromate of potassium and sulphuric acid attacks this body with extreme violence. The products of the reaction are acrolein and allylic acid. The same transformation is effected by platinum-black, though more slowly. Lastly, this alcohol, when treated with potassa and bisulphide of carbon, gives rise to a compound which crystallizes in beautiful yellow needles, resembling xanthate of potassium, and to which analysis will probably assign an analogous formula.

Sulphide of Allyl (*Essence of Garlic*).

When iodide of allyl is allowed to fall drop by drop into a concentrated alcoholic solution of protosulphide of potassium, a very energetic action ensues, the liquid becomes very hot, and an abundant crystalline deposit takes place of iodide of potassium. It is important that the iodide of allyl should only be added gradually, to avoid spirting, by which a part of the product would be lost. As soon as the action ceases, the liquid is mixed with a slight excess of sulphide of potassium; on the addition of water a light yellowish limpid oil separates, possessing a strong smell of garlic. When rectified, this liquid becomes colourless; it boils at 140° C., and gives with nitrate of silver a crystalline precipitate soluble in a boiling mixture of alcohol and water, and separating from this solvent in the form of beautiful white needles, absolutely identical with those which the natural essence of garlic produces. Sulphide of allyl also comports itself with corrosive sublimate and bichloride of platinum, exactly like the natural garlic oil.

On analysis sulphide of allyl gave the following results:—

0·420 grm. of substance yielded 0·975 grm. of carbonic acid and 0·338 grm. of water.

The formula $C_{12} H_{10} S_2 = 2C_6 H_5 S$, requires the following values:—

	Theory		Experiment.
12 equivs. of Carbon	72	63·16	63·3
10 equivs. of Hydrogen	10	8·78	8·9
2 equivs. of Sulphur	32	28·06	——
1 equiv. of Sulphide of allyl . .	114	100·00	

Allylic Mercaptan.

When in the process for the preparation of the preceding compound the protosulphide of potassium is replaced by the hydrosulphate of sulphide of potassium, a volatile product is obtained, having a similar but more ethereal odour. This substance acts with energy upon oxide of mercury, with which it forms a compound dissolving in boiling alcohol and separating from it on cooling in the form of pearly scales of remarkable brilliancy, which present the greatest resemblance to mercaptide of mercury. The liquid obtained in the above-mentioned reaction boils at 90° C., and possesses a composition and an aggregate of properties which closely resemble those of the mercaptan of the ethyl-series, being in fact the allylic mercaptan,

$$C_6 H_6 S_2.$$

Submitted to analysis, this compound furnished the following results:—

I. 0·500 grm. of substance gave 0·896 grm. of carbonic acid and 0·371 grm. of water.

II. 0·300 grm. gave 0·536 grm. of carbonic acid and 0·226 grm. of water.

	Theory.		Experiment.	
			I.	II.
6 equivs. of Carbon . . .	36	48·65	48·86	48·71
6 equivs. of Hydrogen . .	6	8·11	8·23	8·30
2 equivs. of Sulphur . . .	32	43·24	——	——
1 equiv. of Allylic mercaptan	74	100·00		

This compound is powerfully attacked by nitric acid with disengagement of binoxide of nitrogen; the liquid assumes a red colour in the same manner as ethylic mercaptan, the product of the reaction being a peculiar acid, analogous to ethyl-sulphurous acid, which forms with baryta a salt crystallizing in brilliant white scales.

Sulphallylic Acid.

When allylic alcohol is added in small quantities at a time to about its own volume of monohydrated sulphuric acid, the mixture becomes heated without being sensibly coloured. If the acid liquor be diluted with about seven to eight times its volume of water, and then neutralized with carbonate of barium in fine powder, there is obtained on evaporation a substance which crystallizes in brilliant white laminæ.

This salt, which is formed under the very same conditions as sulphovinate of barium, furnished on analysis the following results:—

0·154 grm. of substance yielded 0·087 grm. of sulphate of barium, which represents 33·11 per cent. of barium.

The formula

$$C_6 H_5, Ba, S_2 O_6$$

requires—

	Theory.		Experiment.
6 equivs. of Carbon	36	17·54	——
5 equivs. of Hydrogen	5	2·43	——
1 equiv. of Barium	68·5	33·33	33·11
2 equivs. of Sulphur	32	15·65	——
8 equivs. of Oxygen	64	31·15	——
1 equiv. of Sulphallylate of barium	205·5	100·00	

Oxalate of Allyl and Allyloxamethan.

We have previously stated under what circumstances oxalate of allyl is formed, we will not therefore recur again to its mode of preparation. Purified by a simple rectification, this compound presents itself as a colourless limpid liquid of a specific gravity of 1·05 at 15°·5 C., and boiling constantly between 206° and 207° C. (Bar. 0^{m}·754), possessing an aromatic odour recalling that of oxalic ether, with a faint admixture of the smell of mustard oil. A similar remark in fact applies to all the allyl-compounds; they smell like the corresponding ethyl-compounds, only somewhat more pungent. When mixed with water, oxalate of allyl undergoes gradual decomposition, but it is much more rapidly decomposed when boiled with a solution of caustic potassa. Solution of ammonia attacks it rapidly, and transforms it into oxamide and allylic alcohol. An alcoholic solution of ammonia added with caution so as to avoid the formation of oxamide, produces a substance analogous to oxamethan, which, for this reason, we shall designate by the name allyloxamethan. This substance separates in magnificent crystals when its alcoholic solution is allowed to evaporate spontaneously.

Potassium and sodium energetically attack the oxalate of allyl, especially with the aid of heat, an evolution of carbonic oxide ensues, whilst there passes over on distillation a limpid colourless oil of an aromatic odour, lighter than water; this is in fact the carbonate of allyl.

The analysis of the oxalate of allyl has furnished the following results:—

I. 0·179 grm. of substance gave 0·3695 grm. of carbonic acid and 0·0951 grm. of water.

II. 0·320 grm. of a second specimen gave 0·662 grm. of carbonic acid and 0·170 grm. of water.

The formula $C_{16}H_{10}O_8 = 2C_6H_5, C_4O_8$

requires the following values:—

	Theory.		Experiment.	
			I.	II.
16 equivs. of Carbon	96	56·47	56·31	56·40
10 equivs. of Hydrogen	10	5·88	5·92	5·89
8 equivs. of Oxygen	64	37·65	——	——
1 equiv. of Oxalate of allyl	170	100·00		

The analysis of a perfectly crystallized specimen of allyloxamethan gave the subjoined numbers.

0·532 grm. of substance gave 0·903 grm. of carbonic acid and 0·266 grm. of water.

The formula

$$C_{10}H_7NO_6 = C_6H_3, C_4H_3NO_6,$$

requires the following numbers, which we place in juxtaposition with the experimental percentages:—

	Theory.		Experiment.
10 equivs. of Carbon . . .	60	46·51	46·29
7 equivs. of Hydrogen . .	7	5·42	5·55
1 equiv. of Nitrogen . . .	14	10·84	——
6 equivs. of Oxygen . . .	48	37·23	——
1 equiv. of Allyloxamethan .	129	100·00	

Acetate of Allyl.

Iodide of allyl, when brought in contact with acetate of silver, becomes strongly heated, producing iodide of the metal and a liquid of an aromatic odour which closely resembles that of acetic ether. The crude product obtained in this way is not pure; it always retains a certain quantity of the iodide of allyl, from which it is purified by one or two rectifications over a fresh quantity of acetate of silver.

Thus prepared, acetate of allyl is a colourless liquid, lighter than water, possessing a pungent aromatic smell, and boiling between 98° and 100° C. Potassa decomposes it at the boiling temperature, giving rise to the formation of acetate of potassium and reproducing allylic alcohol.

Submitted to analysis this substance furnished the following results:—

I. 0·500 grm. of acetate of allyl gave 1·100 grm. of carbonic acid and 0·370 grm. of water.

II. 0·450 grm. of acetate of allyl gave 0·988 grm. of carbonic acid and 0·328 grm. of water.

These numbers represent the formula

$$C_{10}H_8O_4 = C_6H_3, C_4H_5O_4.$$

	Theory.	Experiment.	
		I.	II.
10 equivs. of Carbon . . .	60	60·00	59·87
8 equivs. of Hydrogen . .	8	8·24	8·09
4 equivs. of Oxygen . . .	32	——	——
1 equiv. of Acetate of allyl .	100		

Butyrate of Allyl.

Iodide of allyl becomes strongly heated when brought in contact with butyrate of silver; when the mixture is distilled, an amber-coloured liquid is collected which becomes colourless by rectification. This liquid is oily, very limpid, lighter than water, boils at about 140° C., and has an odour like that of butyric ether. Concentrated

solution of potassa, when assisted by heat, decomposes this ether rapidly, producing allylic alcohol and an alkaline butyrate.

On analysis this compound afforded the following results:—

I. 0·400 grm. of substance gave 0·966 grm. of carbonic acid and 0·345 grm. of water.

II. 0·450 grm. of substance yielded 1·081 grm. of carbonic acid and 0·384 grm. of water.

These numbers may be translated into the formula

$$C_{14}H_{12}O_4 = C_6H_5, C_8H_7O_4.$$

	Theory.		Experiment.	
			I.	II.
14 equivs. of Carbon . . .	84	65·62	65·86	65·51
12 equivs. of Hydrogen . .	12	9·38	9·58	9·48
4 equivs. of Oxygen . . .	32	25·00	——	——
1 equiv. of Butyrate of allyl	128	100·00		

Valerate of Allyl.

Valerate of silver comports itself with iodide of allyl exactly like the acetate and butyrate: powerful reaction, attended by separation of iodide of silver, and production of an ethereal liquid which may be obtained pure by repeated rectification over fresh quantities of valerate of silver, washing with slightly alkaline water, and digestion over chloride of calcium.

Thus prepared, valerate of allyl is a colourless liquid which boils at 162°, and the aromatic odour of which recalls that of valerate of ethyl. It is lighter than water, insoluble in this menstruum, but readily dissolved by alcohol and ether. The vapour burns with a luminous flame.

0·384 grm. of substance gave 0·949 grm. of carbonic acid and 0·350 grm. of water.

The formula

$$C_{16}H_{14}O_4 = C_6H_5, C_{10}H_9O_4$$

requires—

	Theory.		Experiment.
16 equivs. of Carbon	96	67·60	67·40
14 equivs. of Hydrogen . . .	14	9·86	10·12
4 equivs. of Oxygen . . .	32	22·54	——
1 equiv. of Valerate of allyl .	142	100·00	

Benzoate of Allyl.

Benzoate of allyl may be obtained either by the action of chloride of benzoyl on allylic alcohol, or by the action of iodide of allyl on benzoate of silver.

It is a nearly colourless liquid with a tinge of amber, heavier than water, in which fluid it is insoluble; it possesses an aromatic odour similar to that of benzoic ether, and boils at 228 C°. Alcohol, wood-spirit, and ether dissolve it freely.

A boiling aqueous solution of potassa decomposes it, reproducing allylic alcohol and benzoic acid which unites with the alkali.

The benzoate of allyl furnished on analysis the following results:—

0·372 grm. of substance gave 0·997 grm. of carbonic acid and 0·209 grm. of water.

The formula

$$C_{20}H_{10}O_4 = C_6H_5, C_{14}H_5O_4$$

requires—

		Theory.	Experiment.
20 equivs. of Carbon	120	74·07	73·10
10 equivs. of Hydrogen	10	6·17	6·24
4 equivs. of Oxygen	32	19·76	——
1 equiv. of Benzoate of allyl	162	100·00	

Cyanate of Allyl.

Cyanate of silver is rapidly attacked even in the cold by iodide of allyl, the heat generated by the reaction being sufficient to carry over nearly the whole of the product.

By this means a colourless very limpid liquid is obtained, boiling at 82°, possessing an extremely pungent odour, which, like that of cyanic ether, produces lacrymation in a high degree. When mixed with ammonia, this substance rapidly disappears, a gentle evolution of heat being perceptible, and the solution furnishes on evaporation a magnificently crystallized body which is the allylic urea. Ethylamine behaves in a similar manner with cyanate of allyl, and on evaporation beautiful prisms of ethyl-allyl-urea are obtained.

When submitted to analysis cyanate of allyl furnished the following results:—

I. 0·438 grm. of substance yielded 0·927 grm. of carbonic acid and 0·244 grm. of water.

II. 0·335 grm. of substance gave 0·926 grm. of carbonic acid and 0·184 grm. of water.

III. 0·381 grm. of substance gave 53 cub. cent. of nitrogen at the temperature of 10° C. and a pressure of 0^m·761.

These results may be represented by the formula

$$C_8H_5NO_2 = C_6H_5, C_2NO_2.$$

	Theory.		Experiment.		
			I.	II.	III.
8 equivs. of Carbon	48	57·83	57·72	57·78	——
5 equivs. of Hydrogen	5	6·02	6·18	6·09	——
1 equiv. of Nitrogen	14	16·86	——	——	16·71
2 equivs. of Oxygen	16	19·29	——	——	——
1 equiv. of Cyanate of allyl	83	100·00			

We have controlled the preceding formula by a determination of the density of the vapour of cyanate of allyl. The following are the details of observation:—

Temperature of the air	12° C.
Temperature of vapour	152° C.
Excess of weight of balloon filled with vapour, over that of balloon filled with air	0·386 grm.

Capacity of balloon	292 cubic cent.
Barometer	0m·762.
Remaining air	0
Weight of litre of vapour	3·961 grms.
Experimental density of vapour	3·045

The weight represented by the formula $C_8 H_5 NO_2$, divided by the experimental density, furnishes the quotient $\frac{83}{3·045}=27·25$.

For a condensation to four volumes of vapour, the more accurate number 28·92 should have been found. Thus it is obvious that the condensation to four volumes observed for most organic substances, obtains also with cyanate of allyl.

The theoretical density of the vapour of cyanate of allyl is $\frac{83}{28·92}=2·87$.

Allyl-urea.

The formation of this substance has already been mentioned. On submitting a solution of cyanate of allyl in ammonia to evaporation, a beautifully crystallized colourless substance is obtained which is soluble in water and alcohol. It contains

$$C_8 H_8 N_2 O_2 = C_6 H_4, C_2 H_4 N_2 O_2,$$

and differs therefore from thiosinnamine only in having the sulphur of this substance replaced by an equivalent proportion of oxygen.

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This compound, which presents the properties of a true urea, and is prepared under conditions which determine the production of this class of compounds, furnished the following results on analysis:—

I. 0·320 grm. of substance gave 0·561 grm. of carbonic acid and 0·234 grm. of water.

II. 0·275 of substance gave 66 cub. cent. of nitrogen at the temperature of 17° and under a pressure of 0m·759.

	Theory.	Experiment. I.	Experiment. II.
8 equivs. of Carbon	48	47·81	——
8 equivs. of Hydrogen	8	8·11	——
2 equivs. of Nitrogen	28	——	27·86
2 equivs. of Oxygen	16	——	——
1 equiv. of Allyl-urea	100		

Ethyl-allyl-urea.

On treating cyanate of allyl by solution of ethylamine, instead of ammonia, a series of perfectly analogous phenomena is observed. On cooling the fluid solidifies into a crystalline mass which dissolves in alcohol, and separates from this solution, on evaporation, in the form of beautiful prisms similar to those of allyl-urea. This substance, which constitutes the ethyl-allyl-urea, yielded on analysis the following results:—

0·500 grm. of substance gave 1·029 grm. of carbonic acid and 0·417 grm. of water. The formula

$$C_{12}H_{12}N_2O_2 = C_4H_5,\ C_8H_7,\ C_2H_2N_2O_2$$

requires—

	Theory.		Experiment.
12 equivs. of Carbon . . .	72	56·25	56·12
12 equivs. of Hydrogen . .	12	9·37	9·25
2 equivs. of Nitrogen . . .	28	21·95	——
2 equivs. of Oxygen . . .	16	12·43	——
1 equiv. of Ethyl-allyl-urea .	128	100·00	

The action of methylamine, of amylamine, and of aniline upon cyanate of allyl, has given us analogous results. We have not submitted these products to analysis; but their resemblance to the preceding compounds, and especially their mode of formation, leave no doubt respecting their composition. These products, as is obvious, belong to that numerous class of compound ureas which have been studied with such remarkable results by M. Wurtz.

Diallyl-urea (Sinapoline).

When cyanate of allyl is heated with water, it assumes a buttery consistence and ultimately solidifies entirely. The product obtained under these circumstances possesses all the properties of sinapoline, which, as is known, is formed by the action of hydrated protoxide of lead upon mustard oil, and is in fact diallyl-urea. The properties of this substance presenting complete identity with those of sinapoline, and its mode of formation being perfectly similar to that of dimethyl- and diethyl-urea, we have not thought it necessary to submit it to analysis. This body is likewise formed when cyanate of allyl is boiled with an aqueous solution of potassa, with the view of transforming it, after the manner of ordinary cyanic ether, into an alkaline carbonate and an ammonia-base. In this case it is necessary to arrest the action at a certain point, for if the ebullition be continued, the sinapoline is destroyed in its turn, producing carbonate of potassium and ammonia-bases.

The formation of sinapoline, by the reciprocal action of cyanate of allyl and water, is represented by the following equation:—

$$\underbrace{2(C_8H_5NO_2)}_{\text{Cyanate of allyl.}} + 2HO = 2CO_2 + \underbrace{C_{14}H_{12}N_2O_2}_{\text{Sinapoline or diallyl-urea.}}$$

Allyl-ammonias.

When cyanate of allyl is boiled with a concentrated solution of caustic potassa until the oily layer entirely disappears, a complete decomposition ensues, and the residue contains only a mixture of carbonate of potassium with an excess of the alkaline hydrate; the volatile products condensed in hydrochloric acid furnish a saline mass which may be

obtained crystallized by evaporation. This substance is always a mixture, of which the quality and quantity of the constituents are variable. The action of potassa on it always produces a base which is readily soluble in water, but also basic oils insoluble in water, the boiling-point of which rises as high as 180° C. Simple distillation of the liberated bases appears to give rise to decomposition, and thus to induce further complication. We have not succeeded in ascertaining with perfect precision the nature of this mixture, but we have found that it invariably contains a considerable quantity of a base which bears the same relation to allylic alcohol that is observed between ethylamine and ordinary alcohol. The formation of this substance, which we propose to designate *allylamine*, is perfectly analogous to the production of ethylamine by means of cyanate of ethyl,

$$\underbrace{C_6H_5NO_2}_{\text{Cyanate of ethyl.}}+2(KO,HO)=2KCO_2+\underbrace{C_4H_7N}_{\text{Ethylamine.}}$$

$$\underbrace{C_8H_5NO_2}_{\text{Cyanate of allyl.}}+2(KO,HO)=2KCO_2+\underbrace{C_6H_7N}_{\text{Allylamine.}}$$

But, as we have already said, the decomposition of the cyanate of allyl is far from presenting the same definite character as that of the corresponding compound of the ethyl-series.

Since it was impossible to separate the mixture of bases by a fractional distillation, we have determined the composition of allylamine by the analysis of the platinum-salt.

Allylamine.

On adding a solution of bichloride of platinum to the hydrochloric solution of the bases resulting from the action of potassa upon cyanate of allyl, there is produced almost invariably a pale yellow precipitate; and the solution filtered off from this precipitate deposits on evaporation a magnificent salt, crystallizing in brilliant orange-coloured needles. The pale precipitate* varies in quantity and composition; but the second salt

* This precipitate appears to be a mixture in variable proportions of the platinum-salts of ammonia and methylamine. The following are some determinations of the platinum in this precipitate:—

I. 0·3405 grm. of the platinum-salt gave 0·1483 grm. of platinum.
II. 0·26375 grm. of the platinum-salt gave 0·1150 grm. of platinum.
III. 0·2945 grm. of the platinum-salt gave 0·1250 grm. of platinum.
IV. 0·321 grm. of the platinum-salt gave 0·1357 grm. of platinum.
V. 0·3345 grm. of the platinum-salt gave 0·1412 grm. of platinum.
VI. 0·2205 grm. of the platinum-salt gave 0·0927 grm. of platinum.
VII. 0·297 grm. of the platinum-salt gave 0·124 grm. of platinum.

Percentages.

I.	II.	III.	IV.	V.	VI.	VII.
43·61	43·6	42·44	42·27	42·21	42·06	41·75

The platinum-salt of ammonium contains 44·22 per cent. of platinum, and that of methylammonium 41·68.

has a perfectly definite and constant composition, it is soluble in water and may be recrystallized from it. The analyses of samples of different preparations gave us the following results:—

I. 0·467 grm. of the platinum-salt gave { 0·216 grm. of carbonic acid. / 0·129 grm. of water.
II. 0·219 grm. of the platinum-salt gave 0·082 grm. of platinum.
III. 0·272 grm. of the platinum-salt gave 0·102 grm. of platinum.
IV. 0·308 grm. of the platinum-salt gave 0·1155 grm. of platinum.
V. 0·2725 grm. of the platinum-salt gave 0·1025 grm. of platinum.
VI. 0·099 grm. of the platinum-salt gave 0·03725 grm. of platinum.

These results, translated into percentages, lead to the following numbers:—

	I.	II.	III.	IV.	V.	VI.
Carbon	12·61	——	——	——	——	——
Hydrogen	3·07	——	——	——	——	——
Platinum	——	37·44	37·5	37·5	37·61	37·62

The formula of the platinum-salt of allylamine

$$C_6H_7N, HCl + PtCl_2 = \left.\begin{matrix} C_6H_5 \\ H \\ H \end{matrix}\right\} N, HCl, PtCl_2$$

requires the following theoretical values:—

		Theory.
6 equivs. of Carbon	36	13·68
8 equivs. of Hydrogen	8	3·05
1 equiv. of Nitrogen	14	5·31
3 equivs. of Chlorine	106·5	40·46
1 equiv. of Platinum	98·7	37·50
1 equiv. of Platinum-salt	263·2	100·00

The same body is likewise formed by the action of ammonia upon the iodide of allyl. On distilling the product of this reaction with potassa, ammonia is disengaged; but towards the end of the operation an alkaline liquid distils over, which no longer contains ammonia, and upon which there floats a small quantity of a basic oil. On saturating the aqueous solution, together with the oil, with hydrochloric acid, and adding bichloride of platinum, the platinum salt of allylamine separates either directly, if the solution be concentrated, or gradually, in magnificent gold-coloured needles, if the liquid be gently evaporated.

0·183 grm. of the salt gave on ignition 0·0685 grm. = 37·43 per cent. of platinum.

Diallylamine.

Impure allylamine, obtained either by the action of potassa on the cyanate, or of

ammonia on the iodide of allyl, when digested with a second quantity of iodide of allyl, is rapidly changed into a mass of hydriodates of new compounds.

This crystalline mass evidently contains a considerable quantity of the hydriodate of diallylamine; but to succeed in separating it from this very complicated mixture would have required a considerable quantity of substance. We have therefore been satisfied to determine by a few experiments the existence of triallylamine and the oxide of tetrallylammonium.

Triallylamine.

The oxide of tetrallylammonium, to which we shall presently allude, submitted to the action of heat, is decomposed with liberation of a basic oil. The experiment was not made on a sufficient scale to admit of the study of the complementary products. The perfect analogy of the oxide of tetrallylammonium with the corresponding ethyl-compound, leaves no doubt respecting the nature of the basic substance which results from its decomposition by the action of heat. This body must obviously be triallylamine, and experiment has confirmed this anticipation. The base saturated with hydrochloric acid and mixed with bichloride of platinum, deposits a yellow platinum-salt, the analysis of which has furnished the following results:—

0·1815 grm. of the platinum-salt gave 0·0515 grm. of platinum.

The formula

$$C_{18}H_9N, HCl + PtCl_2 = \left.\begin{matrix} C_6H_3 \\ C_6H_3 \\ C_6H_3 \end{matrix}\right\} N, HCl + PtCl_2$$

represents the following values:—

	Theory.		Experiment.
18 equivs. of Carbon . .	108	31·47	——
4 equivs. of Hydrogen .	16	4·66	——
1 equiv. of Nitrogen . .	14	4·08	——
3 equivs. of Chlorine . .	106·5	31·03	——
1 equiv. of Platinum . .	98·7	28·76	28·37
1 equiv. of Platinum-salt	343·2	100·00	

Hydrated Oxide of Tetrallylammonium.

The chief product of the action of ammonia upon iodide of allyl is a magnificent crystalline compound, which is deposited from the solution resulting from the reaction. We have treated iodide of allyl with ammonia in sealed tubes, but this is not necessary, for iodide of allyl is rapidly attacked even by an aqueous solution of ammonia at the common temperature. By contact for a few days a large quantity of the iodide dissolves, and the solution sometimes becomes a solid mass. If no deposition of crystals take place from the solution, it is only necessary to add a concentrated solution of potassa; this causes the separation of an oily layer which speedily solidifies. The crystals thus

formed are the iodide of tetrallylammonium, which, like the corresponding compound of the ethyl- and methyl-series, is but slightly soluble in solution of potassa. It is obtained pure by exposing it to the air until the potassa is converted into carbonate, and then recrystallizing it from absolute alcohol.

On analysis, 0·3805 grm. of the iodide gave 0·2905 grm. of iodide of silver.

The formula

$$C_{24}H_{20}NI = \left.\begin{matrix}C_6H_5\\C_6H_5\\C_6H_5\\C_6H_5\end{matrix}\right\}NI$$

leads to the following values:—

	Theory.		Experiment.
24 equivs. of Carbon	144	47·20	——
20 equivs. of Hydrogen	20	6·55	——
1 equiv. of Nitrogen	14	4·59	——
1 equiv. of Iodine	127·1	41·66	41·26
1 equiv. of Iodide of tetrallylammonium	305·1	100·00	

The iodide, treated with oxide of silver, is immediately transformed into the oxide. This forms an alkaline solution, which possesses all the properties of the hydrated oxide of tetrethylammonium. When it is mixed with hydrochloric acid and bichloride of platinum, a pale yellow precipitate is obtained, of which the following is an analysis:—

I. 0·21 grm. of the platinum-salt gave 0·288 grm. of carbonic acid and 0·0987 grm. of water.

II. 0·311 grm. gave 0·0805 grm. of platinum.

These numbers lead to the formula

$$C_{24}H_{20}NCl,\ PtCl_2 = \left.\begin{matrix}C_6H_5\\C_6H_5\\C_6H_5\\C_6H_5\end{matrix}\right\}N\,Cl,\ PtCl_2.$$

	Theory.		Experiment.	
			I.	II.
24 equivs. of Carbon	144	37·57	37·40	——
20 equivs. of Hydrogen	20	5·24	5·19	——
1 equiv. of Nitrogen	14	3·65	——	——
3 equivs. of Chlorine	106·5	27·79	——	——
1 equiv. of Platinum	98·7	25·75	——	25·88
1 equiv. of Platinum-salt	383·2	100·00		

The examination of the allylamines is far from complete; we have not succeeded in separating these bodies from one another with sufficient accuracy to give a precise

account of the physical properties of the different terms of this series; nevertheless our experiments prove the existence in the allyl-series of four compounds analogous to the four bases of the ethyl-series.

Iodide of Tetrallylarsonium.

Iodide of allyl attacks arsenide of potassium, but far less energetically than iodide of ethyl; several liquid compounds, of very fetid odour, are produced, the boiling-points of which, however, extend over so considerable a range of the scale, that we had to abandon our desire of studying them. At the same time a solid crystalline body is formed which is obviously iodide of tetrallylammonium, in which the nitrogen is replaced by arsenic,

$$4(C_6H_5)\,As\,I.$$

On reviewing the preceding results, it is obvious that propylene, a homologue of ethylene, is susceptible of furnishing a mono-acid alcohol and its derivatives, which bears to ethylene the same relation that is observed between methyl-alcohol and marsh-gas, or between benzoic alcohol and toluol, the analogue of marsh-gas among the derivatives of the aromatic acids. The hydrocarbons homologous with, and analogous to marsh-gas, the homologues of ethylene, and indeed probably a very considerable number of other similarly constituted hydrocarbons, may be regarded as starting-points for the preparation of mono-acid alcohols and their derivatives. All these alcohols are formed by the fixation of two equivalents of oxygen, which oxidation, however, cannot yet be effected directly, but is accomplished by a series of substitution-processes.

$$\underbrace{C_2H_4}_{\text{Marsh-gas.}} + O_2 = \underbrace{C_2H_4O_2}_{\text{Methyl-alcohol.}}$$

$$\underbrace{C_{14}H_8}_{\text{Toluol.}} + O_2 = \underbrace{C_{14}H_8O_2}_{\text{Benzoyl-alcohol.}}$$

$$\underbrace{C_6H_6}_{\text{Propylene.}} + O_2 = \underbrace{C_6H_6O_2}_{\text{Allyl-alcohol.}}$$

Olefiant gas and its homologues are capable of furnishing mono-acid alcohols by another reaction, which has not yet been applied to the series of bodies homologous with marsh-gas. In fact, we have learnt by the researches of M. Berthelot, that olefiant gas and its homologues are capable of being transformed into alcohols by the absorption of two equivalents of water; olefiant gas furnishes by this reaction ordinary alcohol; and propylene, propylic alcohol; thus—

$$\underbrace{C_4H_4}_{\text{Ethylene.}} + 2HO = \underbrace{C_4H_6O_2}_{\text{Ethylic alcohol.}}$$

$$\underbrace{C_6H_6}_{\text{Propylene.}} + 2HO = \underbrace{C_6H_8O_2}_{\text{Propylic alcohol.}}$$

Olefiant gas, then, and all its homologues are capable of furnishing *two* mono-acid alcohols, exhibiting slightly different features, but presenting in all their principal characters the most obvious analogies. The alcohols formed by fixation of water, such as ethylic alcohol, have been long well known; in fact the detailed study which has been made during the last quarter of a century of most of their derivatives, has exercised the most important influence upon the development of organic chemistry.

The same remark does not apply to the alcohols produced from these same hydrocarbons by fixation of oxygen. The history of allylic alcohol, which we have endeavoured to trace in the preceding pages, and to which of late important contributions have been furnished by MM. Berthelot and De Luca and by M. Zinin, begins also to throw light upon this second group of alcohols. For this reason, the discovery of allylic alcohol appears to claim the attention of chemists, even had it not assisted in fixing, in the system of organic compounds, the exact position of several very interesting natural products, viz. the sulphuretted oils furnished by the bulbs of the garlic (*Allium sativum*) and the seeds of the black mustard (*Sinapis nigra*), which it had not been possible up to the present time to group around a similar centre.

It is highly probable that by submitting the various hydrocarbons polymeric with propylene to appropriate reactions, other alcohols may be obtained homologous with allyl-alcohol. This much is certain, that by distilling a mixture of cyanate of silver and bromide of amylene, $C_{10} H_9 Br$, a volatile liquid is obtained, which possesses an extremely irritating odour, causing lacrymation like the cyanic ethers. This forms by contact with ammonia a crystallizable urea, which, when decomposed by potassa, yields, like cyanate of allyl, bases, among which the compound $C_{10} H_{11} N$ may be traced.

Allylic alcohol, the history of which we have endeavoured to sketch, forms the third term of a series of alcohols, which may be represented by the general formula

$$C_{2n} H_{2n} O_2.$$

In fact, a group of homologous acids is known, which, according to their composition, bear the same relation to acetic acid and its homologues, that is observed between our new group of alcohols, and the alcohol-series commencing with methylic and ethylic alcohols. In addition to acrylic acid, angelic and oleic acids are the best known terms of this group of acids. It is possible that cyanide of allyl, which is formed by the double decomposition of iodide of allyl and cyanide of silver, but which we have not yet been able to obtain in a state of sufficient purity to admit of analysis, may be transformed by boiling with a solution of caustic potassa into an acid homologous with acrylic acid, in the same way as cyanide of propyl is resolved under similar circumstances into butyric acid.

We conclude this paper with a Table which exhibits in a perspicuous manner the two groups of alcohols and acids.

Alcohols.		Acids.	
$C_2H_4O_2$ Methylic.	$C_2H_2O_2$	$C_2H_2O_4$ Formic.	——
$C_4H_6O_2$ Ethylic.	$C_4H_4O_2$	$C_4H_4O_4$ Acetic.	$C_4H_2O_4$
$C_6H_8O_2$ Propylic.	$C_6H_6O_2$ Allylic.	$C_6H_6O_4$ Propionic.	$C_6H_4O_4$ Acrylic.
$C_8H_{10}O_2$ Butylic.	$C_8H_8O_2$	$C_8H_8O_4$ Butyric.	$C_8H_6O_4$
$C_{10}H_{12}O_2$ Amylic.	$C_{10}H_{10}O_2$	$C_{10}H_{10}O_4$ Valeric.	$C_{10}H_8O_4$ Angelic.
$C_{12}H_{14}O_2$ Caproylic.	$C_{12}H_{12}O_2$	$C_{12}H_{12}O_4$ Caproic.	$C_{12}H_{10}O_4$
$C_{14}H_{16}O_2$	$C_{14}H_{14}O_2$	$C_{14}H_{14}O_4$ Œnanthylic.	$C_{14}H_{12}O_4$
$C_{16}H_{18}O_2$ Caprylic.	$C_{16}H_{16}O_2$	$C_{16}H_{16}O_4$ Caprylic.	$C_{16}H_{14}O_4$
——	——	——	——
——	——	——	——
——	——	——	——
——	——	——	——
$C_{36}H_{38}O_2$	$C_{36}H_{36}O_2$	$C_{36}H_{36}O_4$ Stearic.	$C_{36}H_{34}O_4$ Oleic.

A glance at this Table shows how very few representatives of the series of alcohols $C_{2n}H_{2n}O_2$, and acids $C_{2n}H_{2n-2}O_4$, have as yet been observed. The progress of science will not fail to fill up the numerous gaps.

Even now some bodies are known which are most intimately connected with the preceding series. Indeed, cinnamic alcohol and cinnamic acid belong to a group of substances bearing to the aromatic alcohols and acids the same relation which obtains between the allyl-compounds and their homologues, and the alcohols and acids of the fatty bodies.

Alcohols.		Acids.	
$C_{14}H_8O_2$ Benzoic.	$C_{14}H_6O_2$	$C_{14}H_6O_4$ Benzoic.	$C_{14}H_4O_4$
$C_{16}H_{10}O_2$	$C_{16}H_8O_2$	$C_{16}H_8O_4$ Toluic.	$C_{16}H_6O_4$
$C_{18}H_{12}O_2$	$C_{18}H_{10}O_2$ Cinnamic.	$C_{18}H_{10}O_4$	$C_{18}H_8O_4$ Cinnamic.
$C_{20}H_{14}O_2$ Cuminic.	$C_{20}H_{12}O_2$	$C_{20}H_{12}O_4$ Cuminic.	$C_{20}H_{10}O_4$

From this Table, again, it is manifest how many substances remain still to be discovered in this direction. Cinnamic acid contains two equivalents of hydrogen less than the acid

$C_{18}H_{10}O_4$, which will be discovered between toluic and cuminic acids. It stands to this acid in the same relation as acrylic stands to propionic acid.

The acids belonging to the acrylic group all possess the property of being transformed, by fixation of two atoms of oxygen and two of water, into acetic acid and a second homologous acid, as is shown by the following equation:—

$$C_{2n}H_{2n-2}O_4+O_2+2HO=C_4H_4O_4+C_{2n-4}H_{2n-4}O_4.$$

It is thus that acrylic, angelic, and oleic acids are resolved under the influence of hydrate of potassa at a temperature of 200° C. into acetic acid, and into formic, propionic, and ethalic acids.

Cinnamic acid suffers under the influence of fusing potassa a perfectly analogous transformation. This acid fixes two equivalents of oxygen and two equivalents of water, and yields acetic and benzoic acids.

In carrying out the preceding inquiry, we have been very ably assisted by Mr. Arthur H. Church, to whom our best thanks are due.

XXVIII. *Researches on the Phosphorus-Bases.*

By AUGUSTUS WILLIAM HOFMANN, *LL.D., F.R.S., and* AUGUSTUS CAHOURS, *F.C.S.*

Received June 13th,—Read June 18th, 1857.

IN a note on the action of chloride of methyl upon phosphide of calcium, communicated more than ten years ago to the Institute of France*, M. PAUL THÉNARD pointed out the existence of a series of bodies which correspond to the compounds of phosphorus with hydrogen, which may, in fact, be viewed as hydrides of phosphorus, the hydrogen of which is replaced by an equivalent quantity of methyl.

One of these bodies, a liquid possessing a most offensive odour, spontaneously inflammable and explosive in the highest degree, corresponds to the liquid phosphoretted hydrogen, and appears to occupy, in the phosphorus-series, the same position which belongs to kakodyle among the arsenic-compounds. It is a colourless, somewhat viscid liquid which boils at about 250° C. Exposed to the slow action of the atmosphere this liquid is converted into a compound which is strongly acid and easily crystallizes. This acid is probably analogous to kakodylic acid.

In addition to this liquid body, two solid substances are formed by the action of chloride of methyl upon phosphide of calcium. One of these, according to M. PAUL THÉNARD, corresponds to the solid phosphoretted hydrogen, whilst the other, which is the principal product of the reaction, constitutes the hydrochlorate of a very volatile phosphoretted base. From its composition this body may be viewed as ammonia, in which the nitrogen is replaced by phosphorus, whilst methyl is substituted for the hydrogen. When repeating these experiments in the ethyl-series PAUL THÉNARD arrived at similar results, to which he however only briefly alludes.

At the time when these experiments were first made, the ammonia-bases had not been discovered, and the subject presented obstacles so numerous and varied that the researches of this chemist remained unfinished. Nobody will be surprised at this, who has made himself acquainted with the difficulty of effecting the above-mentioned reactions, and who from his own experience knows the danger which attends the preparation of these compounds, and the horrible odour which some of them possess.

PAUL THÉNARD's remarkable researches did not excite, at the time of their publication, that degree of interest which they really deserved. There were but few facts known, at that period, with which his results could naturally be connected; indeed, until after the ammonia-bases were discovered, the importance of the experiments on these phosphorus-bodies could scarcely be recognized; then it was that M. PAUL THÉNARD's investigations

* Comptes Rendus, t. xxi. p. 144, and t. xxv. p. 892.

attracted that attention to which they were entitled; then it was that the remarkable parallelism of the compounds of phosphorus and nitrogen, more and more distinctly exhibited by these discoveries, began to become an object of general interest to chemists.

It is now many years since M. Paul Thénard abandoned the study of the phosphorus-compounds, for the first knowledge of which we are indebted to him. The unfinished state in which these researches remained, and the rich and abundant harvest collected since that period in all the neighbouring fields of science, necessitated a revision of the subject. The discovery of methylamine, dimethylamine, and trimethylamine, and of the corresponding terms in the ethyl- and amyl-series, had shown that the hydrogen in ammonia may be replaced by binary molecules, such as methyl, ethyl, amyl and phenyl, the newly-formed compounds retaining the basic character of the original ammonia-molecule; whilst the production of triethylstibine and triethylarsine had furnished the proof that the total replacement of the hydrogen in the indifferent antimonietted and arsenietted hydrogen exalts the chemical character of these compounds in a most remarkable manner, the methylated and ethylated bodies exhibiting basic characters scarcely inferior to those of ammonia itself. It remained therefore to be investigated whether phosphorus, which, by its chemical tendencies, stands between nitrogen and arsenic, would exhibit a similar deportment. It remained to be ascertained in what manner the gradual entrance of binary molecules in the place of the hydrogen in phosphoretted hydrogen would change the character of the original compound. The discovery of the tetrethylated ammonium-bases had also opened a new field of research, in which the corresponding terms of the antimony- and arsenic-series were rapidly brought to light. It was indeed possible to predict with certainty, that an appropriately selected method would lead to the production of the analogous derivatives of phosphoretted hydrogen. The time for resuming the study of the phosphorus-bases had in fact arrived.

We have been engaged for a considerable time in the investigation of this subject, and now beg to offer to the Royal Society in the following pages an account of our experiments.

In the first place, we have endeavoured to obtain the bases corresponding to phosphoretted hydrogen, by a method analogous to that followed by M. Paul Thénard.

Recent experience suggested at once the replacement of the gaseous chloride of methyl by the liquid iodide, which is so much more convenient for experiment; and also the substitution, for the phosphide of calcium, of the compound of phosphorus and sodium obtained by the direct union of the elements. On the application of heat, these substances act on one another with great energy, producing combustible and detonating compounds, so that the experiment is not without danger. Often the product of the operation is lost; and if the reaction has taken place without explosion, the separation of the constituents of the very complicated mixture which results can be effected only with the greatest difficulty.

We have convinced ourselves that the product of the action of iodide of methyl upon phosphide of sodium consists chiefly of three different substances, viz. of a liquid which probably is Me_2P, and corresponds to kakodyl; of a second liquid, Me_3P, corresponding to trimethylstibine and trimethylarsine; and lastly, of a beautiful crystalline solid body, Me_4PI, which is the analogue in the phosphorus-series of iodide of tetramethylammonium.

We abstain from a minute description of the experiments made in this direction, since in the further course of the inquiry we have forsaken this method altogether. Indeed this mode of preparation is very uncertain, and the separation of the products formed is attended with almost insurmountable obstacles, not to speak of the difficulty of obtaining pure phosphide of sodium fit for the reaction. The question resolved itself into the discovery of a method which would yield us the desired substances conveniently, without danger, in considerable quantity, and in a state of absolute purity.

It appeared to us that the action of terchloride of phosphorus on zinc-methyl, zinc-ethyl, &c. would enable us to attain the desired result. Experiment has fully confirmed this anticipation.

Dr. FRANKLAND'S remarkable observations on the action of zinc upon iodide of methyl and iodide of ethyl at high temperatures are still fresh in the memory of chemists. Besides the hydrocarbons methyl and ethyl, zinc-methyl and zinc-ethyl are formed in this reaction, which exhibit the deportment of true organic metals, comparable in the intensity of their combining powers with the most electro-positive elements. In the action of a chloride upon such a compound metal, the chlorine was sure to seize upon the zinc, and it was extremely probable that, together with chloride of zinc, methyl- or ethyl-compounds would be formed in definite proportions. In the action of terchloride of phosphorus, the formation of a methyl- or ethyl-compound of phosphorus, corresponding in composition to the terchloride of phosphorus, might be with certainty expected:

$$PCl_3 + 3MeZn = 3ZnCl + Me_3P.$$

These anticipations were in fact fulfilled. The products of these reactions, the bases Me_3P and E_3P, which we propose to call respectively trimethylphosphine and triethylphosphine, remain united with chloride of zinc, and a simple distillation with an alkali alone is necessary to liberate them.

$$3ZnCl, Me_3P + 6KO = 3KCl + 3(KO, ZnO) + PMe_3$$
$$3ZnCl, E_3P + 6KO = 3KCl + 3(KO, ZnO) + PE_3.$$

They are obtained in this way as volatile oils, with a peculiar and strongly-marked odour, and possessing distinctly basic properties.

We found no difficulty in procuring the bodies in question by this method in a state of perfect purity, so as to examine their properties with accuracy.

From the outline which we intend to give, it will be obvious that this group of bodies exhibits the most striking analogies with the ammonia-bases; so much so, in fact, that frequently it will only be necessary to repeat the observations, which were published

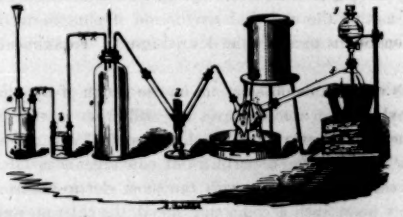
by one of us about eight years ago, regarding the methylated and ethylated derivatives of ammonia*.

The experiments which we have to communicate refer chiefly to the methyl- and ethyl-compounds; though here and there we have used amyl as material. Since we have preferred working in the ethyl-series, we begin with the description of the ethyl-compounds.

EXPERIMENTS IN THE ETHYL-SERIES.

Action of Terchloride of Phosphorus on Zinc-ethyl.

The reaction between these two bodies is very violent, and readily gives rise to dangerous explosions, if the necessary precautions are neglected. We have generally adopted the following arrangement. A tubulated retort is joined to a receiver which in its turn



a. Apparatus for generating carbonic acid. *b*. Wash-bottle containing sulphuric acid. *c*. Reservoir of carbonic acid. *d*. Bent tube containing terchloride of phosphorus. *e*. Receiver. *f*. Retort containing zinc-ethyl. *g*. Dropping apparatus filled with terchloride of phosphorus.

is connected with a wide glass tube bent at an angle of about 130°, and acts like a second receiver. The angle of this tube is filled with terchloride of phosphorus, and the tube is connected with a large cylinder which is supplied by a suitable apparatus with dry carbonic acid. As soon as the carbonic acid has expelled the air from the reservoir, tube, receiver, and retort, an exit-tube from the reservoir, up to that time closed by a caoutchouc cap, is opened to let out the carbonic acid, the evolution of which is maintained during the whole operation. The tubulature of the retort is now connected with the copper digester in which the zinc-ethyl has been prepared; and as soon as the retort has received a charge of the ethereal solution of zinc-ethyl, there is fixed into the same tubulature a little dropping apparatus, consisting of a glass globe, with a tubulature and stopper at the top, and terminating below in a glass tube in which a stopcock is fitted. This apparatus is filled with terchloride of phosphorus, and by appropriately adjusting the stopcock and opening or closing the stopper of the glass globe, any desired flow of the fluid can be maintained with the greatest nicety.

However slowly the action may be accomplished, and however well, moreover, the retort

* Philosophical Transactions, 1850, p. 93; 1851, p. 357.

and receiver may be cooled by water or ice, the action is nevertheless invariably so violent, that all the ether, and with it a large quantity of the zinc-ethyl, passes over into the receiver. By the powerful ebullition which periodically ensues, a portion of the vapour is driven even into the bent tube, and a considerable loss of zinc-ethyl is incurred unless this tube be filled with terchloride of phosphorus, which greedily absorbs every trace of the former compound. This fluid valve, ascending and descending in the tube, in accordance with the progress of the reaction, regulates the function of the apparatus so perfectly, that the operation, which always takes several hours, when once begun continues by itself. Sometimes the absorption is so violent that the terchloride of phosphorus in the tube is sucked back into the receiver, but even then no loss is to be feared, since the tube is connected with the reservoir filled with carbonic acid.

The first drops of terchloride of phosphorus which fall into the solution of zinc-ethyl, hiss like water when coming in contact with red-hot iron. The action becomes by-and-by less violent, and as soon as an evolution of heat is no longer perceptible, the operation is terminated. There remain in the retort, in the receiver, in the bent tube, and sometimes even in the carbonic acid reservoir, two liquid layers,—the one a heavy pale straw-coloured thick fluid, the other a transparent colourless mobile fluid floating on the former.

The heavy fluid, a compound of the phosphorus-base with chloride of zinc, nearly solidifies on cooling, but the viscid transparent mass exhibits no trace of crystalline structure. The light fluid is a mixture of ether with an excess of the terchloride of phosphorus; after disconnecting the apparatus, it is poured off from the viscous fluid, and may be used, after distillation, in a second operation.

In order to liberate the phosphorus-base from its combination with zinc, nothing more than a distillation with potassa is required. To prevent the destruction of the retort, to which the zinc-compound adheres with pertinacity, and the loss of so precious a material, this operation is conveniently performed in the following manner. Solid hydrate of potassa is placed on the hard resinous cake attached to the bottom of the retort, and a slow current of water allowed to flow in by the dropping apparatus, after the air in the retort has been carefully displaced by hydrogen: the heat evolved during the reaction is quite sufficient to volatilize the base with the vapour of the water; it may be condensed by an ordinary cooler. The base, which is now floating on the water of the distillate, is removed by means of a separating funnel; it is allowed to stand for a day over hydrate of potassa, and finally rectified in a current of dry hydrogen gas.

Triethylphosphine.

Thus obtained, triethylphosphine is a colourless, transparent, mobile liquid, which strongly refracts light. The compound is lighter than water, its specific gravity being found to be 0·812 at 15°·5 C.; it is perfectly insoluble in water, but soluble in every proportion in alcohol and ether. Its odour is penetrating, almost benumbing, but still not disagreeable. The intolerable smell which renders it so unpleasant to work with these

phosphorus-compounds generally, arises from other products, which make their appearance in considerable quantities, especially in preparing the phosphorus-base by means of phosphide of sodium and iodide of ethyl. In a diluted state, the odour of the pure triethylphosphine has the greatest similarity to that of the hyacinth*. Long-continued working with this substance causes headache and sleeplessness.

The boiling-point of the triethylphosphine is 127°·5 C., under the barometric pressure of 0·744 m. The determination was made with an ounce of the pure substance. The distillation of the phosphorus-base must be performed in a stream of dry hydrogen, for it attracts oxygen with great energy, especially at high temperatures. It is impossible to pour the liquid from one vessel into another without its becoming perceptibly warm. The product of oxidation formed in this way becomes evident in the last stage of the distillation. When the larger quantity of the base has distilled over, the mercury in the thermometer begins suddenly to rise, and before the temperature has become again stationary, the neck of the retort is found to be coated with a network of beautiful crystals, which are even drawn over into the receiver. These crystals are permanent as long as they are protected from the action of the moist air. After disconnecting the apparatus, it is vain to attempt to collect the crystals, the most minute quantity of water causing them to liquefy to a heavy oil, soluble in water. From these remarks, it is obvious that triethylphosphine must be almost always contaminated with a small quantity of this substance; in fact a bottle containing the base cannot be opened without its being formed. When the phosphorus-base is brought in contact with oxygen, vapours are immediately formed; the liquid frequently becomes so hot that it inflames, and the body is burnt with evolution of dense white fumes of phosphoric acid. If a strip of paper, moistened with triethylphosphine, be introduced into a test-tube containing oxygen and immersed into hot water, the vapour of the phosphorus-base produces with the oxygen an explosive mixture, which detonates after a few moments with considerable violence. When poured into a flask filled with chlorine gas, every drop is inflamed with disengagement of pentachloride of phosphorus and hydrochloric acid, and separation of carbon. Triethylphosphine unites with bromine and iodine, evolving considerable heat, which may give rise to inflammation; but if the action be moderated, crystalline compounds are produced. In cyanogen gas the phosphorus-base is converted into a brown resinous mass. If a piece of sulphur be thrown into a test-tube containing

* There is nothing new in the fact that the odour of a substance may be considerably changed by dilution. Several years ago, when occupied in the preparation of different ethers, which have found numerous applications in perfumery, I had frequent opportunities of observing how the desired aroma, which was absent in the pure substance, was brought out by dilution with alcohol. The hyacinth smell of the dilute phosphorus-base is so characteristic, that one morning I found in my laboratory a large basket filled with hyacinths, the present of a lady friend of mine, who, interested in my labours, had a strong impression that triethylphosphine must be present in the hyacinth. In the interest of science, the entire floral adornment of the garden had been unmercifully sacrificed! It would have been ungrateful not to distil them, but I regret to say, that the anticipation of the amiable donor, who wished to enrich me with so interesting a discovery, proved unfounded. The hyacinth does not contain any phosphorus-base.—A. W. H.

triethylphosphine, it becomes so hot as to fuse the sulphur, which then floats on the liquid base in the form of globules, as sodium does on water, and at last entirely disappears. The clear liquid thus obtained, solidifies on cooling to a magnificent crystalline mass. The experiment must be made with caution, since the vapour of the phosphorus-compound, which rises during the reaction, generally explodes on coming in contact with the air contained in the vessel. Selenium gives rise to similar but less powerful phenomena.

Although triethylphosphine, in its relations to other bodies, possesses all the characters of a well-defined base, it does not exhibit an alkaline reaction. When freshly prepared, it is without action on vegetable colours, but when exposed only for a few moments to the influence of the air, it begins to show a constantly increasing acid reaction. Triethylphosphine unites with acids slowly, but with considerable evolution of heat; with concentrated acid the temperature frequently rises to such a degree as to give rise to inflammation of the liquid. Most of its salts are crystalline compounds, but are very soluble and deliquescent.

The composition of triethylphosphine was sufficiently established in theory, both by its mode of formation and by numerous analogies. Nevertheless we have made an experimental determination.

When freshly prepared and burnt with chromate of lead, 0·369 grm. of triethylphosphine furnished 0·8106 grm. of carbonic acid and 0·4212 grm. of water.

The formula

$$C_{12}H_{15}P = \left.\begin{matrix} E \\ E \\ E \end{matrix}\right\} P$$

leads to the following numbers:—

		Theory.	Experiment.
12 equivs. of Carbon	72	61·01	60·00
15 equivs. of Hydrogen	15	12·71	12·68
1 equiv. of Phosphorus	31	26·28	——
1 equiv. of Triethylphosphine	118	100·00	

The combustion shows a small loss of carbon; but if it be recollected with what avidity this base attracts oxygen, and that the previous manipulations were performed in atmospheric air, this is easily accounted for. It would have been loss of time to repeat this experiment, especially since we shall have to mention in the following pages many analytical determinations, which, although referring to other substances, enable us to infer with equal certainty the composition of the phosphorus-base.

Triethylphosphine forms crystalline compounds with the hydrogen-acids of chlorine, bromine, and iodine, with sulphuric and nitric acids; but all these salts, which can be obtained in the dry state only by the aid of the exsiccator, are but little suitable for analysis. The solution in hydrochloric acid affords a crystalline platinum-salt, which is insoluble in cold water, in alcohol and ether, but which, on account of the

facility with which it decomposes at 100° C., must be dried in the exsiccator. In the water-bath it fuses and is altogether decomposed. The determination of the platinum in the phosphorus-compounds presents considerable difficulties. The platinum in these substances cannot be determined in the ordinary way by simple ignition, because a considerable quantity of platinum is carried off with the phosphorus-vapour, however slowly and carefully the process may be carried out. We unfortunately did not find this out, until a great number of unsuccessful analyses had been made. By heating with a considerable excess of carbonate of sodium in a porcelain crucible on a sand-bath, the temperature of which is gradually raised, the determination succeeds without difficulty. After removal of the portion soluble in water, the platinum-residue, which is always contaminated with silicic acid, is dissolved in nitrohydrochloric acid, the solution evaporated to dryness, and the residue again dissolved in acid: the careful evaporation of this solution furnishes a perfectly accurate result. This method is somewhat tedious, but there is some compensation for this increased complexity by the simultaneous determination of the chlorine.

The analysis of the platinum-salt of triethylphosphine has furnished the following results:—

0·2795 grm. of platinum-salt gave 0·3725 grm. of chloride of silver and 0·0855 grm. of platinum, corresponding to the formula

$$C_{12}H_{15}P,\ HCl,\ PtCl_2 = E_3P,\ HCl,\ PtCl_2.$$

	Theory.		Experiment.
12 equivs. of Carbon	72	22·21	——
16 equivs. of Hydrogen . . .	16	4·94	——
1 equiv. of Phosphorus . . .	31	9·56	——
3 equivs. of Chlorine	106·5	32·85	32·96
1 equiv. of Platinum	98·7	30·44	30·59
1 equiv. of Platinum-salt . .	324·2	100·00	

The preceding experiments sufficiently fix the composition of triethylphosphine. This compound is in fact triethylamine, in which the nitrogen is replaced by an equivalent quantity of phosphorus. The perfect analogy with triethylamine is also shown by the deportment of the phosphorus-base with the iodides of ethyl, methyl and amyl. Triethylphosphine combines with these substances, forming well-crystallized and highly characteristic salts, which may be regarded as iodide of ammonium, in which the nitrogen is replaced by phosphorus and the hydrogen by the radicals of the alcohols.

Iodide of Tetrethylphosphonium.

On mixing triethylphosphine with iodide of ethyl, a violent action ensues after a few moments; the liquid effervesces with almost explosive violence, and then solidifies to a white crystalline mass. If, instead of the pure base, an ethereal solution be employed, the crystals are formed more slowly. This new compound is extremely soluble in

water, less so in alcohol, and insoluble in ether. The aqueous solution crystallizes on addition of potassa-solution, in which this compound, like the iodides of tetramethylammonium and tetrethylammonium, is but slightly soluble. From the alcoholic solution the salt falls, on addition of ether, as a crystalline powder. If ether be added to a cold alcoholic solution, as long as the precipitate first produced is dissolved by boiling, well-formed crystals of the iodide are deposited on cooling.

The mode of formation and analysis leave no doubt respecting the nature of these crystals. They contain the elements of 1 equiv. of triethylphosphine and 1 equiv. of iodide of ethyl.

$$\underbrace{C_{12}H_{15}P}_{\text{Triethyl-phosphine.}} + \underbrace{C_4H_5I}_{\text{Iodide of ethyl.}} = \underbrace{C_{16}H_{20}PI.}_{\text{New compound.}}$$

The following are the results of analyses:—

I. 0·743 grm. of the iodide yielded 0·630 grm. of iodide of silver.

II. 0·648 grm. of the iodide gave 0·555 grm. of iodide of silver.

III. 0·412 grm. of the iodide gave 0·3555 grm. of iodide of silver.

Percentage of iodine:—

I.	II.	III.	Mean.
45·82	46·30	46·62	46·25

The formula

$$C_{16}H_{20}PI = \left.\begin{matrix}E\\E\\E\\E\end{matrix}\right\}PI$$

requires

	Theory.		Experiment.
1 equiv. of the Phosphorus-body . . .	147	53·65	——
1 equiv. of Iodine	127	46·35	46·25
1 equiv. of the Iodine-compound . . .	274	100·00	

Notwithstanding the transparency of the constitution of these compounds, we feel some embarrassment in fixing their nomenclature. We propose to designate the hypothetical compound of one equivalent of phosphorus and four equivalents of ethyl by the name "Tetrethylphosphonium." This term is long, but it leaves no doubt regarding the composition of the body, and marks at the same time its analogy with tetrethylammonium. The iodine-compound is accordingly the iodide of tetrethylphosphonium.

Hydrated Oxide of Tetrethylphosphonium.

The separation of the iodine from the before-mentioned compound presents no difficulties. Oxide of silver removes this element even at the common temperature. A strongly alkaline solution is obtained which retains a small quantity of silver in solu-

tion. This liquid, which is almost without odour, and has a bitter and phosphoric taste, when placed over sulphuric acid, dries up into a crystalline extremely deliquescent mass, the silver separating at the same time in the form of a black powder, or as a brilliant metallic mirror. The mass, when redissolved in water, furnishes a colourless liquid free from silver, but generally containing some carbonic acid. The avidity with which the oxide of tetrethylphosphonium attracts both water and carbonic acid, has prevented us from analysing this body, but its formation, the composition of the corresponding iodide, and the analysis of a platinum- and gold-salt, hereafter to be mentioned, sufficiently warrant the formula

$$C_{16}H_{21}PO_2 = E_4PO, HO.$$

In its deportment with other substances the body in question resembles the oxide of tetrethylammonium; we refer therefore to the detailed description which one of us has given of this compound in a former memoir*. The solution of the oxide of tetrethylphosphonium shows in fact all the reactions of a solution of potassa; the precipitates, such as alumina and protoxide of zinc, dissolve, however, less readily in excess of the phosphorus-compound. The action of heat upon this body gives rise to a peculiar transformation, to which we shall return hereafter.

Tetrethylphosphonium produces crystallizable salts with hydrochloric, nitric, and sulphuric acids. All these compounds exhibit the deliquescent character of the oxide. Like the latter, they are also soluble in alcohol; in ether they are for the most part insoluble. The hydrochloric solution furnishes, with bichloride of platinum and terchloride of gold, difficultly soluble precipitates, which are well adapted for analysis.

Chloride of Tetrethylphosphonium and Bichloride of Platinum.

The pale orange-yellow precipitate which falls on addition of bichloride of platinum to a moderately dilute solution of the chloride, dissolves with difficulty, but without decomposition, in boiling water; it is insoluble in alcohol and in ether. It can be dried at 100° C.

I. 0·1108 grm. of the platinum-salt, when fused with carbonate of sodium, gave 0·135 grm. of chloride of silver and 0·0305 grm. of platinum.

II. 0·5510 grm. of the platinum-salt, treated in a similar way, gave 0·6710 grm. of chloride of silver.

III. 0·5535 grm. of platinum-salt gave 0·1547 grm. of platinum.

IV. 0·3180 grm. of platinum-salt gave 0·089 grm. of platinum.

The formula

$$C_{16}H_{20}PCl, PtCl_2 = \left.\begin{matrix} E \\ E \\ E \\ E \end{matrix}\right\} PCl, PtCl_2$$

* Philosophical Transactions, 1851, p. 357.

requires

	Theory.		Experiment.			
			I.	II.	III.	IV.
1 equiv. of Tetrethylphosphonium	147	41·75	——	——	——	——
3 equivs. of Chlorine	106·5	30·23	30·14	30·12	——	——
1 equiv. of Platinum	98·7	28·02	27·53	——	27·95	27·98
	352·2	100·00				

Chloride of Tetrethylphosphonium and Terchloride of Gold.

The crystalline precipitate, obtained by mixing the two solutions, separates from boiling water in brilliant golden-yellow needles.

0·317 grm. of the gold-salt, dried at 100° C., furnished, after ignition, 0·1275 grm. of gold.

This determination leads to the formula

$$C_{16}H_{20}PCl, AuCl_3 = \left.\begin{matrix}E\\E\\E\\E\end{matrix}\right\} PCl, AuCl_3,$$

as is seen by the following comparison:—

	Theory.		Experiment.
1 equiv. of Tetrethylphosphonium .	147	30·25	——
4 equivs. of Chlorine	142	29·22	——
1 equiv. of Gold	197	40·53	40·22
1 equiv. of Gold-salt	486	100·00	

Action of Heat upon Oxide of Tetrethylphosphonium.

The change which oxide of tetrethylammonium undergoes by the action of heat is well known; this body splits into olefiant gas, water, and triethylamine.

$$E_4NO, HO = C_4H_4 + 2HO + E_3N.$$

We expected an analogous transformation of oxide of tetrethylphosphonium, but experiment has proved that this body suffers a different decomposition. On submitting freshly-prepared oxide of tetrethylphosphonium to distillation, water only passes over in the first instance, but as soon as the solution has attained a certain state of concentration, it suddenly effervesces with evolution of an inflammable gas which may be collected over water. This gas contains carbon and hydrogen, but no phosphorus; it may be left in contact with bromine-water without experiencing the slightest absorption. This experiment shows that the gas cannot contain any ethylene, and with almost the same certainty we may infer the absence in it of all the hydrocarbons of the formula C_nH_n. The evolution of gas ceases long before the whole amount of the liquid contained in the retort has distilled over. On the contrary, it is observed that immediately after the

evolution of gas has ceased, the distillation nearly stops, and commences again only when the temperature has reached 200°; a viscid, nearly inodorous liquid now distils over, the temperature slowly rising, until at about 240° a constant boiling-point is attained: what now distils, generally solidifies to a radiated crystalline mass in the neck of the retort. On fusing this mass with a spirit-lamp, and collecting the liquid in a receiver, it frequently resolidifies instantaneously on cooling; often, however, it remains liquid for months. This body is extremely deliquescent: a crystal exposed to the air only for a few seconds liquefies entirely. It is soluble in water in every proportion, also in alcohol, less so in ether. The aqueous solution is precipitated by potassa; the dissolved body separates in this case in colourless oily drops, which remain liquid even after much concentration, and rapidly dissolve again on diluting the potassa solution with a comparatively small quantity of water. Acids dissolve the oil likewise with facility.

It is obvious at a glance, that the body in question is identical with the product formed by the action of air upon triethylphosphine. A careful comparison of the properties of the two substances places their identity beyond any doubt. It is, moreover, easily proved that the body is a product of oxidation. On boiling triethylphosphine with moderately strong nitric acid, and adding potassa to the highly concentrated liquid, the characteristic oily drops are immediately separated, and disappear again upon addition of water. At the common temperature, oxide of mercury is without action upon triethylphosphine, but on gently warming the mixture, a considerable evolution of heat takes place, metallic mercury is separated, and an oily substance produced which has all the characters of the new compound, and often sublimes in radiated crystals, coating the colder part of the vessel. With oxide of silver exactly the same phenomena are observed. On the other hand, the oily body, when submitted to the action of potassium, instantaneously reproduces triethylphosphine.

It is difficult to obtain this body in a state fit for analysis. It is not affected by solid hydrate of potassa, but on distilling the two substances together, the distillate is nevertheless found to contain a certain amount of moisture, the hydrate of potassa losing a certain quantity of water at the temperature of distillation. If the crystalline mass be separated from the potassa before distillation, it attracts so much water during manipulation, that even now it becomes but imperfectly crystalline after distillation. Distillation with anhydrous phosphoric acid furnishes the compound perfectly dry and solid; unfortunately, however, a portion of the substance is thus decomposed with separation of free phosphorus which contaminates the distillate. Nor have we succeeded in uniting this substance to crystalline combinations; nitrate of silver, bichloride of platinum, and several other reagents were tried in vain.

It is obvious that the preparation of this body in a state of purity is attended with unusual difficulties. These difficulties might certainly have been surmounted, but we believe that the deportment of triethylphosphine with sulphur and selenium, which will be more minutely described hereafter, enables us to infer retrospectively the composition

of the oxide with a degree of certainty scarcely inferior to that furnished by analysis itself. The examination of well-defined sulphur- and selenium-compounds of the composition respectively

$$C_{12}H_{15}PS_2 = E_2PS_2 \text{ and}$$
$$C_{12}H_{15}PSe_2 = E_2PSe_2,$$

sufficiently proves that the body in question is the corresponding oxide, that it is in fact a combination of triethylphosphine with 2 equivalents of oxygen, $C_{12}H_{15}PO_2 = E_2PO_2$; an inference which is moreover supported by the existence of analogous and similarly formed combinations in the antimony- and arsenic-series, viz.

$$C_{12}H_{15}SbO_2 = E_2SbO_2, \text{ and}$$
$$C_{12}H_{15}AsO_2 = E_2AsO_2.$$

The formation of such a binoxide by the distillation of the hydrated oxide of tetrethylphosphonium, is readily intelligible, if we assume that the hydrocarbon simultaneously disengaged consists of hydride of ethyl, an assumption which is in accordance with the general characters of this gas.

$$\underbrace{E_4PO, HO}_{\text{Hydrated oxide of tetrethylphosphonium.}} = \underbrace{E_2PO_2}_{\text{Binoxide of triethylphosphine.}} + \underbrace{EH}_{\text{Hydride of ethyl.}}$$

We should have liked to establish this equation by some analytical numbers, but after some fruitless trials to prepare the substance in a state of purity, we were obliged to desist from the attempt. Even the preparation of a considerable quantity of oxide of tetrethylphosphonium is a long, laborious and expensive operation; but to these numerous impediments a further difficulty is added, which at the first glance appeared altogether inexplicable. Under certain conditions the distillation of oxide of tetrethylphosphonium furnishes either no inflammable gas at all, or only traces; at the same time the formation of the crystalline binoxide either entirely ceases, or takes place only in very minute quantities. We have convinced ourselves that this invariably occurs when the alkaline solution, by exposure to the air, has attracted a considerable quantity of carbonic acid. Instead of hydride of ethyl and binoxide of triethylphosphine, the phosphorus-base itself is obtained in this case, together with another liquid body which contains no phosphorus. By dissolving the distillate in ether, fixing the triethylphosphine by sulphur, and then evaporating the ether, an inflammable aromatic liquid remains, which floats on water. We had not more than a few drops of this oil at our disposal, which precluded the idea of an analysis, but we have no doubt that this liquid is carbonate of ethyl.

$$\underbrace{E_4P, CO_3}_{\text{Carbonate of tetrethylphosphonium.}} = \underbrace{E_3P}_{\text{Triethyl-phosphine.}} + \underbrace{ECO_3}_{\text{Carbonate of ethyl.}}$$

Chloride, Bromide, and Iodide of Triethylphosphine.

Binoxide of triethylphosphine, when treated with hydrochloric, hydrobromic and hydriodic acids, is converted into the corresponding chloride, bromide, and iodide, which closely resemble the oxide in their general properties. They are liquids which gradually solidify in the exsiccator; the crystals fuse at 100° and begin to volatilize, although their boiling-point is very high. The compounds of triethylphosphine with chlorine, bromine, and iodine may also be obtained by the action of these elements in aqueous or alcoholic solutions upon the phosphorus-base itself. Both methods, however, furnish products which are with difficulty purified.

These compounds, as well as the saline compounds which the oxide of triethylphosphine produces with sulphuric and nitric acids, and which in the exsiccator gradually solidify into semi-crystalline masses, have but slightly occupied our attention, because the formation of beautiful sulphur- and selenium-compounds enabled us to gain a sufficiently precise idea regarding the chemical characters of these substances in general.

Bisulphide of Triethylphosphine.

The remarkable phenomena which attend the combination of triethylphosphine with sulphur have been already described. The compound is likewise obtained by distilling triethylphosphine with cinnabar, which in this reaction is reduced to subsulphide or to metallic mercury. Treatment of the oxide with sulphuretted hydrogen or with sulphide of ammonium does not furnish the compound.

The best mode of preparing this beautiful body is the following. Flowers of sulphur are gradually introduced into a dilute solution of the phosphorus-base in ether. The liquid effervesces upon each addition, and the sulphur disappears. As soon as sulphur remaining undissolved indicates the completion of the reaction, the ether is volatilized, and the residuary mixture of free sulphur and bisulphide of triethylphosphine separated by boiling water. On cooling, the liquid deposits the compound in crystals of perfect purity. This sulphur-compound is one of the finest products with which we have become acquainted in the course of our inquiries. Its crystallizing power is such, that by slowly cooling the solution most beautiful crystals may be obtained, even in a test-tube, the liquid column being traversed by an aggregate of thin vertical needles, often five or six inches in length. The difference of the solubility of the compound in cold and boiling water is very great; indeed, but a minute quantity remains in solution at the common temperature. On adding an alkali to the cold solution the mixture becomes turbid, and deposits, after a few moments, small crystals. The sulphur-compound in this respect resembles the corresponding oxide, which is likewise less soluble in alkalies than in pure water. This phenomenon is most strikingly observed by adding potassa to a boiling saturated aqueous solution of the bisulphide; it instantaneously separates in clear oily drops, which rapidly solidify into spherical aggregates of crystals as the liquid cools. The compound is even more readily soluble in alcohol and ether,

and also crystallizes from these liquids, but less beautifully. The solubility in bisulphide of carbon is almost unlimited; from this solvent it crystallizes imperfectly.

The fusing-point of the bisulphide of triethylphosphine is 94° C.; it resolidifies at 88°. When heated beyond 100° C. the bisulphide is volatilized, with diffusion of a white vapour of a disagreeable sulphur-odour, which is but slightly perceptible at the common temperature. When heated with a quantity of water insufficient for its solution, the sulphur-compound rises to the surface as a clear transparent oil, which is copiously volatilized with the vapour of water.

The solution of the bisulphide is without action on vegetable colours. The compound nevertheless appears to possess faintly basic properties. It dissolves more readily in hydrochloric acid, especially when concentrated, than in water, and the solution furnishes with bichloride of platinum a yellow precipitate, which however rapidly cakes into a resinous mass, giving indications of decomposition by the separation of bisulphide of platinum, so that it was not adapted for analysis. The sulphur-compound also dissolves in dilute sulphuric and nitric acids; concentrated nitric acid decomposes it; the fuming acid gives rise to a sort of detonation. The aqueous solution of the bisulphide is not affected by acetate of lead, nitrate of silver, or protoxide of mercury, even at the boiling temperature; the alcoholic solution, on the other hand, is instantaneously decomposed with separation of the sulphide of lead, silver or mercury. The liquid filtered off from the precipitates now contains the oxide of triethylphosphine, either free or in the form of acetate or nitrate, and may be readily separated by the addition of an alkali to the solution. The action of potassium upon this compound instantaneously reproduces the phosphorus-base.

The bisulphide of triethylphosphine has the composition

$$C_{12}H_{15}PS_2 = E_2PS_2.$$

I. 0·1355 grm. of the substance, burnt in a glass tube with a mixture of common salt and chlorate of potassium, gave 0·2190 grm. of sulphate of barium.

II. 0·4007 grm. of the substance yielded 0·623 grm. of sulphate of barium.

	Theory.		Experiment.		
			I.	II.	Mean.
1 equiv. of Triethylphosphine	118	78·67			
2 equivs. of Sulphur	32	21·33	22·14	21·35	21·74
1 equiv. of Bisulphide of triethylphosphine	150	100·00			

The formation of the bisulphide takes place with such facility, and the properties of the compound are so characteristic, that we have frequently used flowers of sulphur as a reagent for triethylphosphine.

Biselenide of Triethylphosphine.

In the action of selenium upon triethylphosphine, the phenomena described in the

preceding paragraphs are repeated. The reaction, however, as might have been expected, is less powerful.

The selenium-compound crystallizes from water with the same facility as the sulphur-compound, but the solution is apt to undergo partial decomposition when exposed to the atmosphere. Even the dry crystals are slowly reddened in the air. The fusing-point of the selenide is 112° C.; it is easily volatilized, undergoing partial decomposition. It contains

$$C_{12}H_{15}PSe_2=E_2PSe_2.$$

In analysing this body, we have availed ourselves of the ready decomposition which it undergoes by nitrate of silver, when dissolved in alcohol.

0·378 grm. of the substance gave 0·557 grm. of selenide of silver.

	Theory.		Experiment.
1 equiv. of Triethylphosphine	118	59·9	——
2 equivs. of Selenium	79	40·1	39·45
1 equiv. of Biselenide of triethylphosphine	197	100·0	

In order to give more completely the history of the phosphorus-bases, we have also examined the compounds which are formed by the action of the iodides of methyl and amyl upon triethylphosphine; but since the products of these reactions resemble in every respect the corresponding ethyl-compounds, we have only to mention the analytical results.

Iodide of Methyltriethylphosphonium.

In treating triethylphosphine with iodide of methyl, all the phenomena mentioned in the case of the analogous experiment with iodide of ethyl are repeated. The action is even still more violent and rapid; and if no ether be added, a portion of the product is readily lost by the explosive effervescence of the liquid. The crystals thus obtained contain

$$C_{14}H_{18}P\,I=(MeE_3)P\,I.$$

0·544 grm. of the compound, dried at 100° C., gave 0·491 grm. of iodide of silver.

	Theory.		Experiment.
1 equiv. Methyltriethylphosphonium . .	133	51·15	——
1 equiv. of Iodine	127	48·85	48·77
1 equiv. of the Iodide	260	100·00	

On treating the solution of this compound, which essentially resembles the simple ethyl-compound, with oxide of silver, a strongly alkaline solution of oxide of methyltriethylphosphonium is obtained. The solution, when saturated with hydrochloric acid and mixed with bichloride of platinum, furnishes a beautiful orange-yellow platinum-

salt, crystallizing in well-defined cubes, truncated by the planes of the octahedron. This salt, which is insoluble in alcohol and ether, may be recrystallized from boiling water without decomposition; it contains

$$C_{14}H_{18}PCl,\ PtCl_2=(MeE_2)PCl,\ PtCl_2.$$

0·4297 grm. of the platinum-salt, fused with carbonate of sodium, gave 0·551 grm. of chloride of silver and 0·127 grm. of platinum.

	Theory.		Experiment.
1 equiv. of Methyltriethylphosphonium	133	39·20	——
3 equivs. of Chlorine	106·5	31·49	31·72
1 equiv. of Platinum	98·7	29·31	29·55
1 equiv. of the Platinum-salt	338·2	100·00	

Iodide of Triethylamylphosphonium.

Iodide of amyl acts but slowly on the phosphorus-base. From a mixture of the two substances in ether, beautiful crystals are deposited after a few days, which may be purified by solution in alcohol and precipitation by ether; they contain

$$C_{22}H_{26}PI=(E_3Ayl)PI.$$

0·577 grm. of the substance gave 0·432 grm. of iodide of silver.

	Theory.		Experiment.
1 equiv. of Triethylamylphosphonium	189	59·80	——
1 equiv. of Iodine	127	40·20	40·45
1 equiv. of the Iodide	316	100·00	

Treatment of this compound with oxide of silver furnishes the free oxide of triethylamylphosphonium, with all the properties characteristic of the class. The corresponding chloride deposits, on addition of bichloride of platinum, a beautiful platinum-salt, crystallizing in prisms, with flat terminal planes. It is insoluble in alcohol and ether, but rather soluble in water. The platinum-salt has the composition

$$C_{22}H_{26}PCl,\ PtCl_2=(E_3Ayl)PCl,\ PtCl_2.$$

0·3494 grm. of the platinum-salt gave 0·3873 grm. of chloride of silver and 0·0880 grm. of platinum.

	Theory.		Experiment.
1 equiv. of Triethylamylphosphonium	189	47·94	——
3 equivs. of Chlorine	106·5	27·02	27·42
1 equiv. of Platinum	98·7	25·04	25·18
1 equiv. of the Platinum-salt	394·2	100·00	

Action of Heat upon the Hydrated Oxide of Triethylamylphosphonium.

On heating this oxide a small quantity of an inflammable gas is evolved, a liquid being formed at the same time, which boils at about 280° C., and obviously corresponds to the binoxide of triethylphosphine. Two distinct changes may occur in this case. Since the oxide contains several radicals, it is possible that either ethyl or amyl may be eliminated in this decomposition, and the liquid simultaneously generated must therefore contain either

$$(E_3, Ayl)\,PO_2, \text{ or } E_3\,PO_2.$$

The higher boiling-point of the compound, and the deportment of the corresponding nitrogen-term (the oxide of triethylamylammonium), which on distillation furnishes diethylamylamine together with water and olefiant gas, are in favour of the first assumption. Accordingly, the inflammable gas would also in this case be hydride of ethyl, and the transformation of the oxide of triethylamylphosphonium under the influence of heat would be represented by the equation

$$\underbrace{(E_3, Ayl)PO,\ HO}_{\text{Oxide of triethylamyl-phosphonium.}} = \underbrace{(E_3, Ayl)O_2}_{\text{Binoxide of diethyl-amylphosphine.}} + \underbrace{EH}_{\text{Hydride of ethyl.}}$$

Experimentally the question remains undecided.

EXPERIMENTS IN THE METHYL-SERIES.

The results recorded in the preceding sections afford a tolerably complete view of the phosphorus-bases. We may therefore be brief in describing the experiments which we have made with the methyl-compounds.

Trimethylphosphine.

This remarkable body is obtained by the same process which we have minutely described for the preparation of the corresponding ethyl-base. Zinc-methyl and terchloride of phosphorus furnish the compound of chloride of zinc and trimethylphosphine, from which the base may be expelled by the action of potassa. All the precautions which have been mentioned as necessary in the preparation of the ethyl-base, are required in a higher degree for the methylated body. Since zinc-methyl attracts oxygen even with greater avidity than zinc-ethyl, the current of carbonic acid must be continuously maintained for a long period. The intensity with which zinc-methyl decomposes terchloride of phosphorus, is not inferior to the violent reaction between caustic baryta and anhydrous sulphuric acid. The mixing cannot, therefore, be too slowly effected. In expelling trimethylphosphine from its zinc-compound, refrigeration by ice is absolutely necessary, since this body is far more volatile than the ethyl-base. The distillation must be made in hydrogen gas, and the current of gas must moreover flow very slowly, otherwise, however perfect and careful may be the arrange-

ments for cooling, a considerable quantity of the body will evaporate in the hydrogen and be lost, not to speak of the diffusion of the almost intolerable odour of the methyl-base in the atmosphere of the laboratory.

Trimethylphosphine is a colourless, transparent, very mobile liquid of an indescribable odour, powerfully refracting light, lighter than water, in which menstruum it is insoluble. The boiling-point of the liquid lies between 40° and 42° C., which agrees with PAUL THÉNARD's observations. Trimethylphosphine has even a more powerful attraction for oxygen than the corresponding ethyl-base. In contact with the air, it fumes and is apt to be inflamed. On distilling even the freshly prepared methyl-base, the neck of the retort becomes coated in the last stage of the operation with a network of beautiful crystals, perfectly similar to those which are observed with the ethyl-base. These crystals may be readily obtained in larger quantity by exposing the methyl-base to a slow current of dry atmospheric air. It is scarcely necessary to mention that these crystals are the binoxide of trimethylphosphine.

In its deportment with chlorine, bromine, iodine, sulphur and selenium, and finally with the acids, the methyl-base exactly imitates the ethylated body. The reactions are however more rapid and energetic.

We have been satisfied to identify trimethylphosphine, prepared by means of zinc-methyl, by the analysis of a platinum-salt.

Hydrochlorate of Trimethylphosphine and Bichloride of Platinum.

The solution of the methyl-base in hydrochloric acid, furnishes with bichloride of platinum, an orange-yellow, indistinctly crystalline precipitate, which, like the corresponding ethyl-compound, is readily decomposed by exposure to 100° C. For analysis it was dried in the exsiccator over sulphuric acid. It contains

$$C_6 H_9 P, HCl, PtCl_2 = Me_3 P, HCl, PtCl_2.$$

0·4021 grm. of the platinum-salt gave 0·1402 grm. of platinum.

	Theory.		Experiment.
1 equiv. of Trimethylphosphine	76·0	26·94	——
1 equiv. of Hydrochloric acid	36·5	12·93	——
2 equivs. of Chlorine	71·0	25·16	——
1 equiv. of Platinum	98·7	34·97	34·86
1 equiv. of the Platinum-salt	282·2	100·00	

Iodide of Tetramethylphosphonium.

The iodide is a white crystalline mass, obtained by the action of iodide of methyl upon an ethereal solution of triethylphosphine. This compound, which may be readily recrystallized from alcohol, is the finest product of the series. Freshly prepared, it exhibits the silvery lustre of sublimed naphthalin. In contact with the atmosphere it assumes a slightly reddish colour.

The composition of this compound is represented by the formula

$$C_8H_{12}PI = M_4PI.$$

On analysis, 0·5455 grm. of the iodide gave 0·5870 grm. iodide of silver.

	Theory.		Experiment.
1 equiv. of Tetramethylphosphonium . . .	91	41·75	——
1 equiv. of Iodine	127	58·25	58·14
1 equiv. of Iodide of tetramethylphosphonium	218	100·00	

On treating the solution of the iodide with oxide of silver, a very caustic solution of the oxide is obtained.

Chloride of Tetramethylphosphonium and Bichloride of Platinum.

The solution of this oxide, mixed with hydrochloric acid and bichloride of platinum, furnishes a platinum-salt which is insoluble in alcohol and ether, but crystallizes from water in beautiful octahedra. It contains

$$C_8H_{12}PCl, PtCl_2 = Me_4PCl, PtCl_2.$$

By analysis, 0·6365 grm. of the platinum-salt gave 0·2120 grm. of platinum.

	Theory.		Experiment.
1 equiv. of Tetramethylphosphonium	91	30·73	——
3 equivs. of Chlorine	106·5	35·95	——
1 equiv. of Platinum	98·7	33·32	33·30
1 equiv. of the Platinum-salt . . .	296·2	100·00	

Chloride of Tetramethylphosphonium and Terchloride of Gold.

The method of preparation, and the properties of the gold-salt of chloride of tetramethylphosphonium, are perfectly similar to those of the corresponding ethyl-body.

When dried at 100° C., 0·3545 grm. left, after ignition, 0·1630 grm. of gold.

The formula

$$C_8H_{12}PCl, AuCl_3 = Me_4PCl, AuCl_3$$

requires the following values:—

	Theory.		Experiment.
1 equiv. of Tetramethylphosphonium	91	21·17	——
4 equivs. of Chlorine	142	33·02	——
1 equiv. of Gold	197	45·81	45·98
1 equiv. of the Gold-salt	430	100·00	

Action of Heat upon the Hydrated Oxide of Tetramethylphosphonium.

Binoxide of Trimethylphosphine.

Precisely similar phenomena as in the ethyl-series: formation of binoxide of trimethylphospine and hydride of methyl (marsh-gas).

$$\underbrace{Me_4PO,HO}_{\text{Hydrated oxide of tetramethylphosphonium.}} = \underbrace{Me_3PO_2}_{\text{Binoxide of trimethylphosphine.}} + \underbrace{MeH}_{\text{Hydride of methyl (marsh-gas).}}$$

The direct formation of the binoxide by the action of oxygen upon trimethylphosphine has been already mentioned.

Bisulphide and Biselenide of Trimethylphosphine.

These bodies likewise resemble the corresponding members of the ethyl-series; they are, however, more soluble and more volatile. The sulphur-compound crystallizes from a highly concentrated aqueous solution in masses of well-formed four-sided prisms, which fuse at 105° C.

The selenium-compound crystallizes exactly like the ethyl-body; its fusing-point is 84° C. In contact with the air this compound blackens with separation of selenium. In this decomposition the characteristic odour of mesitilene is very perceptibly evolved. Even without an analysis we may assign to these compounds the formulæ

$$C_6H_9PS_2 = Me_3PS_2 \text{ and}$$
$$C_6H_9PSe_2 = Me_3PSe_2.$$

In conclusion, the action of the iodides of ethyl and amyl upon trimethylphosphine may be briefly mentioned.

Iodide of Trimethylethylphosphonium.

This substance is rapidly formed by the action of iodide of ethyl upon the ethereal solution of the methyl-base. The compound, which crystallizes perfectly well from boiling alcohol, contains

$$C_{10}H_{14}PI = (Me_3E)PI.$$

0·4435 grm. of substance gave 0·4403 grm. of iodide of silver.

	Theory.		Experiment.
1 equiv. of Trimethylethylphosphonium	105	45·24	——
1 equiv. of Iodine	127	54·76	53·67
1 equiv. of the Iodide	232	100·00	

Chloride of Trimethylethylphosphonium and Bichloride of Platinum.

By treating the iodide with oxide of silver, the caustic oxide is obtained, which gives, with hydrochloric acid and bichloride of platinum, a yellow platinum-salt, inso-

luble in alcohol and ether, but rather soluble in water. From the boiling solution it is deposited in magnificent octahedra.

0·5095 grm. of the platinum-salt gave 0·700 grm. of chloride of silver and 1616 grm. of platinum.

The formula $C_{10}H_{14}PCl, PtCl_2 = (Me_3E)Cl, PtCl_2$

requires the following numbers:—

	Theory.		Experiment.
1 equiv. of Trimethylethylphosphonium	105·0	33·86	——
3 equivs. of Chlorine	106·5	34·33	33·95
1 equiv. of Platinum	98·7	31·81	31·72
1 equiv. of Platinum-salt	310·2	100·00	

Iodide of Trimethylamylphosphonium.

The ethereal solution of the constituents slowly deposits this compound. It is extremely soluble in water, and hence, if the ethereal solution of the iodide of ethyl contain the most minute trace of water, the salt separates in the form of a syrup, which only gradually solidifies. It crystallizes in needles, although with difficulty, from absolute alcohol. Its composition is

$$C_{16}H_{36}PI = Me_3Ayl\,PI.$$

0·5495 grm. of the substance gave 0·4700 grm. of iodide of silver.

	Theory.		Experiment.
1 equiv. of Trimethylamylphosphonium	147	53·65	——
1 equiv. of Iodine	127	46·35	46·22
1 equiv. of the Iodide	274	100·00	

Chloride of Trimethylamylphosphonium and Bichloride of Platinum.

The oxide liberated from the iodide by means of oxide of silver, furnishes, with hydrochloric acid and bichloride of platinum, a very soluble platinum-salt, which crystallizes from boiling water in splendid needles, aggregated in spherules.

I. 0·8275 grm. of this platinum-salt gave 1·009 grm. of chloride of silver and 0·231 grm. of platinum.

II. 0·2455 grm. gave 0·3010 grm. of chloride of silver.

These numbers lead to the formula

$$C_{16}H_{36}PCl,\ PtCl_2 = (Me_3Ayl)\,PCl,\ PtCl_2,$$

as is seen by the following comparison:—

	Theory.		Experiment.	
			I.	II.
1 equiv. of Trimethylamylphosphonium	147·0	41·74	——	——
3 equivs. of Chlorine	106·5	30·24	30·16	30·33
1 equiv. of Platinum	98·7	28·02	27·91	——
1 equiv. of the Platinum-salt	352·2	100·00		

The analysis of this platinum-salt concludes the experimental part of our inquiry; for clearness and comparison we subjoin the following synopsis of the compounds which we have investigated.

α. Methyl-Series.

Trimethylphosphine	Me_3P.
Platino-chloride of trimethylphosphine	$Me_3P, HCl, PtCl_2$.
Binoxide of trimethylphosphine	Me_3PO_2.
Bisulphide of trimethylphosphine	Me_3PS_2.
Biselenide of trimethylphosphine	Me_3PSe_2.
Iodide of tetramethylphosphonium	Me_4PI.
Platino-chloride of tetramethylphosphonium	$Me_4PCl, PtCl_2$.
Auro-chloride of tetramethylphosphonium	$Me_4PCl, AuCl_3$.
Iodide of trimethylethylphosphonium	$(Me_3E)PI$.
Platino-chloride of trimethylethylphosphonium	$(Me_3E)PCl, PtCl_2$.
Iodide of trimethylamylphosphonium	$(Me_3Ayl)PI$.
Platino-chloride of trimethylamylphosphonium	$(Me_3Ayl)PCl, PtCl_2$.

β. Ethyl-Series.

Triethylphosphine	E_3P.
Platino-chloride of triethylphosphine	$E_3PCl, PtCl_2$.
Binoxide of triethylphosphine	E_3PO_2.
Bisulphide of triethylphosphine	E_3PS_2.
Biselenide of triethylphosphine	E_3PSe_2.
Iodide of tetrethylphosphonium	E_4PI.
Platino-chloride of tetrethylphosphonium	$E_4PCl, PtCl_2$.
Auro-chloride of tetrethylphosphonium	$E_4PCl, AuCl_3$.
Iodide of methyltriethylphosphonium	$(MeE_3)PI$.
Platino-chloride of methyltriethylphosphonium	$(MeE_3)PCl, PtCl_2$.
Iodide of triethylamylphosphonium	$(Ayl\,E_3)PI$.
Platino-chloride of triethylamylphosphonium	$(Ayl\,E_3)PCl, PtCl_2$.

On glancing once more over the phosphorus-compounds described in the preceding memoir, a comparison of these substances with the corresponding terms of the nitrogen-, arsenic-, and antimony-series is unavoidably forced upon us. Whether we consider the composition, or whether we review the properties of these groups, the most striking analogies, indeed an almost perfect parallelism, cannot be mistaken; the same formulæ, the same mode of combination, the same decompositions.

This analogy is particularly manifest in the compounds belonging to the ammonium-type. In these remarkable bodies, nitrogen, phosphorus, arsenic, and antimony appear to play absolutely the same part. It is more especially in the oxides of these compound metals that analogy of composition induces a perfect identity in properties, and, indeed, of very salient properties, which may be traced in almost every direction. If we were

satisfied with the study of the reactions of these bodies, we should never suspect, in compounds exhibiting such a close similarity of properties, the presence of elements so dissimilar as nitrogen, phosphorus, arsenic, and antimony; they might, moreover, be confounded with potassa and soda, by which they are scarcely surpassed in alkaline power. Only the deportment of the hydrated oxides, under the influence of heat, distinguishes the derivatives of nitrogen from the corresponding terms of the phosphorus-, arsenic-, and antimony-series.

If we regard, on the other hand, the compounds belonging to the ammonia-type, we observe that the electro-positive character of the substances gradually rises in intensity from the nitrogen- to the antimony-compounds.

Thus, trimethylamine and triethylamine are not capable of uniting with oxygen, chlorine, bromine, and iodine; a power which the corresponding terms of the phosphorus-, arsenic-, and antimony-series possess in a high degree.

Triethylamine unites with the acids producing compounds of the formula

$$E_3N, HCl$$
$$E_3N, HSO_4$$
$$E_3N, HNO_6.$$

The corresponding compounds in the arsenic- and antimony-series do not exist; at all events chemists have not yet succeeded in preparing them. Triethylarsine and triethylstibine only combine *directly* with oxygen, chlorine, sulphur, &c., producing saline bodies, which have the composition respectively—

E_3AsO_3	E_3SbO_3
E_3AsCl_3	E_3SbCl_3
E_3AsS_3	E_3SbS_3.

In the phosphorus-series, lastly, the two classes are represented. Triethylphosphine not only forms compounds analogous to the salts of triethylamine, but also the terms corresponding to the binoxides of triethylarsine and triethylstibine. We have in the first place the terms

$$E_3P, HCl$$
$$E_3P, HSO_4$$
$$E_3P, HNO_6,$$

and in the second place compounds of the formulæ

$$E_3PO_3$$
$$E_3PCl_3$$
$$E_3PS_3.$$

The phosphorus-compounds accordingly hold a position intermediate between the nitrogen-compounds on the one hand, and the arsenic- and antimony-series on the other. It cannot, however, be denied that the phosphorus-compounds stand closer to the arsenic- and antimony-series than to the nitrogen-group.

This cannot surprise us, when we consider the close analogies which phosphorus and arsenic present in many other directions. Both phosphorus and arsenic form well-characterized polybasic acids; the acids of antimony are not yet sufficiently investigated, but the acids of nitrogen, which are better examined, are all found to be essentially monobasic. The equivalent numbers, too, of phosphorus, arsenic, and antimony, present a remarkable connexion, the difference between those of phosphorus and arsenic, and those of arsenic and antimony being virtually the same—

Phosphorus . . .	31	difference . . .	44
Arsenic	75	difference . . .	45,
Antimony	120		

whilst the equivalent of nitrogen stands altogether apart from the rest.

The same relative position of the elements nitrogen, phosphorus, arsenic and antimony, may also be traced in their hydrides,

$$H_3N \qquad H_3P \qquad H_3As \qquad H_3Sb.$$

Ammonia is a powerful alkali;—phosphoretted hydrogen only unites with hydrobromic and hydriodic acids, whilst in arsenietted and antimonietted hydrogen, the power of combining with acids has altogether disappeared. In these hydrogen-compounds the gradation of properties is indeed much more marked than in their trimethylated and triethylated derivatives. On comparing the terminal points of the series, ammonia and antimonietted hydrogen, we cannot fail to be struck by the dissimilarity of properties which at the first glance appears to limit the analogy of the two compounds to a mere parallelism of composition.

In the methylated and ethylated derivatives of these compounds, the intensity of the chemical tendencies in general is so much raised, that the gradation is no longer perceptible to the same extent.

We cannot conclude this memoir without thankfully acknowledging the able and untiring assistance we have received, during this lengthened inquiry, from Dr. A. Leibius in the analyses, and from Messrs. W. H. Perkin and C. Hoffmann in the preparation of the numerous compounds which had to be investigated.

XXIX. *Photo-chemical Researches.*—Part III. *Optical and Chemical Extinction of the Chemical Rays.*

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In order to determine whether the act of photo-chemical combination necessitates the production of a certain amount of mechanical effect, for which an equivalent quantity of light is expended, or whether this phenomenon is dependent upon a restoration of equilibrium effected without any corresponding equivalent loss of light, we must study more specially the phenomena occurring at the bounding surfaces, and in the interior of a medium exposed to the chemically active rays.

A certain large class of bodies permit the chemical rays to pass through them to a greater or less extent, whilst other substances are opake to these rays. The first class, to which the name "diactinous" may be given, includes almost all colourless, blue, and violet media; the second class of "anactinous" bodies consists chiefly of opake, and yellow or red substances. Although the terms "diactinous" and "anactinous," like "diathermanous," "diaphanous," &c., merely represent phases of the same difference, and can therefore bear no strict scientific definition, yet we do not hesitate to employ these terms as a means of avoiding a tedious circumlocution.

When a ray of light penetrates a body which is not photo-chemically sensitive, a part of the light is absorbed, and an equivalent quantity of heat evolved. If the amount of light entering the medium be called I_o, and the amount issuing from the medium I, we have

$$\alpha I_o = I,$$

on the supposition that the extinction and the intensity of the light vary proportionally, when α represents the fraction of the original amount of light which has remained after passage through the medium. If the amount of light αI_o, which has passed through one medium, be now allowed to pass through a second whose coefficient is α_1, the amount of light issuing from the second medium will be $\alpha\alpha_1 I_o$, from a third medium $\alpha\alpha_1\alpha_2 I_o$, &c. The value of the result $\alpha\alpha_1\alpha_2 I_o$ is not affected by any alteration in the order of the media, as such an alteration simply changes the order in which the coefficients $\alpha\alpha_1\alpha_2$, &c. are placed. It is therefore quite immaterial, as regards the amount of light, in what order the various absorbing media are placed between the source of light and the instrument by which the effect is measured.

In order to show that the extinction of the chemical rays is really proportional to the intensity of the light, we have made the following experiments.

The intensity of the chemical rays proceeding from a constant source of light, was measured before and after their passage through a cylinder with plate-glass ends, filled with dry chlorine. After the amount of light I_0 has passed through the first plane glass plate it has become αI_0, after having passed through the chlorine it has diminished to $\alpha b I_0$, and on passing through the second glass plate it will be reduced to $\alpha b \alpha_1 I_0$. The amount of light I which issues from the cylinder, is therefore equal to $I = \alpha b \alpha_1 I_0$ or $\alpha b \alpha_1 = \frac{I}{I_0}$.

If we determine the amount of light I which passes through the cylinder for various intensities of the incident ray I_0, and if we find that for the various intensities the fraction $\frac{I}{I_0}$ remains constant, we may conclude that the amount of transmitted light is proportional to the amount of incident light.

The following experiments show the equality of the fraction $\frac{I}{I_0}$ for amounts of light varying from 1 to 1·9.

Series of Experiments I.

	Exp. 1.	Exp. 2.	Exp. 3.	Exp. 4.	Exp. 5.	Exp. 6.	Exp. 7.	Exp. 8.
I_0.....	13·52	13·20	12·85	13·51	7·21	8·34	12·39	12·84
I	3·63	3·63	3·79	3·79	2·11	2·44	3·69	3·69

Hence the values of $\frac{I}{I_0}$ are—

Number of Experiment.	$\frac{I}{I_0}$	Variation from mean.	Relative intensity of light.
1	0·267	−0·019	1·875
2	0·275	−0·011	1·830
3	0·295	+0·009	1·782
4	0·281	−0·005	1·874
5	0·293	+0·007	1·000
6	0·293	+0·007	1·156
7	0·298	+0·012	1·716
8	0·287	+0·001	1·781
Mean	0·286		

The conclusion derived from these experiments, viz. that the absorption of the chemical rays varies proportionally with the intensity, may serve as the foundation for the general law of the extinction of the optical and chemical rays in transparent media. For, as it is thus proved that the amount of light transmitted through a medium of finite thickness is proportional to the intensity of the incident rays, it may be assumed that the same relation will hold good for an infinitely thin medium. According to this supposition, the relation between the transmitted light I, and the thicknesss of the absorbing medium, is found from the equations

$$I = I_0 . 10^{-As}, \quad \ldots \ldots \ldots \quad (1.)$$

and hence

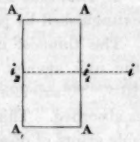
$$\alpha=\frac{1}{h}\cdot\log\left(\frac{I_0}{I}\right),$$

in which I_0 represents the intensity before transmission, I the intensity after transmission through a medium of the thickness h, and $\frac{1}{\alpha}$ the thickness of absorbing medium by passing through which the amount of light has diminished to one-tenth.

The values of α for chlorine, or the chlorine detonating gas, may be determined by allowing the rays from a constant source of light to pass through layers of the gas of various degrees of thickness; whilst the intensity of the incident and transmitted rays is measured. As the gases are enclosed between plane plates of glass, the loss of light in the passage through these plates must be considered. This loss of light is made up of the light reflected from the surfaces of the glasses, and that absorbed in the mass of the glass. For the purposes of our investigation, it is necessary to know what fraction of the loss of light is to be ascribed to the reflexion, and what fraction to the absorption in the mass of glass. The following was the method adopted in order to determine this point.

Let I_q represent the portion of incident light I_0 which enters the medium, and let I_p represent the amount of light which remains after the ray has traversed a thickness h of the medium, and let $\frac{1}{\alpha}$ represent the depth of the medium, by passing through which the original amount of light I_q is reduced by extinction to one-tenth of its intensity, we then have $I_p=I_q.10^{-h\alpha}$, where α represents a constant, which we call the Coefficient of Extinction, dependent upon the nature of the medium. We can only measure the amount of light I_0 falling upon the medium; the light which enters the medium can, however, as the following considerations show, be calculated, at least for rays perpendicular to the reflecting surface. Suppose that the unit amount of light falls in the direction ii_1, fig. 1, perpendicular to the surface AA of the medium, a portion ϱ of this light will be reflected at i_1, and a portion $1-\varrho$ will enter the medium. This ϱ is a constant, dependent on the nature and surface of the transparent medium, which may be called the Coefficient of Reflexion. When the light $1-\varrho$ has passed through the thickness h of the medium, and reached the second surface i_2, the intensity has diminished to $(1-\varrho)10^{-h\alpha}$, in consequence of the extinction which it has suffered. At the second bounding surface the fraction ϱ of this light, that is, $\varrho(1-\varrho)10^{-h\alpha}$, is reflected; so that the amount of light which passes this second reflecting surface is $(1-\varrho)10^{-h\alpha}-\varrho(1-\varrho)10^{-h\alpha}$, or $(1-\varrho)^2 10^{-h\alpha}$. In the case of our experiments, the further reflexions are not considered, as their influence on the result would be inappreciable, and hence we have the following equation representing the relation between the incident light I_0 and the transmitted light I:—

Fig. 1.



$$\frac{I}{I_0}=(1-\varrho)^2 10^{-h\alpha}, \quad \ldots\ldots\ldots\ldots \quad (2.)$$

and similarly for I_0^1, I^1 and h_1,

$$\frac{I^1}{I_0^1}=(1-\varrho)^2 10^{-h_1\alpha}.$$

Hence

$$\alpha=\frac{1}{h_1-h}\cdot\log\left(\frac{I.I_0^1}{I_0 I^1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

$$\varrho=1-\sqrt{(\text{num log}=h\alpha)\frac{I}{I_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

The values for α and ϱ employed in all our experiments were calculated from the equations, by determining the amount of chemical rays which passed through two plates of crown glass, one of which was very thin, and the other much thicker. The following experiments gave the requisite data. I_0 signifies the amount of light, measured by our instrument* before transmission through the glass plate; I signifies the amount of light measured after transmission; and h the thickness in millimetres of the plates employed.

Series of Experiments II.

	Exp. 1.	Exp. 2.	Exp. 3.
I_0.....	19·85	18·42	19·09
I	13·96	13·14	17·21
h	17·1 mm.	17·1 mm.	0·30 mm.

From experiments 1 and 3 the value of α is found to be 0·00642; from experiments 2 and 3, 0·00605, or a mean of 0·00623. From experiment 1 we get $\varrho=0{\cdot}0520$, from experiment 2 $\varrho=0{\cdot}0452$, from experiment 3 $\varrho=0{\cdot}0485$, or a mean of 0·0486.

The following conclusions may be drawn from this experiment:—

1. That 4·86 per cent. of the chemical rays from a flame of coal-gas which fall perpendicularly on a surface of crown glass, are lost by the first reflexion.

2. That nine-tenths of the chemical rays, issuing from the same source of light, are extinguished by passing through 160·5 millimetres of crown glass.

The thinnest of the plates employed by us had a thickness of 0·30 millimetre; the unit amount of light in passing through this plate will be reduced from 1 to $10^{-0{\cdot}3\cdot 0{\cdot}00623}=0{\cdot}9957$. In passing through this plate, therefore, 0·43 per cent. of the light is absorbed. This is so small an amount that it falls within the limits of the unavoidable errors of observation, and we may, without diminishing the accuracy of the result, write $h=0$ instead of $h=0{\cdot}3$. Hence formula (4.) becomes

$$(1-\varrho)^2=\frac{I}{I_0},\ \text{or}\ \varrho=1-\sqrt{\frac{I}{I_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

The value of ϱ calculated according to this formula, from No. 3 of Series I., gives $\varrho=0{\cdot}0506$, which corresponds very nearly to the mean value 0·0486 previously found. This value is probably the more correct, as it is independent of the errors ensuing from the introduction of α.

* Philosophical Transactions, 1857, p. 359, Plate XVII.

As the thick plate of crown glass, which we used for the determination of α, was somewhat dark-coloured, and contained bubbles of air, it was to be expected that the plates of white mirror glass, which we employed for our transparent cylinders, would not give so large a coefficient of extinction. The following experiments confirmed this opinion, showing that the amount of light α, absorbed by the plates of white glass 4·7 millims. in thickness, was not more than that absorbed by the thin plate 0·3 millim. thick; hence that the coefficient of extinction of our plates was so small, that, without incurring any perceptible error, the product $\alpha . 4{\cdot}7$ in formula (2.) becomes equal to 0; or in other words, the absorption in the interior of the glass may be neglected without any appreciable error.

Series of Experiments III.

	Exp. 1.	Exp. 2.	Exp. 3.	Exp. 4.	Exp. 5.	Exp. 6.
I_0	9·58	8·88	9·91	10·36	11·87	11·91
I	8·62	7·99	8·97	9·25	10·85	10·66
h	0·30 mm.	4·70	4·70	4·70	4·70	4·70
ρ	0·0514	0·0514	0·0487	0·5552	0·0439	0·0540

The last horizontal division contains the values of the coefficient of reflexion calculated according to formula (5.), upon the assumption, proved correct by the last experiment, that no appreciable absorption of the rays of light occurs in the plates. The mean value of ϱ is found to be 0·0509. When the coefficient of reflexion for glass is known, the amount of light α, transmitted by n plates, is found from the formula

$$\frac{1-\varrho}{1+(2n-1)\varrho}=\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

If the mean value of ϱ 0·0509, as found in the last experiment, be substituted in this formula, we obtain a value $a_1=0{\cdot}823$ for the amount of light passing through two plates. This value, 0·823, plays an important part in the calculation of the coefficient of extinction, so much so, that we have thought it necessary to corroborate this calculated result by further experiment. For this purpose a series of direct observations were made with the cylinders closed with two plates of glass. In these, as well as the succeeding absorption experiments with these cylinders, the accuracy of the result depended entirely on placing the cylinder before the insolation-vessel (*i*, fig. 2, Part II.), so that none of the rays from the luminous source were obstructed by the edge of the cylinder. This arrangement was effected by illuminating the insolation-vessel, and placing the transparent cylinder in such a position that when the observer moved his eye near the source of light, he saw that the insolation-vessel was covered by the screen, and not by the edge of the interposed cylinder. If the cone of rays from a convex lens was employed, the aperture of the lens was made smaller than that of the cylinder, and lens and cylinder were brought as near as possible together. Having duly observed these precautions, we determined the intensity of the chemical rays proceeding from a flame of coal-gas, before and after their passage through our transparent cylinder.

closed with two plates of glass*. The experimental numbers, and the values of a_1, as calculated with the formula $a_1 = \frac{I}{I_0}$, are thrown into the following Table.

Series of Experiments IV.

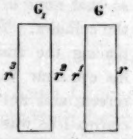
	Exp. 1.	Exp. 2.	Exp. 3.	Exp. 4.	Exp. 5.	Exp. 6.
I_0	17·76	17·18	17·66	18·55	11·39	18·37
I	13·41	13·67	13·69	14·49	9·56	15·04
I_0'	15·84	17·76	17·19	18·26	11·37	18·77
Mean of I_0 and I_0'	16·80	17·47	17·42	18·40	11·38	18·57
a_1	0·798	0·783	0·784	0·788	0·840	0·810

The mean of these values, 0·800, is slightly less than the value 0·823 as calculated from formula (6.). In all the following experiments we shall employ the mean of these two determinations, 0·811, as the most exact value for the coefficient of transmission of our cylinder plates.

At the commencement of this section we have shown that when light passes through a number of media, the amount of transmitted light remains the same, in whatever order the media be placed; thus a film of air between two glass plates would allow exactly as much light to pass through, were the air in front of or behind the plates. This assumption is however only true when all the media possess the same coefficients of reflexion. If the refractive index i, and hence the coefficient of reflexion ϱ, differs according to the order in which the media are placed, the values of i and ϱ vary in a relation dependent upon the arrangement of the media. In this case therefore allowance must be made for the order in which each limiting surface of a medium occurs. This is best explained by an example. We select for this purpose the determination of the coefficient of extinction for water which we have occasion to employ in our subsequent experiments.

Let us first examine what occurs when a ray of light passes through two glass plates (GG_1, fig. 2) surrounded by air, but so thin that, like the glasses of our cylinders, they may be regarded as perfectly diactinic. The fraction i of the unit amount of incident light falling perpendicularly upon the medium

Which passes the first surface at r is . . . $(1-\varrho)$
Which passes the second surface at r_1 is . . $(1-\varrho)^2$
Which passes the third surface at r_2 is . . $(1-\varrho)^3$
Which passes the fourth surface at r_3 is . . $(1-\varrho)^4$.

Fig. 2.

If we suppose that a film of water be placed between the plates instead of the air, the coefficient of reflexion from glass and water will no longer be ϱ, but a new quantity, ϱ_1; so that the unit amount of incident light becomes

* See Philosophical Transactions, 1857, Plate XVII. fig. 1.

On passing the first surface . . . . . $(1-\varrho)$
On passing the second surface . . . . . $(1-\varrho)(1-\varrho_1)$
On passing the third surface . . . . . $(1-\varrho)(1-\varrho_1)^2$
On passing the fourth surface . . . . . $(1-\varrho)^2(1-\varrho_1)^2$.

The total amount of transmitted light is in this case, therefore, $(1-\varrho)^2(1-\varrho_1)^2$; and as ϱ is greater than ϱ_1, the amount of transmitted light is larger than in the case when air was contained between the plates.

If the media are placed in the following order, air, water, glass, air, glass, air; and

If the coefficient of reflexion between air and water $=\varrho_2$,
If the coefficient of reflexion between water and glass $=\varrho_1$,
If the coefficient of reflexion between glass and air $=\varrho$,

a repetition of the foregoing reasoning shows that the transmitted light is not

$$(1-\varrho)^2(1-\varrho_1)^2, \text{ but } (1-\varrho_2)(1-\varrho_1)(1-\varrho)^2.$$

It is easy from these considerations to determine the amount of light transmitted by two diactinous glass plates enclosing a column of water. Let ϱ represent the coefficient of reflexion between air and glass, ϱ_1 between water and glass, h the length of the enclosed column of water, and $\frac{1}{\alpha}$ the length of a column of water by passing through which nine-tenths of the light is extinguished; we then have the unit amount of light reduced,

After passing the surface r (fig. 2) $(1-\varrho)$
After passing the surface r_1 (fig. 2) $(1-\varrho)(1-\varrho_1)$
After traversing the space from r_1 to r_2 $(1-\varrho)(1-\varrho_1)10^{-h\alpha}$
After passing the surface from r_2 $(1-\varrho)(1-\varrho_1)^2 10^{-h\alpha}$
After passing the surface from r_3 $(1-\varrho)^2(1-\varrho_1)^2 10^{-h\alpha}$

If I_0 represent the amount of incident light, and I that of the transmitted light, $\frac{I}{I_0}$ of the unit amount of incident light is transmitted. Hence we have

$$\frac{I}{I_0}=(1-\varrho)^2(1-\varrho_1)^2 10^{-h\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

and

$$\alpha=\frac{\log\left[\frac{I}{I_0}(1-\varrho)^2(1-\varrho_1)^2\right]}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

The value of ϱ for our cylinders filled with air is, as has been shown, 0·0509, and the value of ϱ_1, calculated from the index of refraction from crown glass to water, $i=1{\cdot}1718$, by means of the formula $\varrho_1=\left(\frac{1-i}{1+i}\right)^2$, is found to be 0·006257. An experiment with columns of water of h millimetre thickness gave the following values for I_0 (the light measured without interposed cylinder) and I (with interposed cylinder). The values of α contained in the lowest horizontal line are calculated according to formula (8.).

Series of Experiments V.

	Exp. 1.	Exp. 2.	Exp. 3.	Exp. 4.
I_0.....	16·08	16·08	16·08	10·23
I	13·76	13·99	13·57	8·99
h	83·8 mm.	11·8	27·0	17·3
α	0·0002	0·0008	0·0008	0·0003

From the extremely small values which are thus found for α, it is seen that the coefficient of extinction of the chemical rays employed may, as far as regards the observation, be placed at 0 in columns of water less than 80 millims. in depth; and therefore that the factor 10^{-ah} in formula (7.) becomes $=1$. According to this method it is possible, approximately at least, to determine the coefficient of reflexion of all transparent fluids for the chemical rays. For this purpose a drop of the liquid is brought between two plates of glass, of which the coefficient of reflexion ϱ has already been determined, and the intensity of the light measured before and after transmission through the moistened plates. The values of I_0 and I thus found are substituted in equation (7.), and as the value of h is extremely small, the factor 10^{-ah} vanishes, and we have for the coefficient of reflexion of the liquid,

$$\varrho_1=\frac{1-\sqrt{\frac{I}{I_0}-\varrho}}{1-\varrho}.$$

A simple relation exists, as has been stated, between the coefficient of reflexion ϱ and the refractive index i, which is represented by the equation $\varrho=\left(\frac{1-i}{1+i}\right)^2$, or $i=\frac{1+\sqrt{\varrho}}{1-\sqrt{\varrho}}$. If we substitute the value $\varrho=0{\cdot}0509$, found for crown glass, we obtain for the refractive index $i=1{\cdot}583$. The value of this index of refraction, as obtained by experiment for FRAUNHOFER'S line H lying nearest to the chemical rays, is given in BUFF'S 'Lehrbuch der Physik,' to be 1·5466 and 1·5794, numbers closely approximating to that found from the coefficient of reflexion.

The coefficients of reflexion and the refractive indices for the chemical rays of all substances which can be divided into such thin plates that the absorption may be considered $=0$, may in this way be ascertained. We have thus determined the coefficient of reflexion and the refractive index of North American mica, which we shall make use of in a later part of the investigation.

It has been previously observed that the fraction $\alpha=\frac{I}{I_0}$ (formulæ 1 and 6) of the unit amount of light which passes through n glass plates, is represented by the equation $\frac{I}{I_0}=\frac{1-\varrho}{1+(2n-1)\varrho}$, where ϱ signifies the coefficient of reflexion of the glass. In order to see whether the plates absorb any light in their interior, in which case the formula could not be employed, it is sufficient to make the experiment first with one plate, and then

with a greater number. If both experiments give the same value for ϱ, it is clear that the disturbing influence of absorption within the plate is inappreciable.

The following series of experiments were made with plates of mica, which were only some hundredths of a millimetre in thickness.

Series of Experiments VI.

A. With one plate of Mica.

	Exp. 1.	Exp. 2.	Exp. 3.	Exp. 4.
I_0	12·69	12·37	12·83	12·41
I	10·33	10·33	10·17	10·17
$\frac{I}{I_0}$	0·814	0·835	0·793	0·816
Mean value of $\frac{I}{I_0}=$	0·8143			
$\varrho=$	0·1024			

B. With three plates of Mica.

	Exp. 1.	Exp. 2.	Exp. 3.	Exp. 4.
I_0	12·91	13·08	12·89	12·84
I	7·80	7·80	7·65	7·65
$\frac{I}{I_0}$	0·602	0·596	0·596	0·596
Mean value of $\frac{I}{I_0}=$	0·5969			
$\varrho=$	0·1011			

The mean of the two values found for ϱ is 0·1017, and this gives, when substituted in the formula $\frac{1+\sqrt{\varrho}}{1-\sqrt{\varrho}}=i$, as a mean refractive index of mica for the chemical rays of a coal-gas flame, the number 1·936.

Another important element in the investigation of photo-chemical extinction, is the law according to which the optical coefficient of extinction varies with the density of the absorbing medium. In order to determine this law for the chemical rays we have employed coloured solutions, knowing that for a considerable depth water may be considered as perfectly diactinous. We made use of columns of water of various lengths, each containing the same quantity of a dilute solution of common red ink. The following experiments, VII., show the dependence of the extinguished chemical rays upon the density of the extinguishing medium. I_0 signifies the amount of the chemical rays before transmission through the cylinder, I the amount remaining after transmission, $b=\frac{I}{I_0}$ the transmitted fraction of the unit amount of incident light, h the length of the columns of liquid in millimetres, e the density of the colouring substance, that is, the

amount of colouring matter contained in the unit quantity of water, and l the fraction of the unit amount of light transmitted through the unit length of the several cylinders.

Series of Experiments VII.

	Exp. 1.	Exp. 2.	Exp. 3.
I_0	7·41	8·13	8·22
I	4·28	4·73	4·87
b	0·578	0·582	0·592
h	11·8 mm.	27·4 mm.	83·8 mm.
e	1·000	0·4306	0·1408
l	0·04895	0·02123	0·00707
$\frac{e}{l}$	0·204	0·203	0·200

From the closely approximating values of $\frac{e}{l}$ we may conclude, that with the chemical rays the amount of transmitted light varies proportionally with the density of the coloured medium.

Having determined these preliminary points, we may now return to the question originally proposed, viz.:—

In the combination effected by the light between the particles of chlorine and hydrogen, are the chemical rays expended in a relation proportional to the quantity of hydrochloric acid formed? In order to solve this question, it was necessary to determine experimentally the coefficient of extinction of pure chlorine for a flame of coal-gas.

The chlorine used in the following experiments, prepared from bichromate of potash and hydrochloric acid, carefully washed and dried, was led through the transparent cylinders until all atmospheric air was expelled. Each of the measurements contained in the following Table is reduced from three series of observations, the first of which was made without interposed chlorine-cylinder, the second with chlorine-cylinder, and the third again without the cylinder. The mean of the first and third series gives I_0, or the intensity of the chemical rays before transmission through the chlorine-cylinder, whilst the second series gave I, or the intensity after transmission; h signifies the length of the interposed column of chlorine, P the pressure, and t the temperature on filling the cylinders with chlorine.

Series of Experiments VIII.

	Exp. 1.	Exp. 2.	Exp. 3.	Exp. 4.
I_0	13·51	10·86	11·12	11·40
I	3·79	3·25	3·23	6·39
h	83·8 mm.	83·8	83·8	27·4
P	0·7437 m.	0·7528	0·7528	0·7528
t	16°·0 C.	14·3	14·3	14·3

The amount of light I_0 is not that which enters the column of chlorine. This

amount, according to the former considerations, is that which remains after the quantity lost by passing both the enclosing plates of glass has been subtracted from I_0. I_1 must therefore be multiplied with $a_1=0{\cdot}811$, found from several of the previous experiments, the difference between the coefficients of reflexion between glass and air and glass and chlorine being neglected as imperceptible. The value of $\frac{1}{\alpha}$, which signifies, according to formula (2.), the depth of chlorine to which the chemical rays must penetrate in order to be reduced to one-tenth of their original amount, has been shown by Experiment VII. to vary proportionally with the density of the medium. For chlorine under 0·76 pressure and at 0° C., the value of $\frac{1}{\alpha}$ is found from the equation

$$\frac{1}{\alpha}=\frac{h.P}{0{\cdot}76(1+0{\cdot}00366t)(\log a_1 I_0-\log I)}, \quad \ldots\ldots\ldots (9.)$$

hence we get from Experiment VIII.

$\frac{1}{\alpha}=168$ millims.
$\quad\quad=182$ millims.
$\quad\quad=177$ millims.
$\quad\quad=160$ millims.

Mean . . . . $\overline{171{\cdot}7}$

This mean value, 171·7, gives a value of 0·00582 as the coefficient of extinction for chlorine at 0·76 pressure, and 0° C. for the flame of coal-gas.

In order to prove experimentally that the coefficient of extinction varies, in the case of chlorine, in the inverse proportion of the dilution, another series of determinations was made in which the chlorine was mixed with varying quantities of atmospheric air. For this purpose the caoutchouc-tubes, connected with the cylinders previously used, filled with V volumes of chlorine at $t°$ C., and under the pressure P, were secured by screw-clamps, and into the end of one caoutchouc joining a fine capillary tube was fastened, whilst into the other was placed a chloride of calcium drying-tube. On opening the screw-clamps, air was forced by the mouth through the drying-tube into the chlorine cylinder, whence the gas escaped through the capillary tube into a reversed retort filled with a solution of iodide of potassium, and when sufficient chlorine had been expelled the screw-clamps were again closed. The volume of chlorine expelled, reduced to 0° C. and 0·76 pressure, was determined by volumetric analysis, and this volume subtracted from that originally contained, also reduced to 0° C. and 0·76 pressure, gave the volume V_1 remaining in the cylinder mixed with air. If H represent the depth of the cylinder, measured from the inner surface of one glass plate to the inner surface of the other, and h_1 represent the length of the column of chlorine before displacement, $h_1=\frac{H.P}{0{\cdot}76(1+0{\cdot}00366t)}$, and after displacement, $h_2=\frac{V_1}{V}H$. These various columns of chlorine, h_1 h_2, both of which were expanded to the height H in the cylinder, were placed between the source of light and the insolation-

vessel, and from the diminution of the action from I_1 to I, the values of $\frac{1}{a}$ were calculated as in the previous instance. Chlorine was again expelled from the cylinder into the reversed retort, the cylinder again placed between the flame and the instrument, the values of I_1 and I again determined, and this repeated until the whole of the chlorine had been displaced. The agreement of the sum of all the separate determinations of chlorine with the total quantity introduced, served as a control for the accuracy of the analyses.

The calculation of the volume of chlorine at 0° C. and 0·76 pressure, from the direct observations of the volumetric analyses, was made by means of the formula $c=88{\cdot}012\ a(nt-t_1)$, which we deduced in a former communication.

Experiment gave—

Series of Experiments IX.

(1) Capacity of cylinder filled with chlorine at 16° C. and 0·7453 pressure	190·24 cub. cent.
(2) Contained volume of chlorine at 0° C. and 0·76 pressure	176·24 cub. cent.
(3) Height of the cylinder H	83·8 millims.
(4) Length of the column of chlorine reduced to 0° C. and 0·76 pressure	77·63 millims.

Analyses of the displaced chlorine:—

First portion	$\alpha=0{\cdot}001944$	$t=43{\cdot}1$	$t_1=16{\cdot}3$	$n=7$
Second portion	$\alpha=0{\cdot}001944$	$t=42{\cdot}5$	$t_1=4{\cdot}4$	$n=4$
Third portion	$\alpha=0{\cdot}001944$	$t=42{\cdot}3$	$t_1=17{\cdot}0$	$n=4$
Fourth portion	$\alpha=0{\cdot}001944$	$t=42{\cdot}2$	$t_1=2{\cdot}0$	$n=10$

Hence the volumes of chlorine at 0° C. and 0·76 pressure are found to be—

Original volume of chlorine in cylinder	176·24 cub. cent.
Volume of first portion displaced	48·83 cub. cent.
Volume of second portion displaced	28·34 cub. cent.
Volume of third portion displaced	26·04 cub. cent.
Volume of fourth portion displaced	71·86 cub. cent.
	175·07 cub. cent.

Hence it is seen that the sum of the quantities found in each analysis agrees with the quantity originally taken, 176·24, as nearly as could be expected, considering the number of analyses made. From these numbers the values of h are obtained, that is, the length which the columns of gas would have been, had they been unmixed with air and measured at 0° C. and 0·76 pressure.

1. Height of the total quantity of chlorine	77·6 millims.
2. Height after displacement of first portion	56·2 millims.
3. Height after displacement of second portion	43·6 millims.
4. Height after displacement of third portion	32·2 millims.

The photometric observations with these columns of chlorine, all of which were expanded to a height of 77·6 millims. by dilution with air, gave—

Series of Experiments X.

	Exp. 1.	Exp. 2.	Exp. 3.	Exp. 4.	Exp. 5.	Exp. 6.
I_0.....	8·45	10·93	11·13	11·10	11·39	10·65
I	2·53	3·35	4·39	4·15	5·16	5·51
A	77·6	77·6	56·1	56·1	43·6	32·2

The values of $\frac{1}{a}$, calculated from these experiments, are—

No. of Expt.		
1.	179	millims.
2.	184	millims.
3.	179	millims.
4.	167	millims.
5.	172	millims.
6.	165	millims.
Mean	174·3	millims.

This mean value agrees very closely with that found from the experiments with chlorine undiluted with air, but it is seen that the variation in the separate values is not inconsiderable. The cause of this lies not only in the unavoidable errors of experiment, but more especially in the fact that the formula used in the calculation is, strictly speaking, only applicable to homogeneous light, whilst the chemically active rays emanating from our coal-gas flame were certainly not all of the same degree of refrangibility.

If the intensity I_0 of the light, measured by our instrument, were composed of the components $I_1+I_2+\ldots$ of various degrees of refrangibility, the formula $I=I_0 10^{-Aa}$ would become

$$I=I_1 10^{-Aa_1}+I_2 10^{-Aa_2}+\ldots$$

The values calculated according to the first formula can therefore only give a mean value for the various chemical rays in our flame, and the numbers obtained must vary more considerably amongst themselves than if homogeneous light had been employed. The mean of all the determinations in Series VIII. and X. gives the value of the optical coefficient of extinction of chlorine at 0° C. and 0·76 pressure, for rays from a coal-gas flame, to be $\frac{1}{173·0}=0·00578$.

If the light is not consumed in the act of photo-chemical change, the coefficient just found must remain unaltered when the chlorine and hydrogen mixture is employed; but if, on the contrary, light is not only lost by the optical extinction, but also an amount of light proportional to the chemical action vanishes, experiment must give a larger value for the coefficient. In the first case the light merely liberates the chemical

forces, in the second it suffers a transformation into mechanical energy, effecting the combination of the chlorine and hydrogen. It would seem as if the simplest mode of arriving at a conclusion regarding these two alternatives, were to determine the coefficient of extinction for chlorine and hydrogen, by interposition of the transparent cylinders filled with the mixture: this plan, however, is accompanied by the unavoidable error arising from the impossibility of ascertaining when the mixture under examination has arrived at the point of maximum photo-chemical sensibility. We have therefore preferred to deduce the coefficient of extinction for chlorine and hydrogen directly from the chemical actions effected by a constant amount of light in columns of sensitive gas of various length. For this purpose we have employed the apparatus (Plate XXVII.), which mainly differs from the instrument described in our first communication in the arrangement of the insolation-vessel. The insolation-vessel *iq* consists of a glass tube with as even a bore as possible, 250 millims. in length and 15 millims. in breadth, of which the end *i*, exposed to the light, is a plane surface. The mixture of chlorine and hydrogen, evolved in the manner described in our first communication, enters the insolation-vessel *iq* through the tube *p* melted on to the side of the vessel, and ground air-tight into the glass stopcock *h*; the gas issues through a tube *q* similarly fused into the other end of the insolation-vessel, and thence passes into the observation-tube K. In the inside of the insolation-vessel is placed a semicircular diaphragm, *d*, of black glass, which is melted on to the glass rod i_1, moveable in the caoutchouc cap *c*. By moving this glass rod, on which a scale in millimetres can be etched, backwards or forwards through the caoutchouc cap, the column of gas *id*, exposed to the light, can be shortened and lengthened at pleasure. The tube *iq* contains so much water, that when the diaphragm *d* is turned round to the right, at an angle of 90° from that represented in the figure, the surface of the water just touches the straight edge of the diaphragm. In order to prevent entrance and collection of water in the tube *q* during the alteration of the diaphragm, the rod *i*, must never be moved, unless the curved side of the diaphragm is undermost (as in the position represented in the figure), when the water is not carried back, and no inconvenience is experienced in altering the length of the column of gas. In spite of the number of ground-glass joints, and the consequent fragile nature of the apparatus, all these necessary operations may be executed without danger, if the insolation-vessel is firmly fixed in one position by pressing it against the support at *q* with the finger and thumb. When the apparatus is to be filled with gas, all the precautions must be taken which are described in our first communication, and a thick piece of wood is placed under the board at A, so that the whole arrangement, including the evolution-tube, not represented in the figure, is placed in an oblique position, and all the water collects at the lower end, *i*, of the insolation-vessel, and is completely saturated with the gas which bubbles through from the tube *p*. As soon as the gas has reached the maximum degree of photo-chemical sensibility (determined by experiments made from time to time with the standard flame), the apparatus is again placed in a horizontal position by means of the screws B, B, B.

The following precautions must be observed at the commencement of each experiment. In the first place, the ground-glass joint of the tube p is rendered perfectly air-tight, by slightly pressing the ground edge into its place, and then moving the whole vessel at q up and down as much as the elasticity of the glass joints allows. In the same way, care must be taken that the joint at q is also tight; and in order to preserve these joints secure, pieces of caoutchouc tubing cut lengthways and filled with water are placed over the two tubes where the joint occurs, and the ground-glass surfaces are thus always kept wet. Secondly, the level of water in the tube iq must be so regulated by the screws B B B, at the commencement of each series of observations, that the same section throughout the whole length of the tube is filled with gas. Thirdly, the end of the insolation-vessel which is exposed to the light must be blackened as far as the upper level of water, in order to avoid any alteration in the constitution of the dissolved gases from the action of the light. Fourthly, the insolation-vessel must be placed, before each experiment, exactly in a line with the incident rays which are as nearly as possible parallel. Fifthly, the whole mass of gas contained in the tube, both before and behind the moveable diaphragm d, must be induced before each series of observations. We have always effected this induction by allowing light from a window opposite the door of our darkened room to fall on the tube, by opening the door previous to beginning an experiment, and allowing the action to proceed until it was evident, from the rate of motion on the index scale, that the maximum of the induction was attained. Sixthly and lastly, in order to eliminate all disturbing influence of change of temperature, the instrument was covered, during the experiment, with a tin case C, in which slits were cut, as seen in the figure, so that the case could fit closely over all parts of the insolation-vessel.

From the following considerations, it is seen that in order to obtain the value of α_1, the coefficient of extinction in the chlorine-hydrogen mixture, a determination of the chemical actions W and W_1, in two columns of gas of different lengths h and h_1, is sufficient. Let I_0 represent the intensity of the light on entrance into the gaseous mixture, I the intensity remaining after it has passed through a length z of gas, we have then, according to the former equation, $I=I_0 10^{-\alpha_1 z}$, in which α_1 signifies the coefficient of extinction of the unreduced chlorine mixture. If, also, wdz represents the chemical action effected in the unit of time in a film of gas of the thickness dz, we have $w=NI$, where N signifies a constant, and therefore $w=NI_0 10^{-\alpha_1 z}$. If h is the whole depth of the insolated column of gas, and W the total action occurring therein, we have

$$W=\frac{NI_0}{\alpha_1 \log \text{nat } 10}(1-10^{-\alpha_1 h});$$

or more simply,

$$W=MI_0(1-10^{-\alpha_1 h}).$$

For W_1 and h_1 we get, similarly,

$$W_1=MI_0(1-10^{-\alpha_1 h_1});$$

hence

$$\frac{W}{W_1}=\frac{1-\frac{1}{\text{num log}=\alpha_1\lambda}}{1-\frac{1}{\text{num log}=\alpha_1\lambda_1}}.$$

The value of α_1 is obtained from this equation by a method of approximation.

Experiments conducted in this manner at 23°·3 C., and under a pressure of 0·7589 millims., gave the following elements for the calculation of the values of $\frac{1}{\alpha_1}$ contained in the fifth column. Each value of W and W_1 is the mean of from ten to fifteen observations.

Series of Experiments XI.

W_1.	W.	λ_1.	λ.	$\frac{1}{\alpha_1}$
				millims.
20·97	4·46	194·5	21·5	256
19·38	11·17	193·8	65·5	216
16·26	10·26	114·7	60·9	316
12·34	6·73	153·0	58·6	262
			Mean	262

Had the insolated chlorine and hydrogen mixture possessed the density corresponding to 0° C. and 0·76 pressure, and if it had been perfectly dry, the length $\frac{1}{\alpha_1}=262$ would have diminished, for we have shown that the amount of light absorbed is proportional to the density of the absorbing medium. In a pure dry mixture of equal volumes of chlorine and hydrogen, having a density corresponding to 0° C. and 0·76 pressure, the length of a column of gas through which the light must pass in order to be reduced by optical and chemical extinction to $\frac{1}{10}$th of its original amount, is $\frac{1}{\alpha_2}=234$ millims. Supposing that the hydrogen were replaced by an equal volume of a chemically inactive gas, which, like hydrogen, does not absorb any chemical rays, we should obtain from the coefficient of extinction of pure chlorine $\frac{1}{173}$, a length of the gaseous mixture equal to 2×173 or 346 millims., through which the light would have to pass in order that it should be reduced to $\frac{1}{10}$th. Hence it is seen that for a given amount of chemical action effected in the mixture of chlorine and hydrogen, an equivalent quantity of light is absorbed. For we find that in the standard chlorine and hydrogen mixture, where, together with the thermal effect, a chemical action of the light occurs, the value of the coefficient of extinction is 0·00427; whereas in a chlorine mixture of similar dilution, where the chemical action was absent, the coefficient was 0·00289, or very much smaller.

The difference between these two numbers, 0·00138, represents the coefficient of extinction due to chemical action alone. The consumption of rays from a coal-gas flame, solely employed in the act of photo-chemical combination, is hence represented

by an amount of light which is reduced to $\frac{1}{10}$th by passing through 723 millims. of the standard chlorine mixture, whilst in consequence of optical extinction alone the light is reduced to $\frac{1}{10}$th by passing through 346·6 of the same standard mixture.

It appeared to us of great interest to repeat the experiments with rays from other sources of light. For this purpose we employed the diffuse light of day reflected from the zenith of a cloudless sky. The rays were rendered as nearly parallel as possible by reflexion from a mirror through a blackened tube, 6 feet long and 6 centimetres wide, on to the insolation-vessel of our instrument. All the observations employed in the two next series of experiments (XII. and XIII.), with diffuse light thus reflected, were made between the hours of 9 and 12 in the morning. For the determination of the coefficients of extinction for pure chlorine, a series of experiments similar to Series VIII. was made.

Series of Experiments XII., made on the 4th and 6th of August, 1856.

	Exp. 1.	Exp. 2.	Exp. 3.	Exp. 4.	Exp. 5.
I_0	15·23	12·12	10·84	9·63	10·61
I	2·96	2·62	2·54	2·51	2·28
h	32·4 mm.	27·0	27·0	27·0	27·0
P	0·7567	0·7567	0·7567	0·7567	0·7567
t	26°·3	26·3	26·3	26·3	26·3

The values of the reciprocals of the coefficients of extinction β, are calculated from these numbers according to formula (9.).

No. of Exp.	$\frac{1}{\beta}$
1.	47·4
2.	42·8
3.	45·5
4.	49·7
5.	42·5
Mean . . .	45·6

The coefficient of optical extinction of pure chlorine, at a density corresponding to 0° C. and 0·76 pressure, for diffuse morning light reflected from the zenith, is hence 0·0219, very different from the value 0·00578 found for a coal-gas flame. This result renders it probable that the coefficient of extinction of the standard chlorine and hydrogen mixture is much greater for diffuse morning light than for light from a coal-gas flame. The following experiments made with the chlorine and hydrogen mixture at 23°·8 C. and 0·7535 Barom., with diffuse zenith light, also between the hours of 8 A.M. and 12, gave results resembling Series XI.

Series of Experiments XIII., made on the 3rd, 6th and 7th of August, 1856.

No. of Exper.	W_1.	W.	A_1.	A.	$\frac{1}{\beta_1}$
					millims.
1.	10·31	5·79	125·5	29·0	85·5
2.	12·51	6·03	132·5	23·9	87·0
3.	10·85	7·31	176·9	40·9	85·4
4.	8·32	5·43	187·3	33·8	74·1
				Mean	83·0

The coefficient of extinction for morning zenith light in a dry mixture of chlorine and hydrogen, whose density corresponds to 0° C. and 0·76 pressure, is found to be $=0{\cdot}0136$, and its reciprocal $=73{\cdot}5$ millims. The value of the coefficient of the light consumed in effecting the change is $=0{\cdot}00265$, and its reciprocal 337·3 millims. From this it is seen that the light reflected in the morning from the zenith of a cloudless sky, is much more easily extinguished by passing through chlorine than the rays from a coal-gas flame. Hence it was conjectured that probably the solar light might differ in its properties according to the time of day or year. This supposition has been fully confirmed, as is shown by the following experiments made on the 15th of September, 1856, between the hours of 3 P.M. and $3\frac{1}{2}$ P.M., with zenith light from a cloudless sky.

Series of Experiments XIV.

	Exp. 1.	Exp. 2.
I_0	4·81	6·86
I.....	1·01	1·60
A.....	11·8	11·8
P	0·7597 m.	0·7597
t	17°·5	17°·5

These observations give the following numbers for the reciprocal of the coefficient $\frac{1}{\gamma}$.

No. of Exp.	$\frac{1}{\gamma}$
1.	18·9 millims.
2.	20·5 millims.
Mean . .	19·7

The coefficient of extinction of dry chlorine at 0° C. and 0·76 pressure for the evening light reflected from a cloudless zenith, hence is 0·05076. The optical extinction of reflected cloudless zenith light for chlorine at 0° C. and 0·76 pressure, was more than twice as small on the 4th and 6th of August in the morning, than it was on the 15th of September in the evening. A perfectly analogous difference is observed between the former series of Experiments XIII. and the following, made with the standard mixture, at 25°·7 C. and 0·7565 millim. pressure, on the 2nd of August, 1856, between the hours of $3\frac{1}{2}$ and 5 P.M., with light reflected from the zenith of a cloudless sky.

Series of Experiments XV.

No. of Exp.	W_1.	W.	A_1.	A.	$\frac{1}{\gamma_1}$
					millims.
1.	8·59	7·61	198·5	55·7	59·3
2.	6·61	4·75	185·5	31·0	56·4
3.	8·34	6·45	128·0	48·5	79·8
				Mean	65·2

Hence the value of the coefficient of extinction for the standard chlorine mixture at 0° C. and 0·76 pressure is $\gamma_1 = 0{\cdot}01743$, and its reciprocal 57·4.

A negative value is obtained for the coefficient of chemical extinction calculated from the results of Experiments XIV. and XV. From this we may conclude, that the rays which produced the action in Experiments XIV. possessed a different degree of refrangibility from those which acted in Experiments XV., and that from this reason the two series of experiments are not comparable; this view is easily explained by the fact, that the experiments were made at different times.

The conclusion which we draw from all our observations is, that the coefficients of extinction of pure chlorine for chemical rays from various sources of light are very different. The depth to which such light must penetrate chlorine at 0° and 0·76, in order to be reduced to $\frac{1}{10}$th of its original intensity, is, according to the foregoing experiments,—

(1) A flame of coal-gas . . . . . 173·0 millims.
(2) Reflected zenith light, morning . . 45·6 millims.
(3) Reflected zenith light, evening . . . 19·7 millims.

A variation in the same direction is also seen in the chemical extinction of the light. The depth to which the light must penetrate the normal chlorine mixture, supposing that no optical extinction took place, in order that the original amount of light should be reduced to $\frac{1}{10}$th by the chemical effect produced, is—

(1) A coal-gas flame . . . . . 723·0 millims.
(2) Morning light reflected from a cloudless zenith . . 377·3 millims.

It is thus seen that the chemical rays reflected at different times of the year, and at various hours of the day, not only possess quantitative but also qualitative differences, representing the various coloured rays of the visible spectrum. Had nature endowed our eyes with the power of discriminating the various degrees of refrangibility of the chemical rays, as she has done for the visible rays by impressions of different colours, we should see the rosy tints of morning pass in the course of the day through all the gradations of colour until the warm evening tones at length succeed.

A long-continued series of observations must be made before we are able to understand the influence which these qualitative differences in the chemical rays exert in the

photo-chemical phenomena of vegetation. That this influence must be of the greatest importance, is evident from the varying effects produced in other photo-chemical phenomena by differences in the solar light. We need only call to mind, in proof of this assertion, the fact, well known to all photographers, that the amount of light, determined photometrically, gives no measure for the time in which a given photo-chemical effect is produced, and that the morning light is always preferred to the evening light for the preparation of pictures, even should the light in the latter case be more intense.

XXX. *Account of the Construction of the New National Standard of Length, and of its principal Copies.* By G. B. Airy, *Esq., Astronomer Royal.*

Received May 2,—Read June 18, 1857.

In presenting this Report to the Royal Society, I must solicit their indulgence for many imperfections, arising from the circumstances under which the task of writing it has devolved on me. Through the whole account, I have to record the proceedings of others, and (in a great measure) I have to describe the trains of thought in the minds of others, which have determined those proceedings. The first part of the investigations for restoration of the Standard was made by Mr. F. Baily; but the largest proportion by very far was effected by my much-valued friend the Rev. R. Sheepshanks. These have successively been removed by death from their labours on the Standards. On the eve of the day on which Mr. Sheepshanks was struck down with mortal disease, he was engaged in micrometrical comparisons of Copies of the Standards; on the day following the deadly stroke, the Royal Assent was given to the Bill for legalizing the Official Standards which he had constructed. Under circumstances like these, it is evident that the documents relating to the final operations must have been left in great confusion. In the earlier portions (the papers relating to which had been, in a measure, roughly brought into order), the difficulty is mainly of another kind. Although from official relation and private friendship I was in constant and unreserved intercourse both with Mr. Baily and with Mr. Sheepshanks, and though I was cognizant of every step taken, and sometimes examined the apparatus used, yet I never actually made an observation. It is possible therefore that my want of familiarity with the operation may sometimes make my statements imperfect. I trust, however, that in the main my account will be substantially correct.

It will be necessary for me briefly to revise the history of the British Standards for some time anterior to the work which forms the essential subject of this paper.

I propose to divide my Account under the following heads:—

Section I.—History of the British and of some Foreign Standards, and of the methods of using them in Base-Measures and Pendulum-Measures, anterior to the legalization of the Imperial Standards by the Act of Parliament of 1824; definition of the Standard of Length by that Act; and provision for its restoration in case of loss.

Section II.—Comparisons of Standards, between the passing of the Act of 1824 and the appointment of a Commission for consideration of Standards after the destruction of the Imperial Standard in 1834; with remarks suggested by the advance of collateral theory and experiment in that interval.

SECTION III.—Appointment of the Treasury Commission of 1838; its proceedings and Report. Appointment of the Treasury Commission of 1843; its proceedings to the death of Mr. BAILY in 1844.

SECTION IV.—Proceedings of the Committee and of Mr. SHEEPSHANKS to June, 1847; construction of new Thermometers; and erection of Comparing Apparatus: description of the Apparatus, and of Mr. SHEEPSHANKS' method of comparing.

SECTION V.—Proceedings of Mr. SHEEPSHANKS to the middle of 1850; preparation of Thermometers; adoption and rejection of "Brass 2" as Basis for the new Standard Yard; adoption and rejection of "Split Plug A"; adoption of "Bronze 12"; experiments on thermometric expansion; comparison of Bronze 12 with several bars; first suspicion of personal equation; rejection of Bronze 12 as Basis, and final adoption of "Bronze 28."

SECTION VI.—Operations of important character to the end of 1853; comparison of a small number of bars with Bronze 28; investigation of Personal Equation; investigation of the effect of Inside or Outside position of the bar; investigation of the relative thermometric expansion of Steel, Wrought Iron, Cast Iron, Copper, and Brass, as compared with Bronze; trial and rejection of BAILY'S Apparatus.

SECTION VII.—Comparisons of numerous bars from 1851 to 1855; suspicion of change in Bronze 28; removal of the suspicion; close of the operations on bars defining the Yard by line-measure.

SECTION VIII.—Formation of End-measure Bars.

SECTION IX.—Closing proceedings of Official Character; Extracts from the Final Report of the Commission; Extracts from the Act of Parliament legalizing the New Standard; standard temperatures for the compared bars; disposal of the bars.

SECTION I.—*History of the British and of some Foreign Standards, and of the methods of using them in Base-Measures and Pendulum-Measures, anterior to the legalization of the Imperial Standards by the Act of Parliament of* 1824; *definition of the Standard of Length by that Act; and provision for its restoration in case of loss.*

The principal materials for the history of the Standards are to be found in the following works:—

Philosophical Transactions, 1742–1743, page 185, &c. Account of the laying down of the measure of 3 feet from the Tower Scale upon the Brass Bar of the Royal Society; and comparison of the Yard with the Half-Toise. [The measure thus laid down was marked E, apparently to denote "English measure" as distinguished from the French measure compared with it.]

Philosophical Transactions, 1742–1743, page 541, &c. Comparisons by GRAHAM (under the superintendence of a Committee of the Royal Society) of the Royal Society's Standard, defined by two points; the Exchequer end-standard, reputed to be the National Standard; the Exchequer standard-beds; the Tower standard-scale; the Guildhall standard-beds; and the Clock-Makers' standard-bed. The comparisons were made by the use of various kinds of beam-compasses, not described. It seems that, at the end

of these operations, the equivalent of the Exchequer end-standard was laid down on the Royal Society's bar; it is distinguished by the word "Excheq."

Philosophical Transactions, 1768, page 324, &c. In MASKELYNE'S Account of the Maryland degree, there is incidental mention of the comparison of BIRD'S scale with the Royal Society's brass standard, and of BIRD'S scale with the French toise; giving, however, a result for the relation between the Royal Society's Standard and the Toise which differs sensibly from that found by GRAHAM. The process of laying down the value of an end-measure (as the toise) upon a flat surface appears rather rude.

Philosophical Transactions, 1785, page 402, &c. In General ROY'S Account of the measure of the Base on Hounslow Heath, there is mention of the exact agreement (as ascertained from a comparison made by means of RAMSDEN'S beam-compasses) between 3 feet of the Royal Society's or Tower Standard and 3 feet of General ROY'S Scale (a scale originally the property of GRAHAM, which is said to have been divided by BIRD when in the employ of SISSON); and it seems that, in reliance on the uniformity of the division, General ROY then used 40 inches of his scale without any verification of the supplementary 4 inches. There is also a comparison of the Exchequer Yard (as marked on the Royal Society's bar) with the Tower Yard (similarly marked). And there are experiments on the expansions of various metals.

The lengths of the bars and chain to be used in the actual measure of the base were laid down by means of beam-compasses. Omitting the tentative measure by deal rods, the base was measured in two different ways; namely, by glass rods, and by a steel chain. When the glass rods were used, each rod carried a small slider in its end, which was brought into contact with the end of the next rod; the position of the slider, in respect to its own rod, being noted optically with reference to a graduated scale or vernier attached to its own rod. When the chain was used, the position of a mark near each end of the chain was noted optically with reference to marks carried by a plate of metal attached to a fixed post. The same chain was subsequently used for the measure of other bases.

Philosophical Transactions, 1798, page 135, &c. Sir GEORGE SHUCKBURGH describes his Scale as having been laid down by TROUGHTON from a scale of BIRD'S (apparently not from ROY'S scale, which is separately mentioned). When in use, it was fixed by three screws to a block of mahogany. The comparing instrument was not a beam-compass, but a bar carrying adjustible micrometer-microscopes, differing in no important degree from those now employed. This appears to be the first genuine instance of optical comparison; and, from this time, beam-compasses have never been used in England for comparisons. SHUCKBURGH'S Scale was compared with the Royal Society's Tower Standard E, with the Exchequer Yard on the Royal Society's bar, with ROY'S, with two of BIRD'S, with Henry VII. and with Elizabeth's. BIRD'S Standard of 1758, and apparently also that of 1760 (both made for the House of Commons), were transferred to the scale by beam-compasses, and then compared in the same way.

Base du Système Métrique Décimal, tome troisième. About the end of the last cen-

tury and the beginning of the present century (it is difficult to give a better reference, where a work necessarily occupied a long time, and where no dates are affixed to its separate memoirs), the operations were effected for measuring the bases in the French survey, for ascertaining the length of the seconds' pendulum, and for comparing the adopted mètre with English measure. For the bases: two separate toise-measures were prepared, each equivalent to the toise of Peru, being, like the toise of Peru, end-measures; but there is no description of the process of verifying the equivalence. Then the two toise-measures were placed end to end, so as to give the length of two toises from a fixed block; and a slider with a vernier, carried by a second fixed block, was made to touch the extremity nearest to that second block; the same thing was done with the two-toise measuring-bars; the difference of the vernier-readings, ascertained optically, gave the difference between the length of the measuring-bar and two toises. In the actual use of the measuring-bars, two bars were never permitted to touch: each bar carried a slider or languette (as in General Roy's glass rods), and the ultimate reference was to a vernier. An operation, similar to that for the measuring-rods, was used (except that some subsidiary fractions of the toise were also employed, the sum of their lengths being ascertained by mechanical contact) for the end-bar which was necessary, in Borda's method, for measuring the double-seconds' pendulum; the ultimate determination always resting on optical observation of a vernier. For the length of the mètre (an end-measure) in comparison with the English scale belonging to Professor Pictet of Geneva, a subsidiary piece bearing a fine line was provided: first, this piece was made to touch a fixed block, and a microscope was adjusted on the line; secondly, the same piece was made to touch one end of the mètre while the other end of the mètre touched the same block, and another microscope was adjusted on the line; these two microscopes, thus adjusted, were then used to observe the graduations of the scale.

Philosophical Transactions, 1818, page 33, &c. Kater's Account of experiments for determining the length of the Pendulum vibrating seconds in the latitude of London. The measuring apparatus appears to have been generally similar to Sir George Shuckburgh's, but with microscopes of greater magnifying power. The beam of mahogany which carried the microscopes was elevated by a foot-screw at each end, for focal adjustment of the microscopes. There is no clear account of the way in which the microscope-frame was moved from one standard to another, or of the way in which standards were brought successively under the microscopes. The distances 0—39·4 and 0—42 on Shuckburgh's scale were compared with 0—39·4 and 0—42 on General Roy's scale, and a result was obtained differing considerably from Shuckburgh's. Bird's standard of 1758 (made for the House of Commons) was also compared with 0—36 of Shuckburgh's. The measure of the distance between the knife-edges of the pendulum (which is an operation whose peculiarities, though in the opposite direction, are similar to those of the comparison of end-standards with line-standards) was effected in two ways: one by introducing supplementary blocks bearing lines, and using the microscopes to observe those lines, both when the blocks touched the knife-edges and when the blocks

touched each other; the other by microscopic view of the edges, alternately with white ground and black ground beneath them. The thermometric expansion of the pendulum-bar was ascertained. The determination of the vibrations of the pendulum was made in Mr. BROWNE's house, Portland Place, in the atmospheric air; the correction applied, to obtain the number of vibrations which would have been made at the level of the sea, was computed by increasing the number of vibrations in the proportion of the terrestrial radius at the sea to the terrestrial radius at the station; and the correction, to obtain the number of vibrations which would have been made *in vacuo*, was computed by assuming that the only effect of the air was to diminish the attraction of the pendulum in the proportion of the specific gravity of the air to that of the pendulum.

Philosophical Transactions, 1818, page 103, &c. KATER on the length of the French Mètre, as compared with SHUCKBURGH's scale, 0—39·4. The optical comparison with the Mètre à traits is simple; that of the Mètre à bouts was effected in the same manner as the measure of the pendulum, by the use of supplementary blocks bearing lines.

Philosophical Transactions, 1821, page 75, &c. KATER, Comparison of various British Standards. The standards compared are, the Royal Society's standard (Tower Yard), General ROY's, SHUCKBURGH's (0—36 or 0—40), BIRD's standard of 1760, a standard supposed to represent Colonel LAMBTON's, and a 40-inch bar, supposed to represent RAMSDEN's. It appears that the comparing-apparatus was attached to a bar of mahogany, which was to be placed upon the standard to be observed; the standard, apparently, being laid upon a table. In this paper is stated the reason for recommending BIRD's standard of 1760 for the Imperial Standard; namely, because BIRD's standard of 1760 agrees closely with SHUCKBURGH's scale (0—36), with which the pendulum and the mètre had been compared, and of which a facsimile was known to exist at Geneva.

A general view of the relations between the different standards mentioned above will be found in a "Statement" by Mr. SHEEPSHANKS, given in a subsequent part of this paper.

The Bill for giving legal authority to the late Imperial Standards, for providing for their restoration in case of loss or injury, and for settling various other points relating to standards, received the Royal Assent 1824, June 17. It is intituled, "Anno Quinto Georgii IV. Regis, cap. LXXIV., An Act for ascertaining and establishing Uniformity of Weights and Measures." The following Excerpts contain all that relates to the definition and preservation of the Standard of Length:—

Section I. "... Be it enacted ... that from and after the first day of May, One thousand eight hundred and twenty-five, the Straight Line or Distance between the Centres of the Two Points in the Gold Studs in the Straight Brass Rod, now in the Custody of the Clerk of the House of Commons, whereon the Words and Figures 'Standard Yard, 1760,' are engraved, shall be and the same is hereby declared to be the original and genuine Standard of that Measure of Length or lineal Extension called a Yard; and that the same Straight Line or Distance between the Centres of the said Two Points in the said Gold Studs in the said Brass Rod, the Brass being at the temperature of Sixty-two

Degrees by FAHRENHEIT's Thermometer, shall be and is hereby denominated the 'Imperial Standard Yard.'" . . . . . .

Section III. "And whereas it is expedient that the said Standard Yard, if lost, destroyed, defaced, or otherwise injured, should be restored of the same Length, by reference to some invariable natural Standard: And whereas it has been ascertained by the Commissioners appointed by His Majesty to inquire into the subject of Weights and Measures, that the said Yard hereby declared to be the Imperial Standard Yard, when compared with a Pendulum vibrating Seconds of Mean Time in the Latitude of London, in a Vacuum at the Level of the Sea, is in the proportion of Thirty Six Inches to Thirty Nine inches and one thousand three hundred and ninety-three ten-thousandth Parts of an Inch; Be it therefore enacted and declared, That if at any Time hereafter the said Imperial Standard Yard shall be lost, or shall be in any Manner destroyed, defaced, or otherwise injured, it shall and may be restored by making . . . . . a new Standard Yard, bearing the same Proportion to such Pendulum as aforesaid, as the said Imperial Standard Yard bears to such Pendulum."

SECTION II.—*Comparisons of Standards between the passing of the Act of* 1824 *and the appointment of a Commission for consideration of Standards after the destruction of the Imperial Standard in* 1834; *with remarks suggested by the advance of collateral theory and experiment in that interval.*

In the Philosophical Transactions, 1826, Part II., page 1, &c., is KATER's account of the preparation of the new [Secondary] Standards of Weights and Measures. The only part of this which is important to the present purpose is the following. Four brass bars were prepared, each $1\frac{1}{4}$ inch wide and $\frac{1}{2}$ inch thick, with gold pins bearing points (I refer to the original for the excentric mounting of these points). In the course of the comparisons of these secondary standards with Sir G. SHUCKBURGH's scale, it was found that discordances to the amount of $\frac{1}{6\,0\,0}$ inch were produced by almost undiscoverable inequalities in the surface of the table on which they were laid. In the new secondary standards, this defect was cured by removing the upper half of the thickness of the bar, in those parts where the defining points are planted, so that the defining points are in reality at the middle of the depth of the bar. But it was impossible to apply this remedy to SHUCKBURGH's scale; and consequently all measures made with that scale are liable to some uncertainty.

Philosophical Transactions, 1830, page 359, &c. Captain KATER pursues the subject of curvature of bars produced by the inequalities of the supporting surface. He shows that in one instance (DOLLOND's scale), in which great care had been used, an error of $\frac{1}{6\,0\,0}$ inch had nevertheless been committed. He describes a practical method of testing the planeness of surfaces. He shows that, when the greatest precautions are used, the length of SHUCKBURGH's scale 0—36 appears sensibly equal to the Imperial Standard. Remarking that, when the supporting surface is nearly plane, a bar of any moderate thickness will accommodate its form to the small inequalities, and the errors of divisions

on the upper surface will be greater for a thick bar than a thin one, he proposes as the best form for a scale a very thin bar carried by a stout one.

Philosophical Transactions, 1831, page 345, &c. Captain KATER gives an account of a Copy of the Imperial Yard made for the Royal Society. The principle of construction is that just described, the thickness of the thin bar being $\frac{7}{100}$ inch. The Copy (0—36) is compared microscopically with the Imperial Standard itself. The Copy and the Imperial Standard were laid upon a marble slab, carefully made plane. The apparatus used is the same as that described in the Philosophical Transactions, 1821.

This is the last of KATER'S papers on the subject of standard-comparisons. The same subjects however had been taken up by others both in England and on the Continent; and results of considerable importance, both as to the methods employed in the practical applications of standards, and as to the evaluations applicable to the perpetuation of the Standard, had been obtained. The following account of these will probably suffice.

In various parts of Europe, bases have been measured by the use of end-measures. But in every instance, the principle has been adopted of not permitting the successive measuring-rods to come in contact, but of measuring the interval between their ends. In some instances, this has been done by inserting a glass wedge; in others, by use of a *fühl-hebel* or multiplying-lever: in all cases, reference is ultimately made in some degree to optical observation.

In the extension of the British survey, bases have been measured with a new apparatus: of which the following peculiarities deserve notice. First, the Ordnance Survey Office have used, as standards of reference for field-measures, two 10-feet iron bars; each bar carrying points for microscopic observation on surfaces sunk to the level of the middle of the bar's depth; and supported, not on a table or on immoveable supports, but on two rollers placed each at the distance of one-fourth of the bar's length from one end; the bar being so fixed at the middle as to prevent it from running endways. Secondly, that the measuring-bars (which were so constructed as to be sensibly invariable in length under different temperatures) carried points for microscopic observation; and the space between the point on the following end of one bar and that on the preceding end of the next bar was measured by means of a pair of microscopes firmly connected, but so arranged that the distance between their focal points is sensibly independent of changes of temperature. In this apparatus therefore the measure is purely optical without any reference whatever to contact.

Subsequently to the introduction of this apparatus, new standard-bars and scales were formed (of considerable importance in the present inquiry) and were extensively compared. First, two iron 3-feet bars marked 1A and 2A, bearing points at the middle of the depth of the bars, and usually supported on semicircular blocks of brass, each placed at one-fourth of the bar's length from one end, were prepared for the Department of the Ordnance Survey. These bars were compared immediately with the Imperial Standard in the year 1834: the comparisons were not published till the printing (in 1847) of the Account of the Measurement of the Lough Foyle Base (see pages 82 and [28] of that

work), but were known (by the friendly communication of the Director and Officers of the Survey) to the persons engaged on the National Standards. Secondly, a tubular scale 5 feet long, was prepared in the year 1833 for the Royal Astronomical Society, and was compared with numerous standards in the year 1834. I must refer to Mr. BAILY'S account of the comparisons in the Memoirs of the Royal Astronomical Society, vol. ix.: but I may extract the following points. After experimental proof of the injurious effects of resting the tube upon two fixed supports, it was placed upon two rollers, at the quarter-length distance from each end. A frame carrying two microscopes was prepared, for observing the divisions of the tubular scale and those of other bars or scales which might be brought for comparison. A view of this is given in page 75 of the account to which I refer: its structure was very weak. The scale in question was compared with several similar tubular scales, and with the Imperial Standard. Moreover, lines for observation were cut at the distance 60 inches on four sides of the tube, and their intervals were compared. It is impossible by extracts to do justice to this paper; I can only remark that any person who wishes to become acquainted with the state of the Imperial Standard and the impossibility of observing it accurately, with the errors of comparisons before 1800 and the inaccuracies after that time, must study this memoir. At the same time I remark, that, as will appear in the course of the present communication, on some important points there was still room for additional exactness.

The length of the seconds' pendulum was measured by BESSEL in 1835 (I shall allude shortly to an important correction previously made by him in the theory of the pendulum). The method which he used was that of observing the vibrations of two pendulums whose difference of length is exactly equal to the standard (one toise). This is, so far as I am aware, the only instance in which an end-standard has been used for a physical measure without the essential aid of optical apparatus.

In the same year, BESSEL constructed the new Prussian Standard of three Prussian feet. This is an end-standard; the bar being of steel, and the ends being of sapphire. A line-standard had been made in 1816, but BESSEL'S confidence in line-measures appears to have been shaken, partly by KATER'S discovery of the effects of trifling flexure, mentioned above (though he acknowledges that the means of obviating the injurious effect is extremely simple), and partly by BAILY'S remark on the "personal equation" or difference in the way in which different persons observed the large and distorted point on GRAHAM'S 1760 bar. Moreover, the law of Prussia defined the Prussian Foot in terms of the French Toise, which is represented by an end-standard. This circumstance, in combination with the application of the toise to the indirect measure of the pendulum mentioned above, appears sufficient justification of the introduction of an end-standard.

These are the principal points in the history of standards bearing upon the accuracy of comparison and application of standards. But a discovery had been made in a related subject, which bears importantly upon the legal provision for the restoration of the British Standard in case of its loss. BESSEL discovered that the correction which had been usually applied, and which was applied by KATER, for reducing the vibrations of a

pendulum as observed in atmospheric air to the vibrations in vacuum, ought to be greatly increased. SABINE immediately repeated these experiments, with special reference to the form of pendulums which had been usually employed in England; and after him, BAILY extended the experiments to pendulums of various forms*. The result for British pendulums is, that KATER'S correction ought to have been multiplied by 1·655 nearly. The corresponding error in the length of the yard-standard is a quantity very conspicuous to the naked eye.

There is also great doubt on the propriety of the reduction to the level of the sea.

It may not be out of place to anticipate here an investigation begun by myself in 1843 and completed in 1844 or 1845 (sufficiently early to be used in the observations hereafter to be mentioned), and printed in the Royal Astronomical Society's Memoirs, vol. xv. It relates to the value of the intervals (supposed equal) which ought to exist between different supports of a bar, each support exerting the same vertical pressure upwards, in order that the interval between two points upon the surface of the bar may not be altered by the flexure. The resulting interval of supports $= \frac{\text{length of bar}}{\sqrt{(n^2-1)}}$, where n is the number of supports. It is easily seen that the arrangement of levers by which equal pressures upwards may be exerted at four or eight points, is very simple.

I believe that the matters which I have cited will give a correct idea of the advances made in the science of standards since the Act of 1824. It had been shown that it was imprudent in any case to trust to points or lines traced on the surface of a bar, and therefore for delicate purposes, supposing the standard to be a line-measure, only two lines or points ought to be used, sunk to the middle of the bar's thickness. It had been shown that it was imprudent to lay the bar upon a table or upon fixed supports of any kind, and therefore the bar must be stiff enough to bear to be supported upon a few points at which rollers could be conveniently applied. It had been shown that the physical reference provided in the Act of Parliament of 1824 was erroneous in one particular and doubtful in another; and, as it seemed likely that similar uncertainties might be found in any other physical reference, the conviction was gradually rising that it would be better to trust, for restoration, to attested copies of the Standard. The question of the propriety of adopting line-measure or end-measure for the National Standard, which in this country had been practically decided (without a single opposing instance as regarded accurate standards) in favour of line-measure, had again been raised by BESSEL'S adoption of end-measure.

On 1834, October 16, occurred the fire at the Houses of Parliament, in which the Standards were destroyed. The bar of 1760 was recovered, but one of its gold pins bearing a point was melted out, and the bar was otherwise injured.

On 1838, May 11, the Right Honourable T. SPRING RICE (now Lord MONTEAGLE), Chancellor of the Exchequer, requested a Committee to consider and report upon the course which under all circumstances it would be best to follow.

* See the Philosophical Transactions, 1829 and 1832.

SECTION III.—*Appointment of the Treasury Commission of* 1838*: its proceedings and Report. Appointment of the Treasury Commission of* 1843*: its proceedings to the death of Mr.* BAILY *in* 1844.

The letter of the Chancellor of the Exchequer constituting the Standard Commission was in the following terms:—

"Downing Street, May 11, 1838.

"SIR,—The attention of Her Majesty's Government has lately been directed to the necessity of determining a standard weight and measure to replace those which were destroyed by the burning of the Houses of Parliament.

"Before any steps are taken in this important matter, I am desirous of availing myself of the opinion of yourself and other competent persons; and I therefore request that you will have the goodness to communicate with the gentlemen named in the margin (Mr. F. BAILY, Mr. J. D. BETHUNE, Mr. DAVIES GILBERT, Mr. J. S. LEFEVRE, Mr. J. W. LUBBOCK, Rev. G. PEACOCK, Rev. R. SHEEPSHANKS), to each of whom a copy of this letter will be sent, and report to me the course which you shall agree in thinking best to be pursued under the peculiar circumstances of the case.

"I have, &c.,

"*To the Astronomer Royal.*" "T. SPRING RICE."

The objects of this Commission were evidently intended to be of preliminary character, and therefore indefinite and wide. It was proper, for instance, to consider, not only whether the Parliamentary enactment for the restoration of the Standard of Length ought to be repealed, but also whether a new fundamental measure or a new scale of measures, implying a National Standard of different length from the old one, should be recommended. The Commission collected much evidence, and digested it under the heads

A. Basis (arbitrary or natural) of the system of Standards.
B. Construction of Primary Standards.
C. Means of restoring the Standards.
D. Expediency of preserving one measure, &c. unaltered.
E. Change of Scale of Weights and Measures.
F. Alteration of the Land-Chain and the Mile.
G. Abolition of Troy Weight.
H. Introduction of Decimal Scale.
I. Assimilation to the Scales of other Countries.
K. Construction of Secondary Standards.
L. Construction of Local Standards.
M, N, O relate to legal and practical details.

On 1841, December 21, the Commission presented their Report, of which the following are the parts which relate to the Standard of Length.

Chapter I. describes the ruined state of the Imperial Standards.

Chapter II. ..... With respect to the definition, we are of opinion that the definition

contained in the Act 5 Geo. IV. cap. 74, sect. 1, ... by which the standard yard ... is declared to be a certain brass rod therein specified, is the best which it is possible to adopt. With respect to the provisions for restoration, we are not prepared to recommend the adoption of that prescribed by section 3 of the same Act, whereby it is directed that, in case of the loss of the metallic standard above-mentioned, the yard shall be restored by taking the length which shall bear a certain proportion to the length of the pendulum, vibrating seconds of mean time, in the latitude of London, in a vacuum, at the level of the sea. .... Since the passing of the said Act, it has been ascertained that several elements of reduction of the pendulum-experiments therein referred to are doubtful or erroneous: thus it was shown by Dr. Young, Philosophical Transactions, 1819, that the reduction to the level of the sea was doubtful; by Bessel, Astron. Nachr., No. 128, and by Baily, Philosophical Transactions, 1832*, that the reduction for the weight of air was erroneous; by Baily, Philosophical Transactions, 1832, that the specific gravity of the pendulum was erroneously estimated, and that the faults of the agate planes introduced some degree of doubt; by Kater, Philosophical Transactions, 1830, and by Baily, Astronomical Society Memoirs, vol. ix., that very sensible errors were introduced in the operation of comparing the length of the pendulum with Shuckburgh's scale, used as the representative of the legal standard. It is evident, therefore, that the course prescribed by the Act would not necessarily reproduce the length of the original yard. ..... Several measures, however, exist, which were most accurately compared with the former standard yard (in particular the Royal Astronomical Society's Scale, described in their Memoirs, vol. ix., and the iron bars belonging to the Board of Ordnance, in the custody of Colonel Colby,) .... and by the use of these the values of the original standards can be respectively restored without sensible error. And we are fully persuaded that, with reasonable precautions, it will always be possible to provide for the accurate restoration of standards, by means of material copies which have been carefully compared with them, more securely than by reference to any experiments referring to natural constants. With these views we recommend the following course:—

1. That so much of the Act 5th Geo. IV. cap. 74, as provides for the restoration of the standards in the manner therein mentioned, be repealed; and that the standard of length be defined, in subsequent enactments of the legislature, either by the whole length of a certain piece of metal or other durable substance, supported in a certain manner, at a certain temperature (to the construction and dimensions of which, further allusion will be made in this Report); or by the distance between two points or lines engraved upon the surface of a certain piece of metal or other durable substance, supported in a certain manner, at a certain temperature; but that the standard be in no way defined by reference to any natural basis, such as the length of a degree of meridian on the earth's surface in an assigned latitude, or the length of the pendulum vibrating seconds in a specified place.

* This reference appears to have been transcribed by mistake from that which follows. It ought to have been Sabine, Philosophical Transactions, 1829.

5. That four sets of copies (to be denominated the Parliamentary Copies) of the standards of length and weight be made, in all respects similar to the legal standards of length and weight (which are hereinafter denominated the Parliamentary Standards), and as nearly as possible equivalent to them; their difference from the Parliamentary Standards, if appreciable, being carefully ascertained; and that the several Parliamentary Copies be distinguished by proper marks.

6. That one set of the Parliamentary Copies be enclosed in a case hermetically sealed, and imbedded within the masonry of some public building; and that the place in which it is enclosed be pointed out by a conspicuous inscription on the outside; and that this set of Copies be not disturbed, without the sanction of an Act of Parliament.

17. That the superintendence of the construction of the new Parliamentary Standards be entrusted to a committee of scientific men, to be named by Her Majesty, subject to the general instructions which follow.

18. That the Superintending Committee be instructed to combine, in the way which may appear most advantageous, the evidence afforded by the Royal Astronomical Society's scale, by the Royal Society's scale No. 46, and by the 3-feet bars of the Ordnance Survey; and by the use of these to construct a new standard of length, representing as nearly as possible the same measure as the lost standard.

19. That the whole length of the bar, or the distance between two points or lines marked upon it, be adopted as the Parliamentary Standard of length; and that the bar bear no other divisions.

20. That the question of adopting the whole length or the distance between two points or lines, as well as the choice of material, and the general construction of the standard-bar, be left to the decision of the Superintending Committee.

21. That the length of the new Parliamentary Standard be one yard; there appearing to us no sufficient reason for departing from the length hitherto adopted for the standard.

22. That the thermometrical expansion of the standard-bar be determined by actual experiment on it.

23. That the temperature at which the length defined by the standard-bar represents the yard be fixed by the Superintending Committee.

The Report contains other recommendations, relating to periodical comparisons of the Standards and Copies, collection of ancient standards, &c., but it does not appear necessary to allude further to these in the present memoir.

This Report was presented to the Chancellor of the Exchequer, and after a time the following authority for further proceedings was received:—

"Treasury Minute, dated 20th June, 1843.

"My Lords read the Report of the Commissioners for the restoration of the Standards of Weight and Measure, presented to both Houses of Parliament in 1841, and advert more particularly to the recommendation of the Commission, that a committee of scientific men should be appointed to superintend the construction of the new Parliamentary Standards.

"It appears to my Lords to be very desirable that the gentlemen who acted as Commissioners for the Restoration of the Standards should afford their assistance, if they can be prevailed upon to do so, in directing the executive measures necessary for carrying out the recommendations contained in the report now before my Lords.

"Their Lordships therefore deem it expedient to propose to the undermentioned noblemen and gentlemen to undertake the task; viz.—

The Astronomer Royal,
The Marquis of Northampton,
The Lord Wrottesley,
F. Baily, Esq.,
J. E. D. Bethune, Esq.
J. G. S. Lefevre, Esq.,
Sir J. W. Lubbock, Bart.,
The Rev. G. Peacock, Dean of Ely,
The Rev. R. Sheepshanks,
Sir J. F. W. Herschel, Bart., and
Professor Miller."

(The name of the Earl of Rosse was subsequently added.)

Before this time, the Lords of the Treasury had ascertained that it would be agreeable to the members of the Committee named, that the work of Standards of Length should be entrusted to Mr. Baily, and that of Standards of Weight to Professor Miller, and had engaged the services of those gentlemen; Mr. Baily declining to accept remuneration.

The first meeting of the Committee was held on 1843, July 11, when it was reported that the following measures were collected:—

The scale of the Royal Society, No. 46.
The scale of the Royal Astronomical Society.
The two 3-foot bars of the Ordnance Survey.

It was resolved (provisionally), That the Parliamentary Standard of Length be a measure *à traits*; that the material be bell-metal (including any variations in the proportion of tin, &c.) or steel, as Mr. Baily should think best; that the Standard be about an inch square; that its ends be notched away to half its thickness, and that the marks defining the standard length be made upon pins (if necessary) inserted in the surface parallel to the length of the bar thus exposed; that the marks be adjusted to represent the length of one yard at the temperature of 61° or 62°, as nearly as possible; and that the bar be supported by a series of supports distributed over many points through its whole length, so as to destroy the effect of gravity in bending the bar as completely as possible.

It was, I believe, at this meeting, that the formation of the planes for receiving the defining-marks, by sinking wells to half the thickness of the bar instead of cutting the bar in the form of a notch, was sanctioned, but not by formal resolution.

Mr. BAILY proceeded immediately to make experiments on different materials, as applicable to the formation of Standard Bars. Mr. BAILY, I believe, entertained with me the opinion that, if a bell-metal or gun-metal mixture could be found possessing the properties most essential to the present use of a standard bar, it would be greatly preferable to iron or steel for the construction of a standard intended to last through many ages, by reason of its almost perfect immunity from rust. Therefore, although a bar of steel was prepared at the same time as bars of different compositions of gun-metal, and though it is expressly recorded that experiments were made on it, they do not appear to have been made with the same detail as those on gun-metal bars; at least I find no such details in Mr. BAILY'S Manuscript Book or in his correspondence with me. The entries most worthy of note are the following:—

"1843, August 18.—Mr. SIMMS having procured the five* bars before mentioned [all 40 inches long and 1 inch square], and also a brass tube, I this day attended at his house to make experiments on the curvature of the same, by attaching various weights to the centre, when the bar was supported at the ends. The steel bar was the first tried.

"August 21.—Attended again at Mr. SIMMS'S, and experimented on the brass bar No. 1.

"August 23.—Experimented on the gun-metal bar No. 2.

"August 24.—Experimented on the gun-metal bar No. 3.

"August 25.—Experimented on the gun-metal bar No. 4.

"August 26.—Experimented on the brass tube, similar to Ast. Soc. Standard.

[It appears, from a letter of Mr. BAILY to myself, dated 1843, August 30, that the greatest weight was 21 lbs.]

"August 31.—Let fall all the brass bars from an elevation of 5 feet on a stone pavement. The gun-metal No. 4 was the least affected, but none of them broke.

"September 9.—Loaded No. 4 at the centre with ½ cwt. weights till the total weight was 5½ cwt., when it snapped with a loud report, like a loud shrill bell violently struck. It was very elastic whilst the weights were being applied. [The bar was evidently showing signs of elasticity during the operation, and sprung up and down like a piece of whalebone; and I really believe that the bar would have resumed its straightness, if the weights had been removed immediately before the last was put into the scale. The pieces, on breaking, sprung up, by reason of their elastic force. The fracture was precisely in the middle, which indicates the soundness and uniformity of the structure, and the granulation is very good. As the weight of the scale, ropes, and iron hook, must have exceeded ½ cwt., the total weight applied was above 670 lbs.—*Letter to myself, dated* 1843, *September* 11.]"

* Messrs. MAUDSLAYS and FIELD, by whom the brass and gun-metal bars were cast, have favoured me with the particulars of their composition. They are as follows:—

No. 1. Copper 32, zinc 12.
No. 2. Copper 32, tin 1, zinc 3.
No. 3. Copper 32, tin 3, zinc 2.
No. 4. Copper 32, tin 5, zinc 2.

The following is extracted from the Minutes of the Meeting of the Committee, 1843, November 10:—"Mr. BAILY reported that he had made trials of various materials for the Standards of Length; namely, steel, brass, and four kinds of gun-metal. One of the gun-metals, composed of 28 lbs. of copper to 4 lbs. 6 oz. of tin [I believe that which is called No. 4 above, though the statement of the component parts is not quite correct], was very brittle, and the bar had been broken in his experiments. The next [No. 3 I believe], composed of 28 lbs. of copper to 2 lbs. 10 oz. of tin, was very elastic, returning perfectly to its original state after having been sensibly bent by heavy weights; and on being chipped, was found by Mr. BAILY and Mr. DONKIN to be much less brittle than the former. Mr. BAILY intimated that he was disposed to recommend the use of this metal."

The following is extracted from Mr. BAILY's MS. book:—"November 13.—Attended at Mr. SIMMS's, and submitted the gun-metal bar No. 3 to the same process as on September 9. After putting 5½ cwt. in the scale, the bar was so much bent that it slipped off the support; and on measuring the curvature it was 4·1 inches. I then submitted a copper bar of the same dimensions to a similar process. After putting 4 cwt. into the scale, the bar was bent 3·8 inches."

The experiment is farther explained by the following extract of a letter to myself, dated 1843, November 15:—"I have tried the effect of applying a great weight to the bar which Mr. DONKIN and myself seemed to prefer to that which had been previously tried. The result however has completely altered my opinion. I found that 5½ cwt. bent the bar so much that it slipped off the support. The curvature (or versed sine) was 4·1 inches, which was permanent, and did not recover itself. It certainly was a tougher metal, which was what Mr. DONKIN, and I believe all artists or mechanics, delight in. But this is not the quality that suits us; and I therefore prefer the more brittle metal, which was elastic as long as it would hold together, and would rather break than bend."

Mr. BAILY adds: "Instead of having the ends of the bar cut off as a notch in KATER's mode, I propose to have a cylindrical hole drilled, ⅜ths of an inch in diameter, half-way through the bar. I find that I have as much light as on the open surface, and my objection to weakening the bar is thereby removed. Mr. SIMMS says he can cut as fine a line at the bottom of this well, as he can on a plane surface. He is trying his hand at this experiment, and if it succeeds, I trust there will not be any objection to the method. It will be much safer as to strength, and the lines will be well protected."

Mr. BAILY's illness and death prevented him from laying these results before the Committee, but I communicated them at the meeting of 1844, November 30. The gun-metal No. 4 has been adopted in all the standard bars subsequently made (except where a difference is specially mentioned), and it appears to be an admirable alloy for all similar purposes. Its exact composition is—

Copper	16
Tin	2¼
Zinc	1

as stated by Mr. SIMMS, who superintended the construction, and by Messrs. MAUDSLAYS, who performed the actual casting. The construction of bars with well-holes for the defining marks has also been adopted in all the subsequent instances; but the bars have been reduced to the length of 37 inches by Mr. BAILY, and 38 inches by Mr. SHEEPSHANKS.

After the first experiments applying to the fitness of metals, Mr. BAILY proceeded with experiments in verification of some made by Captain KATER on the influence of small inequalities of the surface supporting a bar. The experiments were made at his own house, No. 37, Tavistock Place. The bar of metal No. 3 was notched to half its depth at each end, and at each end there was cut a line for microscopic observation on the upper surface (the interval of the two lines being 38 inches), and a line for microscopic observation on the surface at mid-depth (the interval of the two being 39 inches). The bar was laid upon a very even bar (whose section is T) in the comparing-apparatus of the Royal Astronomical Society, and the microscopes were adjusted sometimes to one pair of lines, sometimes to the other. At the commencement of this and of other series of experiments, it is noted that the microscopes were made parallel. The value of one division of the micrometer $=0{\cdot}00005$ inch. A wire, or pair of wires, described as "No. 2 brass wire in my Cavendish experiment," was placed in different positions under the bar. [Only one brass wire is mentioned in the 'Account of the Cavendish Experiment,' page 29; its thickness was 0·0134 inch.] Sometimes the bar was placed on rollers. The following are variations in the measure of the intervals of the lines, under different circumstances. Each number is the mean of several experiments.

			div.
1843, Sept. 14.	Lines on upper surface	Bar lying flat	0·00
		Wires under the ends	−14·35
		Wires under the middle	+ 6·05
	Lines on surface at mid-depth	Bar lying flat	0·00
		Wires under the ends	+ 2·50
		Wires under the middle	− 2·50
Sept. 15.	Lines on upper surface	Rollers 8 inches from centre	+24·68
		Rollers 10 inches from centre	+21·92
		Rollers 12 inches from centre	+17·94
		Rollers 6 inches from centre	+25·12
		Rollers 10 inches from centre	+19·76
		Rollers 14 inches from centre	+12·80
Sept. 16.	Lines on upper surface	Bar lying flat	0·00
		Wires 8 inches from centre	+ 6·65
		Wires 10 inches from centre	+ 5·30
		Wires 12 inches from centre	+ 3·17
		Wires 14 inches from centre	− 0·17
		Wires 16 inches from centre	− 5·45

The following experiment is on the middle 3 feet of Mr. BAILY's tubular 5-foot scale, No. 5. The lines on the upper surface. [It would seem that it took a faulty bearing when laid flat on the bar.] Single readings.

		div.
1843, Sept. 17.	Bar lying flat	+37·8
	Pins at 13½ inches and 46½ inches...	+28·0
	Rollers at 14½ inches and 45½ inches	+32·0
	Pins at 15½ inches and 44½ inches...	+32·5
	Pins at 16½ inches and 43½ inches...	+36·0
	Pins at 17½ inches and 42½ inches...	+35·5
	Pins at 18½ inches and 41½ inches...	+38·5
	Bar lying flat	+38·0

The bar of gun-metal No. 3 was then reduced to the length 37 inches, and lines were cut (evidently on its upper surface) at the distance 36 inches, and five experiments were made on 1843, Sept. 28 and 29. The following is the mean of results:—

	div
Bar lying flat	+ 2·41
Wires under the centre	+ 5·85
Wires 2 inches from the centre	+ 9·51
Wires 4 inches from the centre	+ 9·19
Wires 6 inches from the centre	+ 7·39
Wires 8 inches from the centre	+ 4·49
Wires 10 inches from the centre	+ 1·19
Wires 12 inches from the centre	− 0·75
Wires 14 inches from the centre	− 4·11
Wires 16 inches from the centre	− 8·69
Wires 18 inches from the centre	−11·61

The same bar was then supported on rollers at different intervals, and eight experiments were made on 1843, October 10 and 11; of which the following are the means:—

	div.
Rollers 0·75 inch on each side of centre	0·00
Rollers 4 inches on each side of centre	+ 0·06
Rollers 6 inches on each side of centre	− 2·71
Rollers 8 inches on each side of centre	− 4·99
Rollers 10 inches on each side of centre	− 7·45
Rollers 12 inches on each side of centre	− 9·81
Rollers 14 inches on each side of centre	−12·93
Rollers 16 inches on each side of centre	−16·03
Rollers 18 inches on each side of centre	−21·34

On 1843, October 16, I supplied Mr. Baily with the formulæ defining the intervals at which rollers ought to be placed beneath a bar in order that the flexure of the bar might not influence the measure of the distance between two lines upon the upper surface (the rollers being supposed to exert equal pressures upwards). With two rollers, the interval ought to be $\frac{\text{length of bar}}{\sqrt{3}}$; with four rollers, each interval $= \frac{\text{length of bar}}{\sqrt{15}}$; with eight rollers, each interval $= \frac{\text{length of bar}}{\sqrt{63}}$*. Lever-frames were afterwards prepared, each with eight rollers.

It has been remarked above, that Captain Kater had constructed for the Royal Society a standard on the principle proposed by himself, of carrying a slender scale in a groove cut in a stout bar; as it appeared probable that the effect of a given degree of

* See Mem. Astron. Soc. vol. xv.

flexure in altering a measure cut upon the upper surface would be much less for a thin scale than for a thick one. In order to test this, Mr. BAILY caused two slight marks to be cut upon the stout bar of the Royal Society's scale, and made five experiments on 1843, December 24, of which the following numbers are the means. The wires used were common large pins.

		div.
Measures of intervals upon the slender scale	Bar laid flat	0·0
	Wire under the middle	+ 0·1
	Wires under the ends	− 6·0
Measures of intervals upon the surface of the stout bar	Bar laid flat	0·0
	Wire under the middle	+ 5·2
	Wires under the ends	−42·0

In order to collect together the observations which may be considered as applying to the reduction of comparisons of standards, before entering upon the comparisons themselves, I shall interrupt the chronological order (as I have already done) to give the results of experiments on thermometric expansion made by Mr. SIMMS, at 138 Fleet Street, under the general instruction of Mr. BAILY, with some suggestions by myself.

A tin trough was prepared, 44 inches long, 5 inches wide, and 5 inches deep, supported on four screws. Glass windows were provided in proper places for admitting lateral illumination. A lever-frame with eight rollers (on the principles to which I have already adverted) was placed in the trough, and the bar which was the subject of experiment was placed upon the rollers. The trough was filled with water, of any required temperature, which could be heated by spirit-lamps below the trough. The microscopes were fixed to a stone ledge built into the wall of the room. The object-glasses of the microscopes (which were common object-glasses of unusually short focal length) were immersed in the water. At all ordinary temperatures the vision was quite distinct, even with the trough uncovered: when the temperature was high, the vision was indistinct, except the trough was covered, but was very good when it was covered. The thermometers were placed in the water, on the surface of the bar. I imagine that these experiments may challenge comparison with any that ever were made, in every respect except one, viz. that the thermometers employed, though probably as good as any then in existence, did not possess the authority which was acquired by thermometers subsequently made by independent graduation.

A screw-apparatus was provided, acting in opposition to a spring, by which the line at one end of the bar could be brought under the microscope at that end; and the micrometer-movement of that microscope was never used. The value of one division of the micrometer which was employed was ascertained to be 0·000025287 inch. The bars which were the subject of examination were evidently not the same as those already mentioned; for the gun-metal bars are all described at this time (1844, February) as being 40 inches long; whereas, of the formerly-mentioned bars, No. 4 had been broken and No. 3 had been shortened before the end of 1843.

On my application to the Lords Commissioners of the Treasury, the ruined Imperial Standard was in March 1844 placed in the hands of Mr. Baily; and the experiments on its expansion are here detailed after the others.

It will be remarked that some comparisons are made with the temperature rising and some with the temperature falling. There appears to be no sensible difference in the results.

Description and length of bar.	Thermometer-reading.		Change of length.	Modulus for 1° Fahr.	Modulus deduced from the compound expansion.
	First.	Second.	div.		
Gun-metal (?) No. 1, 40 inches.	46·90	98·25	780·00	·0000095970	
	47·20	94·37	716·00	95959	
	33·70	94·65	933·50	96823	
	94·65	67·05	437·50	100209	·0000094859
	67·05	46·50	285·00	87674	
Gun-metal No. 2, 40 inches.	46·90	97·15	771·50	·0000097059	
	97·15	42·25	881·00	101447	
	41·55	95·55	799·70	93621	
Gun-metal No. 3, 40 inches.	38·35	98·30	935·30	·0000098627	
	96·95	45·35	817·60	100168	
	32·00	97·85	1005·00	96482	
Gun-metal No. 4, 40 inches.	33·65	99·05	994·03	·0000096086	
	99·05	42·95	818·23	92204	
Brass tube, 40 inches.	32·70	98·52	1047·75	·0000100632	·0000102345
	98·52	170·25	1179·10	103932	
	170·25	32·65	2253·85	103548	
	35·15	98·67	1020·50	101564	102433
	98·67	171·50	1188·83	103206	
	171·50	118·10	858·83	102170	103745
	118·10	40·95	1283·60	105174	
	34·35	106·35	1181·50	103738	105092
	106·35	183·75	1302·10	106351	
	183·75	99·02	1442·95	107661	105554
	99·02	37·20	1004·00	102699	
	33·65	97·23	1033·67	102778	105892
	97·23	176·17	1353·60	108405	
	176·17	100·45	1295·17	108132	
Brass bar, 40 inches.	40·60	98·91	890·55	·0000096550	
	98·91	45·35	828·75	97818	
	45·00	99·72	858·26	99154	·0000101079
	99·72	177·95	1267·49	102425	
	177·95	98·35	1307·65	103852	101155
	98·35	37·45	940·50	97629	
	35·05	100·17	1004·40	97491	100310
	100·17	176·92	1246·72	102690	
	176·92	88·75	1446·27	103697	100890
	88·75	35·80	805·90	96217	
Steel bar, 40 inches.	37·25	97·13	559·47	·0000059056	
	97·13	43·75	520·07	61592	
	33·25	89·22	516·35	58321	·0000062284
	89·22	176·10	891·05	64836	
	176·10	100·58	787·24	65900	62935
	100·58	34·67	620·73	59537	

TABLE (continued).

Description and length of bar.	Thermometer-reading. First.	Thermometer-reading. Second.	Change of length.	Modulus for 1° FAHR.	Modulus deduced from the compound expansion.
			div.		
Copper bar, 40 inches.	37°25	98°60	865·30	·0000089164	·0000094103
	98·60	177·95	1229·10	97921	
	177·95	100·42	1231·50	100418	95993
	100·42	38·10	892·05	90489	
	40·65	94·05	782·56	92643	94914
	94·05	170·70	1169·96	96493	
	170·70	99·66	1103·22	98174	94750
	99·66	43·27	806·70	90437	
	112·40	183·45	1084·72	96514	
	183·45	99·63	1302·92	98267	95547
	99·63	51·50	691·38	90811	
Old Imperial Standard, 36 inches.	47·35	99·50	752·50	·0000101356	·0000104456
	99·50	176·27	1164·67	106318	
	176·27	99·20	1187·92	108267	104469
	99·20	48·70	710·40	98811	
	50·80	38·80	165·65	96963	
	38·80	99·27	847·75	98474	102843
	99·27	178·45	1196·92	106180	
	178·45	99·87	1205·72	107778	104648
	99·87	50·45	703·70	100018	
	50·45	41·50	130·90	102733	
	52·85	39·90	180·45	97877	
	39·90	97·85	800·70	97053	103205
	97·85	168·70	1091·75	108163	
	168·70	99·72	1057·59	107693	103734
	99·72	54·15	651·06	100552	
	54·15	37·50	228·94	96583	
Glass bar, 36 inches.	53·90	96·60	300·95	·00000495	·0000051525
	96·60	174·60	584·42	526	
	174·60	97·86	583·22	534	51157
	97·86	49·65	326·80	476	
	50·00	38·43	80·92	492	
	38·43	99·35	412·80	476	50591
	99·35	176·02	578·19	530	
	176·02	99·16	583·02	533	51157
	99·16	43·47	382·35	482	
	51·45	99·20	340·33	501	50404
	99·20	174·92	545·67	506	
	174·92	98·70	564·93	522	
	55·50	38·28	116·34	473	
	38·28	98·75	416·67	484	51339
	98·75	168·93	538·23	539	
	168·93	98·67	529·46	529	51938
	98·67	57·00	303·20	511	
	57·00	41·43	110·10	497	

It will be remarked that the modulus of expansion, in every instance, is greater for the high temperatures than for the low ones. As the same thermometers were used throughout, it is possible that this may arise from an error in the graduation of the thermometers; but it is not improbable that it is a real peculiarity in the relation between the expansion of the solid metals and that of quicksilver. Admitting the truth

of the apparent difference, it becomes necessary to define the temperature near which any numerical value of the modulus is supposed to apply. Mr. SHEEPSHANKS, after discussing these experiments, fixed upon the following as representing the linear expansion for 1° of FAHRENHEIT at temperatures differing little from 62° FAHRENHEIT, the length of the bar being 36 inches:—

	inch.
Metal No. 1 (Brass)	0·000343
Gun-metal No. 2	0·000351
Gun-metal No. 3	0·000354
Gun-metal No. 4	0·000339
Brass Tube	0·000367
Brass Bar	0·000350
Copper	0·000326
Steel	0·000212
Glass	0·000178
Brass of the Imperial Standard	0·000355

I now proceed with the Comparisons of Standards, made by Mr. BAILY at his house, 37, Tavistock Place, with the purpose of ascertaining the state of the materials available for the restoration of the length of the lost Imperial Standard.

Besides the SHUCKBURGH Scale (to which, on account of its imperfections and the doubts and discordances in its comparisons, little importance was attached), it was known that five existing standards, of superior construction, had been compared with the Imperial Standard; namely, the Royal Society's 3-foot brass scale (constructed by Captain KATER), the Royal Astronomical Society's brass tubular scale, two iron bars belonging to the Ordnance Department and preserved in the office of the Trigonometrical Survey, and a 3-foot brass scale (constructed by Captain KATER and similar to that of the Royal Society), in the possession of a private person. The loan of the last-mentioned standard was refused to the Commission; but I had obtained the four first-mentioned from their owners and custodiers, and delivered them on 1843, December 5, to Mr. BAILY.

After some preliminary observations, Mr. BAILY made the following comparisons:—

The Royal Society's scale compared with the middle 3 feet of the Royal Astronomical Society's scale. The Royal Astronomical Society's scale supported on its rollers, at quarter-length distance from the centre. The Royal Society's scale lying flat on the T-bar of the frame, and nearest to the observer. There appears to have been no difficulty in the observation, except from the size of the dots on the Royal Society scale, the breadth of each being 77 divisions of the micrometer. The Royal Astronomical Society scale is longer than the Royal Society's scale,—

		inch.
1844, Jan. 2,	from 20 comparisons at temperature 43°·21 FAHR., by	0·000564
3,	from 20 comparisons at temperature 38·20 FAHR., by	438
4,	from 20 comparisons at temperature 40·90 FAHR., by	699
5,	from 20 comparisons at temperature 46·34 FAHR., by	664
6,	from 20 comparisons at temperature 50·20 FAHR., by	637
	Mean by 100 observations . . .	0·000600

The Ordnance Bar 1A compared with the middle 3 feet of the Royal Astronomical Society's scale. It appears that the Bar 1A was put in the place of the Royal Society's scale. The bar is notched at each end to half its depth, and the defining points are upon gold pins in the surfaces thus exposed. In the first and second series, the iron bar was laid on two semicircular brass pieces 18 inches apart (not on rollers), in the same manner as when the bar was compared with the Imperial Standard *. In the third, fourth and fifth, it was placed on rollers. A doubt on the temperature exists, for the following reason. After the comparisons had been made, Colonel COLBY informed Mr. BAILY that the thermometer of 1A gave indications too low by 1°. Mr. BAILY however states that in the first five comparisons the thermometer of 2A was used. Yet the simultaneous readings of thermometers seem to make it probable (but not certain) that the thermometer of 1A was really employed. If so, the mean temperature in the following Table ought to be increased 0°·5. The Ordnance Bar 1A is longer than the Royal Astronomical Society's scale,—

		inch.
1844, Jan. 7,	from 20 comparisons at temperature 47°·32, by	0·001300
8,	from 20 comparisons at temperature 43·82, by	1754
9,	from 20 comparisons at temperature 42·62, by	1999
10,	from 20 comparisons at temperature 41·85, by	2067
11,	from 20 comparisons at temperature 43·10, by	1872

The Ordnance Bar 2A compared with the middle 3 feet of the Royal Astronomical Society's scale. In the first two series, the bar was supported on its semicircular brass pieces; in the others, on rollers. The thermometer of 2A was undoubtedly employed. The Ordnance Bar 2A is longer than the Royal Astronomical Society's scale,—

		inch.
1844, Jan. 12,	from 20 comparisons at temperature 43°·72, by	0·002108
13,	from 20 comparisons at temperature 44·60, by	2035
14,	from 20 comparisons at temperature 43·52, by	2148
15,	from 20 comparisons at temperature 40·72, by	2519
16,	from 20 comparisons at temperature 36·97, by	2958

* Royal Astronomical Society's Memoirs, vol. ix., and Measure of the Base on Lough Foyle.

The Ordnance Bar 1A was now put in the place of the Royal Astronomical Society's scale, and 1A and 2A were both supported on rollers. 2A is longer than 1A,—

		inch.
1844, Jan. 17,	from 20 comparisons at temperature 38°·25, by	0·000423
18,	from 20 comparisons at temperature 41·65, by	367
20,	from 20 comparisons at temperature 42·55, by	311
22,	from 20 comparisons at temperature 44·10, by	359
23,	from 20 comparisons at temperature 43·12, by	313

Three bars (the Royal Astronomical Society's scale, 1A and 2A) were now mounted together; the Royal Astronomical Society's scale being furthest from the observer, and 2A nearest to him. 1A and 2A were supported each on its own semicircular pieces, which were carried by the same supporting bar. The Ordnance Bar 1A is longer than the Royal Astronomical Society's scale,—

		inch.
1844, March 13,	from 10 comparisons at temperature 42°·67, by	0·002203
14,	from 10 comparisons at temperature 42·13, by	2520
15,	from 10 comparisons at temperature 43·82, by	2140

And Ordnance Bar 2A is longer than the Royal Astronomical Society's scale,—

		inch.
1844, March 13,	from 10 comparisons at temperature 42°·67, by	0·001875
14,	from 10 comparisons at temperature 42·13, by	2055
15,	from 10 comparisons at temperature 43·82, by	1751

Here 1A appears longer than 2A; whereas the previous observations had made 2A longer than 1A. The only circumstance with which this discordance could be connected was this: that when (as in this instance) the two iron bars were supported by brass semicircular pieces resting upon the same T-bar, it was necessary to place cards under the brass semicircles of 1A, in order to bring them to the same height.

The Ordnance Bars 1A and 2A were now mounted, each on its own semicircular pieces, and on separate T-bars. The Bar 2A is longer than 1A,—

1844, March 16, from 20 comparisons at temperature 46°·25, by 0·000211 inch.

The arrangement was then altered, merely by placing cards under the semicircular pieces of 1A and depressing its T-bar properly. The Bar 1A is then longer than 2A,—

1844, March 18, from 10 comparisons at temperature 44°·0, by 0·000054 inch.

Then the semicircular pieces (those of 1A having cards beneath them) were placed on the same T-bar. The Bar 2A is longer than 1A,—

1844, March 19, from 10 comparisons at temperature 45°·0, by 0·000318 inch.

I shall depart from chronological order, for the purpose of inserting two series of

observations connected with the same subject, but at a higher temperature in a very steady season.

The Royal Astronomical Society's Standard and Ordnance Bar 1A were mounted as formerly (1A upon its semicircular pieces), and the Royal Astronomical Society's Standard was longer than 1A,—

1844, May 14, from 20 comparisons at temperature 65°·5, by 0·001185 inch.

Then 2A was inserted in place of 1A, and after waiting some time for steady weather, the following comparison was made, giving the Royal Astronomical Society's Standard longer than 2A,—

1844, June 17, from 20 comparisons at temperature 65°·47, by 0·001222 inch.

This is the last comparison made by Mr. Baily.

In the month of April Mr. Baily was occupied with comparisons intended to verify the invariability or variability of the Royal Astronomical Society's Standard. They consisted, partly of comparisons of the middle 3 feet of that scale with the middle 3 feet on other tubular scales, of exactly similar construction, with which it had been compared in the year 1835; and partly of comparisons of different measures on the same tube, which also had been compared in 1835.

The Royal Astronomical Society's scale was found to be shorter than Mr. Baily's tubular scale No. 5,—

		inch.
1844, April 8,	from 20 comparisons at temperature 50°·67, by	0·000120
9,	from 20 comparisons at temperature 52·61, by	161

In 1835, the Royal Astronomical Society's scale was longer than Mr. Baily's by 0·000046 inch.

Mr. Simms's tubular scale No. 4, of construction exactly similar, was substituted for Mr. Baily's. The Royal Astronomical Society's scale was found to be shorter than Mr. Simms's No. 4,—

		inch.
1844, April 14,	from 20 comparisons at temperature 56°·33, by	0·000180
15,	from 20 comparisons at temperature 56·90, by	163

In 1835, the Royal Astronomical Society's scale was longer than Mr. Simms's by 0·000102 inch.

Mr. Baily's scale No. 5 and Mr. Simms's No. 4 were then mounted together, each on its own rollers; and Mr. Baily's was found to be shorter than Mr. Simms's,—

		inch.
1844, April 16,	from 20 comparisons at temperature 57°·11, by	0·000087
17,	from 20 comparisons at temperature 56·76, by	116

In 1835, Mr. Baily's was longer than Mr. Simms's by 0·000042 inch.

Mr. BAILY then made the following comparisons of measures defined by lines cut on different parts of the Royal Astronomical Society's Tube. Near to the two extremities of the 5-foot tube, transversal lines had been cut (known by the letters A, *a*) crossing the continuation of the line of the divided scale; then, supposing the tube to be turned round its axis through 90°, two similar transversal lines B, *b*, were cut; turning the tube 90° further, two similar lines C, *c*, were cut, which therefore were on the side of the tube opposite to A, *a*; turning the tube 90° further, two similar lines D, *d*, were cut. The comparison of the longitudinal intervals between these was the object of the following observations:—

1844, April 21, at temperature 60°, by six comparisons in each case, the excesses of the various measures above A*a* were as follows:—

	inch.
B*b* exceeds A*a* by	−0·000009
C*c* exceeds A*a* by	+0·000126
D*d* exceeds A*a* by	−0·000015

In 1835 the corresponding differences were

	−0·000056
	+0·000056
	−0·000169

This was the last special inquiry on which Mr. BAILY was engaged.

Mr. BAILY died on 1844, August 30.

The state of the problem at this time, as regarded the restoration of the Standard, was nearly as follows. The object might be considered: either to reproduce in a satisfactory form the value of the Scientific Standard (SHUCKBURGH's), which Captain KATER considered as a scientific representative of the National Standard of Length; or to reproduce the Legal Standard. With regard to SHUCKBURGH's scale: the experiments of Mr. BAILY had confirmed the remarks which originated with Captain KATER himself, that minute circumstances, whose effect was totally unsuspected in the earlier comparisons of this scale, produced a disturbance in the apparent length, which might have deranged in different directions the results of its comparisons for physical measures, and whose magnitude was intolerably great; that there was no security whatever that, in retaining documentary or numerical expressions of measure founded on this scale, we were referring to a consistent system; and that in future it would be better to lay aside this scale entirely. With regard to the Legal Standard: the principal difficulty, inherent in the standard itself, arose from the size and irregular form of its defining dots, which indeed made accurate comparison impossible. With regard to the existing scales which had been compared with the Legal Standard: the Tubular Scale of the Royal Astronomical Society (complicated in its mechanical structure) was found to have changed, in itself, and by comparison with similar scales, in a way and to an amount which ruined its credit; the Ordnance Bars (of firm and simple structure) appeared in themselves trust-

worthy, but the injurious effects of their supports, and an uncertainty in the temperature referred to one thermometer, introduced great irregularities in their comparisons and doubts in the reductions of them. The Royal Society's Scale had been carefully constructed in reference to one apprehended cause of error; but it seemed likely that a construction so complicated might introduce other errors of equal magnitude; and it appeared certain that the form already partially introduced (a stiff bar supported at many points by free roller-motion) would be far less liable to error. Finally, there ran through the whole of these comparisons one specific cause of uncertainty: nobody knew the authority for any one of the thermometers used, no one doubted that they might be very sensibly in error, and two thermometers inclosed in the same box and intended to be used with the two Ordnance Bars actually differed an entire degree.

It was plain therefore that, either in the scientific or in the legal sense, the restoration of the Standard was indeterminate. The formation of a New Standard must be an operation *de novo*; the length must be confined within certain limits (wide in the scientific, narrow in the commercial sense), but within these it might have any definite value; and when that definite value was fixed, it must in no way be again referred to the old standards or scales, whether original or intermediary. The principal object now was, to ensure constancy and definiteness to the new Standard and its copies, and means of reducing without sensible error the comparisons which might be made with them. As far as depended on the Standard itself, it was hoped that the construction adopted gave sufficient security. As regarded the means of making comparisons, a far firmer apparatus than had hitherto been used was requisite. As regarded the effects of temperature, it was necessary to create an entirely new system of thermometers, founded upon the natural constants, to be determined by appropriate physical experiments; and to use them in new determinations of thermometric expansion.

Such is the state in which the problem was left to Mr. BAILY'S successor.

SECTION IV.—*Proceedings of the Committee and of Mr.* SHEEPSHANKS *to* 1847, *June; construction of new Thermometers, and erection of Comparing Apparatus; description of the Apparatus, and of Mr.* SHEEPSHANKS' *method of comparing.*

The Committee at their meeting of 1844, November 30, received with pleasure and thankfully accepted Mr. SHEEPSHANKS' offer to continue the work commenced by Mr. BAILY. Intimation of this arrangement being given to the Lords Commissioners of the Treasury, a request to the same effect was addressed by their Lordships to Mr. SHEEPSHANKS and assented to by Mr. SHEEPSHANKS; Mr. SHEEPSHANKS, like Mr. BAILY, declining to accept pecuniary remuneration. I had received from Mr. BAILY'S representatives the four bars on which the restoration of the standard was to be founded, as also the MS. books of comparisons, and I placed them in Mr. SHEEPSHANKS' hands.

At the Meeting of 1845, June 6, Mr. SHEEPSHANKS communicated a Report to the following effect:—

"The object proposed appears to be the restoration of the length of the lost Parlia-

mentary Standard representing the Imperial Standard Yard. The composition of this bar is not known. The defining points were large and irregular dots (upon gold pins, melted out by the fire of the Houses of Parliament), which did not admit of accurate observation. This bar, and other bars compared with it (Sir GEORGE SHUCKBURGH'S, the Royal Society's, and Colonel COLBY'S), were usually laid for comparisons (it is believed) on a flat surface, in which arrangement, as is now known, the metal is in constraint, and the comparisons are liable to considerable error. On the completion of the Royal Astronomical Society's scale, it, as well as other scales of similar construction, were compared with the Parliamentary Standard: though these comparisons were more accurately made than those already cited, yet the mounting of these scales is liable to some objection, and sufficient attention was not given to their thermometric expansions at the temperatures in which they were commonly used. A more serious difficulty is this: when, for the scales of the construction similar to that of the Royal Astronomical Society, the same method of supporting is used as was used in their first comparisons, the difference between the lengths of the different scales is not found to be the same as it was at those first comparisons; the discordance being $\frac{1}{5000}$th of an inch, or perhaps twice the probable error of one micrometer-reading. Moreover, the intervals between divisions engraved on four sides of the Royal Astronomical Society's scale do not preserve the same relation as formerly, the discordance being about $\frac{1}{10,000}$th of an inch. The difference between the lengths of Colonel COLBY'S bars appears to be sensibly the same as it was formerly; but there are other objections to the accuracy of comparisons derived from these bars: they are of different metal from the Standards; their expansions have not been ascertained; their thermometers do not correspond; they were not properly supported. It is (in Mr. SHEEPSHANKS' opinion) an advantage that they are made of a pure metal. References to the determinations of the length of the pendulum may be rejected at once, not only because they are inaccurate, but also because they are in fact expressed in terms of SHUCKBURGH'S scale, and not in terms of the Parliamentary Standard. Mr. SHEEPSHANKS considers therefore that our materials for restoration are generally unsound; and that with our endeavours to give the best evidence in our power as to the past, we must combine, what is now more important, the discussion of methods for avoiding similar difficulties for the future. For this purpose it appears necessary to investigate anew the questions of stability and thermometric expansion of standards, and to devise better methods of comparing standards than have hitherto been used. And in the pursuit of these inquiries Mr. SHEEPSHANKS proposes to reject all considerations but those of accuracy and permanence; to adopt the most complete and convenient mode of experimenting, regardless of time, trouble in preparation, and (to a certain degree) of expense. Mr. SHEEPSHANKS then gave the following sketch of his proposed plan of operations:—

"The place of experiments to be in the lower cellars of Somerset House, in the apartments of the Royal Astronomical Society.

"The supports of the microscopes to be built of large squared stones.

"The bars when under examination to be placed in a drop-box surrounded by a large mass of water.

"For ascertaining actual thermometric expansion, the drop-box of one bar is to be placed in ice, and that of the expanding bar in water of different temperatures, and these are to be compared repeatedly by the microscopes. [This arrangement was not precisely carried out.] A similar method will be used for ascertaining whether after repeatedly heating and cooling one bar, it returns to the same length. [This was not specially done.]

"Bars of iron, copper, and cast steel, have been prepared, in addition to the bars of gun-metal prepared under Mr. BAILY's direction. Mr. SHEEPSHANKS proposes to compare carefully bars of these metals and alloys, and also to determine their minimum time of vibration, and after some time to repeat these observations. The object of this is to ascertain the permanency of the bars. For the vibration observations, apparatus for swinging *in vacuo* is now in preparation. [This was not effected.] All these operations ought to precede the comparisons for restoration of the lost standard.

"Mr. SHEEPSHANKS then remarked that (for reasons already given) it appears impossible to restore the standard with reasonable accuracy, and he suggested therefore for consideration whether it might not be better to adopt the length which has been used in the measurement of Colonel COLBY's base for the Trigonometrical Survey of Ireland. This appears permissible, because there is sufficient latitude in the vagueness of comparison of the old standard to allow it; and it appears worthy of recommendation, because (as Mr. SHEEPSHANKS understands from Colonel COLBY) the length adopted in the Irish base appears to correspond very exactly with the length adopted in General ROY's base on Hounslow Heath. Logically there is no inconsistency in this course, because the divisions of the 10-feet iron bars used by Colonel COLBY were laid down from TROUGHTON's scale, from which also the divisions of Sir GEORGE SHUCKBURGH's scale (judged by Captain KATER to be equivalent to the old Parliamentary Standard) were laid down. Mr. SHEEPSHANKS proposed therefore to make a 3-foot iron bar, to compare it with the scales which were directly compared with the old Parliamentary Bar (for general information), and to multiply it by $\frac{10}{3}$, and then to compare it with Colonel COLBY's 10-feet bars; and from this latter comparison to infer the length proper for the new National Standard.

"A subsequent process, but still distant, would be the preparation for comparisons with measures *à bouts* and for pendulum experiments."

The following are further extracts from the Minutes of the same meeting (1845, June 6):—

"Mr. AIRY stated, in reference to the question of the use of alloys or pure metal, that he had just heard that great difficulty had been experienced by Messrs. MAUDSLAYS in casting a bar of pure copper; the cast being honey-combed and full of bubbles.

"On consideration of Mr. SHEEPSHANKS' report, it was Resolved,—

"That the thanks of the Committee are due to Mr. SHEEPSHANKS for the spirit with which he has entered into the investigation of all the difficulties connected with the formation and perpetuation of a standard of length.

"That the Committee approve most fully of the steps taken by Mr. SHEEPSHANKS in the preparation of new apparatus (as described by him), for the examination of all the causes of error, for the investigation of the thermometric correction, for the ascertaining the permanency of standard bars, and for the accurate comparison of bars.

"The question of adoption of the measures derived from the Survey was deferred."

Mr. SHEEPSHANKS now turned his attention energetically to the practical preparation of the various apparatus to which his Report alludes. Of the details of the incessant labour of the next two years, few traces remain except in Mr. SHEEPSHANKS' correspondence with myself. But to these labours there was little interruption. Old apparatus was to be tested and new to be planned and carried into execution; instrument-makers much occupied with other business were to be kept closely to the work; manipulations of various kinds were to be learnt; and all was done under the disadvantage of Mr. SHEEPSHANKS' domestic residence being at Reading, while the work of every kind was executed in London. The most tedious of the operations actually carried on appears to have been the preparation of original thermometers (or rather the acquisition of the experience necessary for their preparation, for I do not think that a thermometer was really finished before the end of 1847). It would seem that at this time there was no workman in London who could be trusted, without the closest superintendence even in the minutest points, to make a thermometer. At length Mr. SHEEPSHANKS found himself so far prepared that he deemed it necessary to apply to the Committee for further instructions. The following are extracts from the Minutes of the Meeting of 1847, June 4:—

"Mr. SHEEPSHANKS made a statement to the following effect:—

"The tardiness in the proceedings connected with the Standard of Length has arisen from the delays incidental to the planning of new machinery, and of having it executed when all the workmen are busy.

"I conceive the ultimate object to be, to procure a permanent standard, which can be easily multiplied and widely distributed, which is to be connected with our best existing measures, with the base of the Trigonometrical Survey, and with the standard measures of other countries.

"But the immediate want is for an actual type of the standard measure. Therefore, although it will be desirable (in the scientific order) to end with establishing the standard, I believe it will be best for public convenience to begin with it.

"The first considerations which occupied my attention were the *permanency* of bars and the mode of perpetuating any measure. I propose to ascertain whether and how much a bar has altered in the following manner. Bars of different metals, of the same dimensions as those proposed for the standards, are to have each a yard marked on it,

and are besides to be converted into pendulums, each oscillating on a knife-edge placed where it makes the time of vibration a minimum. If the period of vibration of the same bar at the same place is unchanged by time, gravity being supposed unchanged, then the whole length of the bar is unchanged; and the distance of the divisions marked on it, supposing the bar to continue straight, is unchanged. The bar must of course be guaranteed from rust or change of shape. As the test of actual length is required to be very exact, the series of experiments should be continued without interruption as long as possible. I have therefore had an apparatus constructed for swinging these pendulums *in vacuo*, which, so far as I can judge, will give almost any degree of accuracy to this part of the inquiry, with very little trouble to the observer; this was in fact my first step, and until this was secured I did not think it worth while to apply myself to the rest of the apparatus.

"The next point is, the very accurate comparison of two bars with each other, the bars being supposed of the same material. Connected with this is, the determination of the law of expansion of the metal that may be used, at usual and moderate temperatures. The preparations for this apparently simple operation will perhaps appear complicated. I should however be unwilling to give up any of them; so important do I hold this solidity of the apparatus to be.

"The real difficulties in the practical problem have been, so far as I can judge, the instability of the optical beam compass employed, the unsatisfactory mode of supporting the bar, the want of due verification of the axes of the microscopes, and the variations of temperature, which were not indicated by the measuring means employed. Thus, in Mr. Baily's apparatus [the apparatus of the Royal Astronomical Society], which was certainly an improvement on all that preceded it, the observer's body had a sensible effect on the bar which carried the microscopes, and probably on the direction of their axes; it also affects differently the temperature of the two bars and that of the thermometers. These effects may be partly eliminated by interchanging the bars, &c.; but there is another cause of error which cannot be eliminated, namely, that the beam which carries the microscopes is affected by the motion of the frame which carries the bars. [Mr. Sheepshanks then described an experiment of the comparisons of three iron bars, in which every single reading of the microscopes was very satisfactory, yet the result as relating to the difference of length of the three bars was unsatisfactory.]

"Having thus arrived at the conviction of the absolute necessity of a steady stand and an immoveable beam compass, I have tried to obtain them in the manner which you will see.

"The principle I have had constantly in my mind has been this. We do not know where the difficulties may be: let us not throw away a chance; let us take care to provide for all known contingences, regardless of cost, and trust to combating the unknown difficulties when they are found. It seemed to me not suitable to the Committee, nor indeed to our national character, to try any but the best conceivable means, and I have acted for the Government as I would do for myself.

"I will now say a few words on the question of measuring temperature, leaving the measurement of the expansion of the bars aside.

"A little consideration showed me that there was no use in making most accurate comparisons, unless I had most accurate means of measuring the temperature also; and that I required to know the reading of the thermometer to at least the $\frac{1}{10}$th of a degree of FAHRENHEIT. I soon found that there was no reasonable expectation of procuring such thermometers, and I proceeded as follows:—

"Mr. SIMMS assigned me a very intelligent and zealous workman, who, after some disappointment of external aid, undertook to graduate the thermometers himself. The tubes were divided by the dividing engine (probably as well as it could be done) to 180 equal parts, from somewhere below freezing to about boiling. He then bit-in the divisions by fluoric acid.

"The thermometer with the tube thus prepared comes into my hands. By a column of mercury occupying about 90 parts, the whole length is bisected and the error of the 90 divisions ascertained, and also the error of 30 relative to 120, and that of 150 relative to 60. Next with a column of mercury occupying about 60 parts, which is carried in three steps from 0 to 180, the errors of 60 and 120 are found, and thus (as above) those of 30 and 150 are known. Also by carrying the column of 60 parts backwards and forwards from 90, new measures of the errors of 30 and 150 are obtained; if these agree with the former, the errors of 30 and 150 are well known. Then with a column of about 50 parts, which is carried in three steps from 0 to 150 and from 30 to 180, the divisions 50, 100, 80, 130 are corrected; and the length of this 50 column being now well known, it is used by application to the former divisions to subdivide to every 10. A column of 45 parts will then subdivide all to every 5, and this subdivision is checked by the comparison of each 5 with the mean of the adjacent 10. I have learned the manipulation of a thermometer so as to break the column myself, and have thus saved much trouble. [Mr. SHEEPSHANKS then described the practical method of performing this operation.]

"The next thing is to settle the boiling- and freezing-points (as all the determinations already described are referred to degrees 0 and 180, which may be erroneous). There is no difficulty as to the freezing-point; a great number of observations showed me that with proper precautions I could rely on it to about $\frac{1}{100}$th of a degree. The boiling-point presented some difficulty, until the following method was adopted. The water is boiled in a close vessel having apertures for the escape of steam under control, and these are regulated till there is just no sensible pressure on a mercurial siphon communicating with the interior. [The bulb, as I understand, was surrounded by steam.] The thermometer may now be considered as completely made.

"But there is an anomaly which offers some difficulty. Take an old thermometer, ascertain its freezing-point, boil the thermometer, ascertain its freezing-point again, and you will find it has fallen from one-third to one-fourth of a degree. I found a sensible though not a large change by raising the temperature to 100°. I believe

that the freezing-point rises in time to its old place, to be again displaced by a fresh boiling.

"This difficulty I propose to meet thus. I take properly graduated thermometers which are of some age, and of which the freezing-points are carefully ascertained. I compare these with standard thermometers which have just been verified, at temperatures as high as I can get by natural means (say as high as 70° or 80°). Then those (*i. e.* the old thermometers) are good so long as the freezing-point does not alter; and I expect that it will not alter, for these thermometers will not be exposed to severe changes of temperature. If it should alter, the correction of the freezing-point will be common to all the readings. I intend to make these comparisons by putting the standard and other thermometers into the iron trough, and reading with the telescope.

"The experiments for expansion will be made for those temperatures which naturally occur for such experiments, and not much higher.

"What I have now to ask is, that you approve or disapprove of what I have done, and sanction the payment of the expense incurred, except so much as refers to the thermometers. This is not very large, and is as yet incomplete; nor can it be said whether some portions of the apparatus may not belong rather to the artist than to Her Majesty's Government.

"I now come to the point on which I particularly require direction.

"I intend to inquire into and to leave means for inquiring into the best material for a standard bar. But this will be a work of time. The question which presses is, what material will you adopt now? Mr. Baily wished to have bronze, and a number of bronze bars are made. I have had half a dozen bars of several materials, viz. cast steel, cast iron, Swedish iron, and Low Moor iron: I ordered some of copper, but the casting failed; Mr. Donkin has however given me a copper bar. I propose to keep all the bases of any measure which may be made in the hands of the Committee, so as to have the whole subject accessible, and to preserve the power of completing it.

"If the Committee will specify the material, my mode of proceeding will be as follows. Suppose that cast steel were proposed I should take a steel bar, mark the divisions determining a yard pretty nearly, find its expansion, compare it with other authorities, and then define the yard in terms of it.

"Having thus done, I should direct the artist to make me five copies, *mutatis mutandis*, as near as might be to the definitive yard. These copies would be the Standard and four Parliamentary copies A, B, C, D. I should now make a very careful comparison of the primary Standard with my generator, and also with A, B, C, D. The Standard and A, B, C, D may now go to the Government for legislation or adoption, and for conservation. The *generator* I should retain, and upon this should found all my future inquiries, and in fact all copies for publication. The process is the same whatever metal is employed, and in each metal I propose to have a scale and a pendulum, for perpetuation and for comparison with other measures.

"If the Committee will determine as to the metal and the authority for the length, I

think I need give them little trouble for some time. But I think it would be better that, for the present, they should confine themselves to the metal.

"If experience should show that there is a manifest advantage in some other metal, I will apply to them again for direction.

"With regard to the authority for length, I must own that I do not think the comparison with former measures (except the 10-foot and 6-inch Ordnance bars) of much value; or that the Parliamentary Standards will be of much use. But I rely upon being able to furnish at small trouble and no great cost, measures which shall have a given relation to a certain measure within $\frac{1}{50,000}$th of an inch, and which can be perpetuated without much risk. I expect to produce a standard yard per day, by devoting an hour and a half or two hours to the readings of one bar and the adjustment of a second in its box; which second is to be compared next day, or in 48 hours, and have its place taken by a third."

The Committee then inspected sundry apparatus under the guidance of Mr. SHEEPSHANKS.

First, in one of the chambers of the floor next below the ground floor, they inspected the comparing apparatus of the Royal Astronomical Society; being that which Mr. BAILY had proposed to use for the Parliamentary Standards, &c., and to which attention is called in Mr. SHEEPSHANKS's communication.

Secondly, they saw the vacuum apparatus for pendulum vibration, consisting of a large brass truncated cone supported by three spreading claws, with proper windows below for observations of vibrations, and with proper agate planes at the top for the support of the knife-edges: the cap closing the top was also exhibited.

Thirdly, in one of the vaults two storeys below the ground floor, they saw the principal parts of the new comparing apparatus. Upon a large low pier surmounted by a stone are planted two stone pillars, carrying a connecting top stone, and two stones crossing it; to these crossing stones the microscopes are attached; upon the platform there travels a large three-wheeled carriage or moveable platform, upon which is placed the large box (intended to contain the water, &c. by which temperature will be given), in which is a drop-box intended to carry the bars, each in its proper case and supported by its proper lever-frame. The illumination is effected by an Argand lamp placed at a considerable distance, whose light is directed by proper reflectors.

The Committee then resumed. It was then

Resolved,—That the Committee approve entirely of the course followed by Mr. SHEEPSHANKS, and request that Mr. AIRY will take measures for the examination and discharge of the instrument-maker's accounts.

The Committee then considered the question of material for the Standard. Mr. AIRY read a private letter from Mr. FARADAY, of which the following is an extract:—"I do not see any reason why a pure metal should be particularly free from internal change of its particles, and on the whole should rather incline to the hard alloy than to soft copper, and yet I hardly know why. I suppose the labour would be too great to lay

down the standard on different metals and substances; and yet the comparison of them might be very important hereafter, for twenty years seem to *do* or *tell* a great deal in relation to Standard measures."

After deliberation it was

Resolved,—That at present the Committee see no reason for departing from their resolution of the 11th of July, 1843—"That the material employed for the construction of the Parliamentary Standard of Length be bell-metal or steel, as Mr. BAILY may think best,"—as explained by Mr. BAILY at the meeting of the 10th of November, 1843, when Mr. BAILY recommended an alloy of copper and tin.

I may here appropriately give some description of the apparatus which Mr. SHEEPSHANKS employed.

First, for the thermometers. The apparatus for the freezing-point (the usual small vessel for containing pounded ice or snow, admitting of drainage of superfluous water), and the apparatus for the boiling-point (a small covered saucepan, with a mercurial gauge, and a hole for the thermometer-stalk), require no further description. In the operations for breaking the column of quicksilver, Mr. SHEEPSHANKS never trusted to subdividing the graduations of the thermometer-tube by eye. The tube was laid in a horizontal position; and a microscope pointing downwards, mounted in a frame which admitted of horizontal motion above the tube, was directed successively to each end of the column of quicksilver; and at each end three readings of the microscope-micrometer were taken, namely, one for the end of the quicksilver-column, and one for each of the graduations preceding and following that end of the quicksilver. A numerical computation was then required for determining the fractional part of the tube-graduation corresponding to the position of the end of the quicksilver-column. This operation, though very exact, occupied much time; it might without doubt have been facilitated by proper mechanical arrangements, of which however Mr. SHEEPSHANKS never availed himself.

Secondly, for the comparison of 3-foot bars. The apparatus established in the cellar of the Royal Astronomical Society's Apartments in Somerset House will be understood from the diagrams in Plates XXVIII., XXIX. and XXX., to which the following explanations apply.

Figure 1, Plate XXVIII., is a ground-plan of the apparatus. The trough for enclosing the case which actually contained the bars is entirely omitted. The observer's station (or what may be considered the front of the apparatus) is the lower part of the diagram.

A, A, &c. is a wooden enclosure, covering the back and (partially) the sides, and extending from the floor nearly to the ceiling of the room.

B, B, &c. is the general platform of masonry.

C, C are the two stone piers.

D, D, a horizontal slab extending right and left; resting on the top of the piers.

EE, EE, two solid transoms of stone, resting upon DD; they carry the two microscopes.

FF, a slab resting upon the two transoms.

G, G, the two micrometer-microscopes. They pass through channels cut in the sides of the transoms; each is carried by two horizontal Y's which are cemented into the back of the channel, and is pressed into the Y's by bars of brass whose ends are fixed by screws to the points of the Y's.

H, H are two lamps for throwing light upon the reflectors of the microscopes, and thus illuminating the defining-points of the standards under observation.

II, II, two perforations in the slab DD, through which the light of the lamps passes to the microscope-reflectors.

J, J, narrow plates of brass, attached I believe for no other purpose than to support the plates, which cover the open side of the channels in which the microscopes are fixed, and which protect the microscope-tubes from violence and from radiation.

KK, KK, broad plates of brass below the transoms, which I believe were intended to intercept radiation between the transoms and the standards.

L, L, cords which supported the plates K.

M, the great travelling platform, on which the box carrying the standards was placed. The box is not shown in this diagram.

N, N, the two side-wheels of the travelling platform.

O, their common axletree.

P, the tail of the travelling platform, with a wheel below its end, which is not seen in this diagram. This wheel, as well as N, N, roll on the masonry B.

Q, Q, Q, Q, four screws for vertical adjustment of the box which is placed on the platform.

R, R, R, R, four screws for lateral adjustment of the box.

S, S, two screws for longitudinal adjustment of the box. They were sometimes turned by hook's-joint handles.

T, T, &c., eyes for the fixing of cords, for conveniently raising the platform.

Figure 2 is a front view of the apparatus.

D' is a small block of stone cemented to D and to E, apparently to connect them more firmly.

E' is a larger block cemented to D and to both the transoms E, apparently for the same purpose.

The Y's which carry the microscopes G are shown (concealed by stone) in this view.

U is the third wheel of the travelling platform, carried by its tail P.

V is the large cast-iron box or water-trough carried by the platform; which was nearly filled with water, and into which was dropped the box containing the Standards to be compared.

W, eyes for the fixing of cords.

X, a tap for the discharge of the water.

Figure 3, Plate XXIX., is a side view of the apparatus.

J' is the plate which covers the microscope-tube.

The bars which press the microscope-tube into the Y's are shown in this view, concealed only by the plate J'.

Figure 4 is a plan of the water-trough, as mounted on the platform, and carrying the drop-box, in which the two Standards to be compared are supposed to be placed.

The upper surface V of the water-trough is cast in a pannelled form.

Y is the drop-box, made of brass, resting by its flanges all round upon the upper surface of V.

Z is the cover of the drop-box.

a, a are two handles of the cover.

b, b are two long slits, for observing the stalks of two thermometers enclosed in the drop-box.

b′, b′, two slits through which light is thrown from the upper reflecting prism to the lower reflecting prism, to illuminate the thermometer-scales.

c, c, c, c are four holes, through which the defining points of two Standards are viewed by the microscopes G.

Figure 5 is a longitudinal section of the water-trough, the drop-box, and the quicksilver-box, showing one of the bars as partially floating in the quicksilver, for observation with the microscopes. (The bars intended to be placed in quicksilver were all covered with gold-beater's skin.)

d, d are two blocks supporting the quicksilver-box.

e is the quicksilver-box of cast iron.

f is a swinging stirrup, on which the middle of the bar rests very slightly.

g, g, adjusting screws which act on the immediate support of f.

h, the Standard Bar under observation, floating almost freely in the quicksilver.

i, i, the well-holes in the bar, sunk to the depth of half its thickness; in the bottom of each well-hole is fixed a gold pin, on the surface of which the defining mark is traced. In all the bars ultimately constructed by Mr. Sheepshanks, the defining mark is a fine line, transversal to the bar; accompanied by two lines parallel to it, one on each side at the distance 0·01 inch; which were intended to give the means of ascertaining the value of the divisions of the micrometer. There are also two longitudinal lines; and the point of the transversal line intended to define the Standard Measure is that which is midway between the longitudinal lines.

j, j, two bent thermometers. It is to be remarked that there is a longitudinal partition in the quicksilver-box, and that two large and deep notches are cut in this partition, in which the bulbs of the thermometers are lodged, and through which the quicksilver flows. The stalks of the thermometers rest on the edge of the partition. (In a few bars, holes were sunk in the bars themselves, and bent thermometers with small bulbs were placed in these holes, in quicksilver, I believe).

j′, j′, two long rectangular glass prisms, for turning a beam of light, thrown vertically through the slits b′, b′, into a horizontal direction, by internal total reflexion, in order to illuminate the thermometers jj.

j″, j″, two paper cards in an inclined position, beyond and below the thermometers, upon which was thrown the beam of light passing horizontally from j′, j′, and which formed a white ground for the vertical view of the thermometer-tubes.

Figure 6, Plate XXX., is merely an enlarged view of the adjustment of the stirrup-support in figure 5.

Figure 7 is a transversal section of the water-trough, drop-box, quicksilver-box, and bars.

k is the longitudinal partition of the quicksilver-box.

Figure 8 is a plan of two standards as placed in the quicksilver-trough for comparison. In figure 4 they are concealed by the cover Z.

Figure 9 is an enlarged plan of the adjustment of the stirrup-supports in figure 8.

Figure 10 is a longitudinal section of the mounting employed when the standard-bars are supported by "block-and-stirrup."

l is a block, planted on the platform M.

m, a wooden box.

n, n, two blocks.

o, a brass plate resting upon them.

p, a small metallic block (with foot-screws), on which the standard-bar immediately rests. This is the "block" of the "block-and-stirrup" apparatus.

q, one of two uprights supported by a pedestal similar to that of p.

r, a stirrup swinging with pivots on the top of q. This is the "stirrup" of the "block-and-stirrup" apparatus. The standard-bar rests in this stirrup.

Figure 11 is a plan of the instrumental arrangement for the comparison of the length of one standard-bar defined by line-measure with the sum of the lengths of two sections of two standard-bars defined by end-measure. The principle concerned in this operation will be explained in the latter part of this Memoir.

s and t are two end-bars, with their ends in contact. The line-bar h, and each of the end-bars s and t, is supported upon a lever-frame, to be described shortly.

w, w, &c. are collets, fixed near the extremities of the end-bars by stiff-friction only.

x, x are spiral springs attached to the collets, drawing the ends of the bars into contact, with a pressure of about 3 lbs.

Figure 12 is a longitudinal section of the same arrangement, showing however only the two end-bars upon their lever-frames.

v, v, v, v are insertions of pieces of agate in the ends of the metallic bars.

For understanding of the supporting-mechanism, it must be remarked that each bar rests upon two lever-frames; and that the form of each lever-frame is the following. The foundation-block supports the centre of motion at the centre of the length of a lever of first order; this lever consists of two parallel plates separated by a much larger space than the breadth of the bar, and the parallel plates are tied by cross-ties. This lever carries the two centres of motions of two similar but smaller levers of second order, each consisting of two parallel plates, falling between the parallel plates of the lever of

first order, but yet more widely separated than the breadth of the bar. Each lever of second order carries two rollers, upon which the standard-bar rests.

y is the support of a lever of first order.

z, z, the cross-ties of a lever of first order.

a, *a*, the centres of motion of levers of second order.

b, *b*, the cross-ties of a lever of second order.

c, *c*, the rollers carried by a lever of second order, on which the standard-bar immediately rests.

It will be seen that the pressures on the four different rollers *c*, *c*, *c*, *c*, carried by one lever-frame, are necessarily equal; and that if the two lever-frames are planted symmetrically under the bar, the pressures on the eight different rollers are necessarily equal.

Figure 13 contains a view and section, on an enlarged scale, of the ends of two end-bars in contact. The inclined lines show how the angles of the square bar are chamfered off so as to form an octagon; which is turned at the extremity into a cylindrical form.

Figure 14 represents the microscope commonly employed to read the graduations of the thermometers. The small pieces at the bottom of the frame embraced the edge of the flange of the drop-box, permitting the frame to slide longitudinally. The mode of illuminating the thermometer-tubes will be understood from this figure: *d* is a small hand-lamp, by which light was thrown horizontally into the reflecting prism *e*. By this reflexion, the light was thrown downwards upon the reflecting prism j'; and by the second reflexion the light was thrown horizontally upon the inclined card j'', which served as a ground under the thermometer-tube j when viewed by the microscope from above. In order to facilitate the reading of the comb in the field of view, a card *f* was mounted, which, when the micrometer-wire had been brought to the end of the column of quicksilver or to the nearest division on the tube, was inclined to the position *f'*, and received the light of the lamp, and fully illuminated the field of view.

This, perhaps, will be a proper place for inserting a memorandum by Mr. W. Simms, Jun., on Mr. Sheepshanks' practice in observing the Standard Yards, with which Mr. W. Simms favoured me after the death of Mr. Sheepshanks:—

"The two bars to be compared were placed upon their supports within the trough, at a distance of about 2 inches between their centres, and as nearly parallel as practicable, care being taken that each division should pass nearly under the axis of the micrometer-microscope when the trough was wheeled up to its place.

"There is a considerable amount of adjustment provided in the trough-carriage; but Mr. Sheepshanks preferred having the apparatus so carefully arranged, that but little adjustment might be required at the time of comparison.

"I believe no observations were ever made on the same day on which the bars were arranged; I frequently placed them for Mr. Sheepshanks in the evening, and he observed the following day; so that generally eighteen hours were allowed to intervene at the least. The trough was always covered up when left, small shutters being placed over the holes in the covers through which the divisions were seen.

"The observations were made thus. The trough [upon the travelling platform M] being wheeled by the hand, so as to bring the divisions on one of the bars under the micrometers, the trough-adjusting screws were used finally to adjust the division, so that the micrometer-readings should be on opposite sides of the zero and nearly to an equal amount: three readings were now taken of A [one of the microscopes], and entered together; then three of B [the other microscope] were made and registered; three more of B under these; and finally, three of A. Mr. SHEEPSHANKS considered this method of registering six observations of each division to be a check upon any start which might take place in the apparatus; and three readings were easily kept in memory, so that they could be written down at once.

"The trough being wheeled up, and the other bar being similarly adjusted by the trough-adjusting screws, the micrometer-readings were taken in precisely the same manner.

"The lamps were then changed; that which illuminated micrometer A being applied to B, and *vice versâ*.

"The trough remaining undisturbed, a similar set of observations were made of the same bar, a slight change of reading being frequently found to arise from a change of light. Then the first bar was brought up to position and observed. This completed a set.

"At an early period of the observations the bars were made to change places in the trough, and another set was taken the following day, as Mr. SHEEPSHANKS considered that the position of the trough might affect the distance between the micrometers.

"In Mr. SHEEPSHANKS' register of observations, the bar marked 'inwards' is that farthest from the observer.

"At a later period the bar of comparison was left undisturbed, and a number of other bars was compared with it; then that bar was placed in the other position, and the comparisons again made; this was to avoid having to arrange two bars every time.

"Mr. SHEEPSHANKS commonly made several sets of observations at one visit, but each set was always kept separate."

SECTION V.—*Proceedings of Mr.* SHEEPSHANKS *to the middle of* 1850; *preparation of Thermometers; adoption and rejection of "Brass* 2*" as Basis for the new Standard Yard; adoption and rejection of "Split Plug A;" adoption of "Bronze* 12*;" experiments on thermometric expansion; comparison of Bronze* 12 *with several bars; first suspicion of personal equation; rejection of Bronze* 12 *as Basis, and final adoption of "Bronze* 28*."*

I now commence an account of the measures made by Mr. SHEEPSHANKS with the various apparatus which I have described. And here I must state generally, that the magnitude of the work makes it impossible for me to enter into great details. From actual numeration of a large portion of the observations, and from estimation of the rest, I find that the number of micrometer-readings amounts to nearly two hundred thousand. I can therefore only give abstracts or results.

The work which obviously was first to be undertaken, was the preparation of thermometers. It will be convenient here to describe all that was done, to the time of Mr. SHEEPSHANKS's death.

Thermometers were procured, generally in Series of twelve at a time, and each Series was distinguished by a letter or mark; as A, B, C, Flat S, Flat Bore, Flat Notched, Round Bore, Round Notched, Long S, Bent. After all destruction by breaking, the number of thermometers still in my hands is 66. It was, I suppose, the intention of Mr. SHEEPSHANKS, that each of the Standards which he proposed to construct should be accompanied by a thermometer. Of these thermometers, a few were never examined at all; but the greater part were examined, some for the freezing-point, some for the boiling-point, some for the subdivision, some for two or all three of these elements. Some were merely compared with others of the series whose errors were determined by the original process (as, being intended for use through a limited range of temperature, they did not admit of independent determination of the boiling-point).

The following thermometers, still remaining, were completely corrected by the original process:—A, 2, 5, 6, 8, 9; Long S, 1, 2, 3, 6. The following were corrected by comparison:—B, 4, 5, 6, 7, 8, 9, 10; C, 1, 2, 4, 6, 7, 8, 9, 10; Flat S, 1, 2, 3, 5, 7 (all by comparison with A, 8, 9); Flat S, 6, 8; Flat Bore, 1, 2; Bent, *l*, *s*; also BAILY'S A (by comparison with others of the A series); Flat S, 9, 10, 11, 12 (by several of the A series and several of the Long S series); Bent, 0 Notch, 2 Notches (by comparison with the whole of the A series and the whole of the Long S series). The thermometers described as Bent 0 Notch and Bent 2 Notches are otherwise marked R and L, and are the thermometers which were inserted in the quicksilver trough, and on which in fact almost the whole of Mr. SHEEPSHANKS's operations in the comparison of his own standards depend, so far as they are influenced by temperature. In some few instances, however, the temperatures depend on *l* and *s* (bent thermometers with small bulbs which were inserted in holes in the bars). The freezing-points and the inequality of graduations of L, R, *l*, *s* were determined independently, but not the boiling-points.

I am inclined to think that the papers containing some of the early comparisons of thermometers have never reached me. Thus it would appear that for Mr. SHEEPSHANKS's "Statement," dated 1848, July 18, some comparisons of the Ordnance Bar Thermometers and of BAILY'S Thermometer A must have been made; but I find none, in the papers which I possess, anterior to 1855. Even for the important thermometers L and R, I find no comparisons anterior to 1850. In one of Mr. SHEEPSHANKS's papers I find mention of "two good old thermometers," and I conjecture that comparisons had been made with some thermometers supposed to possess superior authenticity.

It would seem that, as soon as some progress was made in the preparation of thermometers, Mr. SHEEPSHANKS proceeded to determine the value of one new bar, in terms of the Lost Imperial Standard. The bar first employed for this purpose was called by him "Brass 2." This bar was compared with SHUCKBURGH'S Scale, the Royal Society's Standard, and the Ordnance Standards, and with others not essential to the present

purpose. To facilitate the comparison with the Ordnance Standards, two bars of wrought iron (Low Moor and Swedish) were prepared, similar in form to the bronze and brass bars, and were used as intermediaries; some comparisons were however made directly with the Ordnance Bars. Then it was necessary to express the length of SHUCKBURGH'S Scale and of the Royal Society's and Ordnance Standards in terms of the Lost Imperial Standard. For SHUCKBURGH'S and the Royal Society's Scale, Mr. SHEEPSHANKS adopted KATER'S and BAILY'S values. For the Ordnance Bars, a very troublesome investigation was undertaken. On account of the difference of metal, it was necessary to examine the graduation of the thermometers; and the result was, that MURPHY'S thermometer L required the correction $-1^{\circ}\cdot12$, MURPHY'S R $-0^{\circ}\cdot95$, 1A $+0^{\circ}\cdot7$, and 2A $-0^{\circ}\cdot4$. BAILY'S thermometer appeared nearly correct. Then there was a discussion, at great length, of the personal equations of the different observers, in the observation of the different dots of the Imperial Standard, and of the Ordnance Standards. Long study would be necessary to enable any person to exhibit all the steps of these operations in a clear and logical form. The values which Mr. SHEEPSHANKS appears to have adopted for the lengths of the two Ordnance Bars in terms of the Imperial Standard, at the temperature 62° FAHRENHEIT, are 1A=35·999716 inches, 2A=35·999892 inches. Finally, from these various sources, Mr. SHEEPSHANKS inferred the value of Brass 2, and communicated it to the Members of the Committee in the following paper.

"*Statement of the means employed for obtaining an Authentic Copy of the Lost Imperial Standard Yard. By the Rev.* R. SHEEPSHANKS.

"As I have now completed the first set of observations for restoring the lost Standard, I wish to present the results to the Committee.

"I have taken a brass bar (marked No. 2) as my term of comparison. This is held by its middle, and floats as nearly as possible in mercury. The comparisons have been made carefully with the best *primary* evidence of the lost Imperial Standard.

"I am very much surprised, and not a little mortified, that this evidence should be so scanty; still I trust that it is sufficient for my purpose, which is simply to procure a copy of the Imperial Standard, which is at least as accurate as any results derived from, or referred to it. A short notice of our measures of length will not be out of place.

"All the accurate scales which have been made within the last hundred years can be traced back, errors excepted, to the yard laid off by GRAHAM, on a bar made for the Royal Society in 1742. I shall call this GRAHAM'S bar, to distinguish it from other scales belonging to the same Society. Upon this bar, GRAHAM laid down a yard taken from a Standard kept at the Tower (this yard is marked E on GRAHAM'S bar, and is called by me GRAHAM'S *Tower Yard E*), but not with any particular care*; and when

* It is stated in the Philosophical Transactions, 1742–3, p. 185, that "Mr. GRAHAM did with the greatest care lay off the length of three English feet from the Standard of a Yard kept in the Tower of London." As the Tower Standard was a flat scale, this could be done with great accuracy. When both were compared

the authority of this original yard was called in question, he laid down, subsequently, a second yard marked *Excheq.*, which he took with great care* from an old Standard of Elizabeth kept at the Exchequer as the legal Standard. These two yards were found by GRAHAM to be thus related:—

Tower Yard E = Exchequer Yard *Excheq.* + $0^{in.}\cdot0075$.

"This Tower Yard E is generally called the 'Royal Society's Standard Yard,' by the earlier authorities.

"I do not know when BIRD made his original scale, but from his method of dividing mural quadrants, a well-divided scale must, I think, have been in his possession before the graduation of the Greenwich Mural Quadrant. In 1758 and 1760, he made two Standard bars for a Committee of the House, which are known by those dates, the Standard of 1760 being in 1821 declared to be the legal Imperial Standard, which I am attempting to restore. I find, from some notices by MASKELYNE, in the Philosophical Transactions for 1768, pp. 324, &c., that BIRD's *own* scale was just $0^{in.}\cdot001$ in 3 feet *shorter* than GRAHAM's Tower Yard E†; and I believe, though I have no absolute evidence of the fact, that his own scale was the type followed by BIRD in constructing his Standards of 1758 and 1760, and in fact all the scales which issued from his hands; so that BIRD's scales are, errors excepted, $0^{in.}\cdot001$ shorter than GRAHAM's Tower Yard E. It is certain that TROUGHTON, in laying down his own scale, took the fundamental length from a scale by BIRD (Philosophical Transactions, 1798, p. 137). Hence, as a general rule, all scales, having BIRD or TROUGHTON for their authors, will be found to be about $0^{in.}\cdot001$ shorter in 36 inches than GRAHAM's Tower Yard E; although accidental errors, to half that amount at least, may occur in any of the older scales which have been examined by SHUCKBURGH, KATER, &c.

"If this account be, in the main, correct, we see how it comes to pass that SHUCKBURGH's scale by TROUGHTON should be *nearly* of the same length as BIRD's Standard of 1758 and 1760, since they are all of the same family. The statements of SHUCKBURGH and KATER as to the *identity* of the measures must be qualified, and TROUGHTON's belief in the perfection and similarity of his own scales cannot be fully admitted‡.

"Another family of scales, viz. General ROY's scale and RAMSDEN's scale, are said to have been carefully compared with GRAHAM's Tower Yard E, and to have agreed *exactly*

with the Exchequer Standard, pages 548 and 550, their difference appeared to be $0^{in.}\cdot0111 - 0^{in.}\cdot0075 = 0^{in.}\cdot0036$. But the Exchequer Standard (a rude end-standard) could not be very exactly compared; and it seems probable that the origin of the discordance is not in the inaccuracy of the transfer, but in the inaccuracy of comparison through the Exchequer Standard.—G. B. A.

* That is, with as great care as the nature of the case permitted. The construction of the Exchequer Standard did not admit of accurate transfer.—G. B. A.

† "These are not MASKELYNE's very words, but his meaning applied to my own nomenclature."

‡ "From SHUCKBURGH's measures of the 6-inch spaces of his scale, we may gather that the yard $0^{in.}\ldots36^{in.}$ exceeded the yard $0^{in.}\ldots42^{in.}$ about $0^{in.}\cdot00045$, and the 6-inch space $30^{in.}\ldots36^{in.}$ exceeded the 6-inch space $36^{in.}\ldots42^{in.}$ by $0^{in.}\cdot00053$."

with it. From these scales, the measuring apparatus was constructed, with which the bases of the English Trigonometrical Survey were determined. I refer to KATER's Memoir, Philosophical Transactions, 1821, p. 75, and to Appendix No. 1, with respect to this operation.

"RAMSDEN's scale is lost. ROY's scale was compared by Sir G. SHUCKBURGH with his own scale, very unsatisfactorily*, and KATER has shown the result to be worthless. KATER's comparison, in 1821, agrees very well with ROY's own account of it, viz. that it was made for GRAHAM, who would probably follow his own Tower Yard E more scrupulously than BIRD thought it necessary to do. I have not compared ROY's scale, as I consider KATER's comparisons with the Imperial Standard in 1821 not satisfactory by his own showing in the Memoirs of 1826 and 1830: BAILY did not compare ROY's scale at all with the Imperial Standard.

"I have not looked after KATER's standards of 1826, because they only profess to be exact copies of SHUCKBURGH's scale, and he has thrown doubt on the relation of SHUCKBURGH to the Imperial Standard in his memoir of 1830. I object strongly to his mode of measurement, *i. e.* resting the apparatus upon the scale to be measured. On this account also I reject DOLLOND's scale, though it was compared directly with the Imperial Standard. The most favourable type of KATER's yard is the scale which he made for the Royal Society in 1831.

"The Royal Astronomical Society's tubular scale, and two other scales of the same construction for the Danish and Russian governments, were compared by BAILY with the Imperial Standard. I give, in Appendix No. 2, reasons why I have not admitted the Royal Astronomical Society's scale in these comparisons.

"The scales I have actually compared with the Brass bar No. 2 are as follows:—

		in.	in.
Sir G. SHUCKBURGH's scale		0	36
Sir G. SHUCKBURGH's scale		10	46
All on the same bar	GRAHAM's Tower Yard E.		
	GRAHAM's Exchequer Yard.		
	Yard nearly same as E, but anonymous.		
	KATER's Royal Society's scale.		
	Ordnance Iron Bar, A1.		
	Ordnance Iron Bar, A2.		

"The direct comparisons with the Imperial Standard made by BAILY and others in 1834 are not very satisfactory. The comparisons between the Imperial Standard and

* "He compares the successive feet of ROY with the 12 inches of his own scale, $12^{in.}..24^{in.}$, and finds, assuming his own foot to be correct, that

ROY 36 inches = $35^{in.}\cdot9999$ SHUCKBURGH.

But this foot is, according to his own account, less by $0^{in.}\cdot00006$ than the mean foot of his scale, and less by $0^{in.}\cdot00013$ than the mean foot of the yard from $0^{in.}..36^{in.}$. He also says, p. 178, that

ROY 42 inches = $42^{in.}\cdot00010$ of SHUCKBURGH.

KATER, on the contrary, makes ROY 36 inches = SHUCKBURGH $36^{in.}\cdot0009$, and is, I doubt not, nearly right."

Shuckburgh's scale are not very numerous nor very consistent. Those between the Imperial Standard and the Ordnance iron bars, A1 and A2, are singularly discordant.

"I ought to mention that I selected my terms of comparison with the Imperial Standard, *à priori*, on my own opinion of their value from their history. I have not compared any scale which appeared to me suspected, for I should not have known how to estimate the weight, and mere discordance would not have justified rejection.

"Some details and some steps of my comparisons will be found in the Appendix No. 3. From these I deduce the following approximate value of the lost Standard:—

"1st. From Shuckburgh's scale $10^{in.}$ .. $46^{in.}$:—

	in.	in.	in.	
Brass bar 2 = Shuck.	10	— 46	+ 0·000222	, 100 Obs. (Appendix No. 3.)
Shuck. $10^{in.}$ — $46^{in.}$ = Imp. Stand.			+ 0·000058	(Mem. R. A. S. vol. ix. p. 120.)
∴ Brass bar 2 = Imp. Stand.			+ 0·000280	

"2nd. From Shuckburgh's scale $0^{in.}$ .. $36^{in.}$:—

	in.	in.	in.	
Brass bar 2 = Shuck.	0	— 36	− 0·000006	, 120 Obs. (Appendix No. 3.)
Shuck. $0^{in.}$ — $36^{in.}$ = Imp. Stand.			+ 0·000090	(Phil. Trans. 1830, p. 377.)
∴ Brass bar 2 = Imp. Stand.			+ 0·000084	

"I have taken Kater's last value as most worthy of confidence, but I do not much trust it, on account of the objectionable mode of resting the microscope apparatus upon the scale.

"3rd. From Kater's scale, made for the Royal Society 1831:—

	in.	in.	
Brass bar 2 = Kater's scale	36	+ 0·000849	, 110 bisections.
Kater's scale $36^{in.}$ = ...		35·999380	(Phil. Trans. 1831.)
∴ Brass bar 2 = ...		36·000229	

"I have given the values of Brass bar 2 in terms of Imperial Standard from comparisons with A1 and A2 in Appendix No. 3.

"Hence Brass bar 2 =	in. 36·000280	by Shuckburgh	in. 10 .. in. 46 and comparisons in 1834,
=	·000084	Shuckburgh	0 .. 36 and Kater's authority.
=	·000229	Kater's scale.	
=	·000303	A1, compared in 1834.	
=	·000275	A2, compared in 1834.	
=	36·000234	by a mean of all.	

"This should be pretty near the truth; but I prefer 36·00025, if in such a matter such a difference be worth notice.

"I propose, therefore, in constructing the new standards, to assume that

Brass bar 2 = $36^{in.}$·00025 of lost Imperial Standard at 62° Fahrenheit,

and to make that the base of my future proceedings.

"*July* 18, 1848. "R. Sheepshanks.

"APPENDIX No. 1.

"*On the Scales used in the Trigonometrical Survey, and especially* ROY'S *Scale.*

"The Irish Survey depends on a base which is ultimately referred to an iron bar of 10 feet, and a brass bar of 6 inches. These were laid off, I believe, from the 5-feet brass measures of TROUGHTON or SHUCKBURGH, as nearly as could be readily managed, but they have no *immediate* relation to any *yard*, standard, or otherwise. It will be necessary to ascertain the value of these units in terms of the new standard, when that is fixed; but there is nothing to be extracted from them which would assist us in restoring the Imperial Standard destroyed in 1834.

"At first sight, one might have expected some help from the scales employed in the English Survey, viz. from ROY'S scale and RAMSDEN'S scale, which were, or were intended to be, the units of the bases of Hounslow Heath, Romney Marsh, &c.; but a careful examination of the memoirs relating to those bases, and of KATER'S memoir (Philosophical Transactions, 1821, p. 75), extinguishes hope of any aid from this quarter.

"ROY'S scale (Philosophical Transactions, vol. lxxv. p. 401) was purchased by him at SHORT'S sale. 'It was originally the property of the late Mr. GRAHAM, the celebrated watchmaker; has the name of JONATHAN SISSON engraved upon it, but is known to have been divided by the late Mr. BIRD*, who then worked with SISSON.' ROY, aided by RAMSDEN, found that 36 inches, taken from 'GRAHAM'S Tower Yard E,' exactly reached to 36 inches on his scale, the temperature being 65°. GRAHAM'S Exchequer Yard was found $0^{in.}\cdot0069$ shorter. From the 40 inches of ROY'S scale, the deal rods and glass rods used by ROY in measuring the Hounslow Heath base were laid off, not directly, but by 40-inch spaces marked off on a large plank. By these points laid down on the large plank, the chain used in the Romney Marsh base was verified.

"RAMSDEN'S scale was also compared with GRAHAM'S Tower Yard E, 'when they were found to be precisely of the same length.'—Philosophical Transactions, 1795, p. 435. This must have been a comparison of 36 inches only. From his own scale, RAMSDEN laid off 40-inch spaces on the 21-foot iron bar which he used for ascertaining the actual lengths of the chains A and B, by which the base of Hounslow Heath was remeasured, &c.

"If, then, the operations had been correctly performed, all the base measurements of the English Ordnance Survey would have been expressed in terms of GRAHAM'S Tower Yard E.

"But it is impossible to trust, implicitly, random expressions that two measures *agree exactly*: this cannot be ascertained by a beam compass. ROY and RAMSDEN are both silent as to the manner in which they ascertained the relation of their 40 inches to 36 inches; and KATER'S account of the 21-foot iron bar does not give a very high idea of RAMSDEN'S exactness in transferring his 40-inch spaces to his iron bar. (See his memoir, Philosophical Transactions, 1821, p. 71.) Thus, according to KATER,—

* "The fundamental length was very probably laid off by GRAHAM himself."

"Roy's scale, 36 inches = Graham's Tower Yard E − 0·00047 in., which is pretty well.

"But a mean yard of Ramsden's bar = Graham's Tower Yard E + 0·00114

"Roy's yard, according to Kater, is about $0^{in}\cdot0009$ longer than the Imperial Standard.

"In fixing the Imperial Standard, Kater very wisely, I think, dismissed all the supposed units of the Survey. Ramsden's scale is lost; and I have not used Roy because it has no independent value,—professing merely to be a copy of Graham's Tower Yard;—because the comparisons made of it by Shuckburgh are very indifferent and inaccurate; and because those by Kater in 1821 are inferior in number, and very inferior in quality, to those made by him in 1831 between the Imperial Standard and the Scale of the Royal Society, which I consider his most complete work in linear measurement.

"Appendix No. 2.

"*The Royal Astronomical Society's Scale and the Tubular Scales discussed.*

(The purport of these remarks has already been given.)

"Appendix No. 3.

"*Comparisons between Brass Bar* 2 *and other Scales.*"

(These are now unnecessary.)

Mr. Sheepshanks now commenced comparisons of Brass 2 with several Bronze bars, but was much distressed by the imperfect definition and varying position of the defining lines, as viewed under the microscopes. Numerous attempts were made to correct these faults; at length, recourse was had to another construction which he called "Split-Plug." Instead of cutting the line upon the end of a gold pin, a plug of gold, included in one of bronze, was split longitudinally; the surfaces of separation were smoothed; the two halves of the plug (each half consisting of two metals) were then united by pins; the compound plug was forced from below into a hole of the bar; and its end was ground off level with the bottom of the well, and was polished. The division of the gold plug presented a well-defined dark line, scarcely affected by change in the direction of the light. It appears to have been first used on 1849, March 12. But about the same time another improvement was made, namely, the use of pierced glass prisms instead of annular mirrors, for reflecting the light of the lamps upon the defining lines. The custom was also introduced of interchanging the lamps after every series of observations. These innovations seem to have been so far satisfactory, that no desire was afterwards entertained to extend or retain the use of the Split-Plug.

Split-Plug A was however compared fourteen times (five observations in each series) with Brass 2 between 1849, March 12, and May 14; the mean gave

At temperature $49^{\circ}\cdot1$, Split-Plug A = Brass 2 + $0^{mm}\cdot1044$.

One revolution = $0^{in}\cdot00370$.

Split-Plug A was then used as a bar of reference with which fourteen bronze bars were compared.

It was also compared directly with the Royal Society's Scale; but the result of this direct comparison, 36$^{in.}$·000485, did not exactly agree with that of the indirect comparison through Brass 2. After remarking this discordance, Mr. SHEEPSHANKS adds, "There was another objection to making the split-plug bar my standard. The construction is not mechanically sound; the difference from the true yard leaves too much influence to the micrometer; and I have upon it no scale by which I can control the micrometer. I found too that by my better arrangement of the light, the superiority of the split-plug was no longer evident. So I looked over the values of my bars compared with Split-Plug A, and selected Bronze 12 as generating standard to which everything is to be referred."

After this decision, no farther use was made of the results of comparisons of bars with Split-Plug A. The divided plugs were removed from A and from another of similar construction called Split-Plug B, and they were then fitted with gold plugs and incised lines in the same manner as the others, and were called Bronze 39 and Bronze 40.

In the month of September 1849 Mr. SHEEPSHANKS proceeded to compare Bronze 12 with the original standards (omitting SHUCKBURGH's scale). The results are the following:—

(1^r=0·003576).

By 100 observations at 56°·04 FAHR., Bronze 12=Ordnance 1A−0^r·0582
=36$^{in.}$·000219

By 100 observations at 56°·5 FAHR., Bronze 12=Ordnance 2A−0^r·1275
=36·000093

Also,

By 75 observations, Low Moor A=Ordnance 1A+0·0663=35·999962

By 55 observations, Low Moor A=Ordnance 2A+0·0102=35·999930

Mean . . . . =35·999946

and

By 200 obs. (reduced separately) at 62° FAHR., Bronze 12=Low Moor A+ ·000149
=36·000095

In a similar way, through a bar of Swedish iron,

Bronze 12=36·000063

By 160 observations at 57°·58 FAHR., Bronze 12=R. Soc.+0^r·1727=35·999982.

From these, Mr. SHEEPSHANKS concluded

Bronze 12=36·000043

at the fundamental temperature 62° FAHRENHEIT.

(Different reductions gave 36·000034 and 36·000048.)

In the winter of 1849–1850 Mr. SHEEPSHANKS made a series of experiments for the thermometric expansions of Bronze, Brass, and Low Moor Iron. The different temperatures were given to the bars by pouring in water at different temperatures into the external box: this, I believe, is the first occasion on which it was so used. The quan-

tity of water employed at once was 11 gallons. The corrections to the thermometers L and R were still obtained, I believe, from old thermometers: but, as the comparison of L and R with new original thermometers followed closely, I am not quite certain on this point. Each number below is the mean of about thirty comparisons:

$$1^r = 0^{in}\cdot 003587.$$

For relative expansion of Bronze 12 and Low Moor Iron.

Day, 1849.	Temperature.	Reading for Bronze 12.	Reading for Low Moor Iron.
	°	r.	r.
November 30....	68·76	201·5598	201·2976
	64·41	201·0627	200·9355
December 1 ...	51·50	199·8867	200·1899
	58·88	200·5319	200·5998
	57·16	200·3766	200·4895
December 3	67·65	201·3903	201·1642
December 5	78·81	202·5139	201·9086
December 6 ...	54·82	200·2089	200·4075
	61·15	200·7820	200·7572
December 8 ...	48·46	199·6225	200·0354
	75·45	202·1283	201·6404
December 10 ...	68·92	201·5656	201·2943
	64·13	201·0967	200·9797

From these, the excess of expansion of 36 inches of Bronze above Iron for 1° FAHRENHEIT was inferred to be $0^r\cdot 03318$.

For absolute expansion of Low Moor Iron, Mr. SHEEPSHANKS compared the higher and lower temperatures on each of the following days: Nov. 30, Dec. 1, Dec. 6, Dec. 8, Dec. 10, as given in the last Table. From these he obtained, for 1° FAHRENHEIT,

Absolute expansion of 36 inches of Low Moor Iron $= 0^r\cdot 06095$.

For absolute expansion of Bronze 12 and Brass 2.

Day, 1850.	Temperature.	Reading for Bronze 12.	Reading for Brass 2.
	°	r.	r.
January 24.....	36·06	198·8437	198·8780
	51·75	200·3481	200·4152
	67·82	201·8598	201·9467
January 25.....	70·14	202·0599	202·1320
	45·85	199·7360	199·7850
January 26.....	41·95	199·2963	199·3419
	70·46	202·0152	202·0959
May 4.....	54·23	200·0187	200·0701
	65·87	201·1187	201·1719

From these were obtained, for 1° FAHRENHEIT,—

Absolute expansion of 36 inches of Bronze = 0·09507
Absolute expansion of 36 inches of Brass = 0·09601;

and by combining the absolute expansion of Bronze with the excess of expansion of Bronze above Iron,

Absolute expansion of 36 inches of Iron = 0·06189.

In the spring of 1850, from February to April, Mr. SHEEPSHANKS determined the scale of the two thermometers L and R by comparison with five original thermometers of the A series; and in 1849 and 1850 he compared *l* and *s* with two of the same series. From this time, I conceive, every observation possesses extraordinary value.

In the summer of 1850, seventeen bronze bars and two copper bars were compared with Bronze 12. The excesses of their lengths over that of Bronze 12 were as follows:—

		r.
Bronze	2	−0·0252
	6	+0·0042
	7	−0·0272
	8	−0·0018
	18	−0·0347
	19	+0·0014
	20	−0·0146
	21	−0·0034
	22	+0·0033
	23	−0·0054
	24	+0·0107
	25	+0·0840
	26	+0·0379
	27	−0·0010
	28	−0·0155
	29	+0·0364
	30	+0·0386
DONKIN'S Copper		+0·0397
MAY'S Alloy (32 copper, 1 tin)		+0·0547

In the course of these observations, Mr. SHEEPSHANKS saw reason for abandoning Bronze 12 as Basis, and adopting Bronze 28 in its stead. He remarked that the length of Bronze 28 (=Bronze 12 − 0ʳ·0155 = 36ⁱⁿ·000048 − 0ⁱⁿ·000056 = 35ⁱⁿ·999992) was as nearly as possible that of the Imperial Standard, that the bar floated evenly in the quicksilver, and that it was nicely divided. Bronze 12 was not nicely divided, and did not float evenly; and one of its gold dots was much tarnished while the other was quite bright; and the tarnished dot was much disfigured by a mercurial stain. Another consideration, which afterwards led to extensive series of observations, was the following.

A portion of the comparisons of bars abstracted above was made by Mr. W. Simms, jun., and a part by Mr. W. Ellis (of the Royal Observatory). On comparing the results for ten bars, it was found that in every instance Mr. Sheepshanks assigned to these bars a greater length (in reference to Bronze 12) than was given by Mr. W. Simms. The mean value of the discordance was about $0^{in}\cdot000054$; by which, apparently, Mr. Sheepshanks read the length of Bronze 12 shorter than Mr. W. Simms read it. Mr. Sheepshanks and Mr. W. Ellis agreed nearly; and Mr. Sheepshanks and Mr. W. Simms agreed nearly when they observed other bars. It was thus that the suspicion of personal equation, in the observation of lines apparently well cut, first arose: and it appeared evident that the divisions of Bronze 12 were peculiarly liable to this personal equation. For this reason, in addition to others mentioned above, Bronze 12 was rejected as Basis. Bronze 28 was adopted in its stead, and was retained as Basis in all the subsequent observations.

Section VI.—*Operations of important character to the end of* 1853: *comparisons of a small number of bars with Bronze* 28; *investigation of Personal Equation; investigation of the effect of Inside or Outside position of the Bar; investigation of the relative thermometric expansion of Steel, Wrought Iron, Cast Iron, Copper, and Brass, as compared with Bronze; trial and rejection of* Baily's *Apparatus.*

Observations made in the year 1850 had led Mr. Sheepshanks to think it possible that a very small error might depend on the position of the bars as near to or far from the observer. On examining the plans of the comparing apparatus, it will be seen that this could arise in no other way than the following: that the travelling carriage is not exactly in the same position for the observation of the two bars, that the foundation stones will therefore be differently pressed in the two cases, and that therefore there may be (in some way not easily conceived) a strain of the transom bars which carry the microscopes. The first experiments seemed to support this idea; but the more complete experiments of 1852 practically negatived it. The following Table contains the epitome of the observations made at the end of 1850 and the beginning of 1851. Each set of observations is the mean of five comparisons. The excess is in every case increased by $1^{in}\cdot0000$, in order to make it always positive. The computations of means and of probable errors were made by Mr. W. Ellis at the Royal Observatory.

Bars compared with Bronze 28.	Position of Bar Inside or Outside.	Number of sets of observations.	Excess of Bar above Bronze 28 increased by 1·0000.	Probable Error of one set.	Probable Error of Mean.
			r.	r.	r.
Bronze 1	Inside.	4	1·0236	0·004406	0·002203
Bronze 1	Outside.	4	1·0180	0·004088	0·002044
Bronze 2	Inside.	12	0·9986	0·004292	0·001239
Bronze 2	Outside.	12	0·9935	0·004367	0·001261
Bronze 7	Inside.	12	1·0001	0·003359	0·000970
Bronze 7	Outside.	12	0·9977	0·003307	0·000955
Bronze 11	Inside.	8	1·0219	0·008207	0·002902
Bronze 11	Outside.	8	1·0172	0·003534	0·001336
Bronze 12	Inside.	16	1·0137	0·003807	0·000952
Bronze 12	Outside.	16	1·0128	0·004123	0·001031
Bronze 31 (reversed)...	Outside.	16	1·0250	0·003929	0·000982
Bronze 15	Inside.	27	0·9737	0·004888	0·000978
Bronze 15	Outside.	18	0·9678	0·004795	0·001130
Bronze 20	Inside.	12	1·0105	0·003013	0·000870
Bronze 20	Outside.	12	1·0040	0·004775	0·001378

Remarking that the probable error will be the same for the Mean Excess above Bronze 28 and for the Increase of Excess depending upon the Compared Bar being Inside, the results from these observations are the following:—

Name of Bar.	Mean Excess above Bronze 28.	Increase of Excess when compared Bar is Inside.	Probable Error.
	r.		r.
Bronze 1	1·0208	+0·0028	0·00150
Bronze 2	0·9960	+0·0025	0·00088
Bronze 7	0·9989	+0·0012	0·00068
Bronze 11	1·0195	+0·0023	0·00160
Bronze 12	1·0132	+0·0004	0·00070
Bronze 15	0·9708	+0·0030	0·00075
Bronze 20	1·0072	+0·0032	0·00081

These results undoubtedly support the idea that the bar appears too long when in the Inside position by a quantity not differing much from $0^{r}\cdot0022 = 0^{in}\cdot000008$.

Much time was employed, in the latter part of 1851 and the beginning of 1852, in the observations of numerous bars, the results of which will be given hereafter. In many of these observations Mr. W. Simms, jun. took a part, and thus there was frequent comparison of the results given by two different observers. In the autumn of 1852, the feeling of insecurity, with regard to personal equation and the influence of the bar's position, pressed so strongly on Mr. Sheepshanks, that, after consulting with me, he determined on making an extensive series of observations with the assistance of several observers who were familiar with micrometrical observations, and whose accuracy could not be questioned. The persons who thus aided Mr. Sheepshanks were, Mr. Henderson and Mr. Dunkin of the Royal Observatory, Mr. Simms, Mr. W. Simms, jun., and Mr. Warren De la Rue. The following Table gives the immediate results of the operation. The element exhibited is the apparent excess of each bar above Bronze 28.

except in the last division of the Table, where Bronze 7 is the standard of reference. The "division" is one-hundredth part of a revolution, or $0^{in}\cdot0000358$. Each "observation" is the mean of three comparisons.

Excess of each Bar above Bronze 28.

Name of Bar.	Observer.	Position of Bronze 28.	Excess when Bronze 28 is outside.	Excess when Bronze 28 is inside.	Number of observations.	Probable Error of Mean.	Symbol for Excess, independent of position but dependent on observer.
			d.	d.		d.	
Bronze 10	Sheepshanks	Outside.	−1·382		78	0·0643	x_3
		Inside.		−1·131	64	0·0642	
	Henderson	Outside.	−1·830		20	0·1396	x_4
		Inside.		−1·735	20	0·1528	
	Dunkin	Outside.	−1·515		20	0·1196	x_6
		Inside.		−1·240	10	0·0940	
	Simms	Outside.	−0·788		8	0·1593	x_8
		Inside.		−1·669	16	0·1138	
	W. Simms, jun.	Outside.	−1·656		16	0·0849	x_{10}
		Inside.		−2·350	8	0·1168	
	De la Rue	Outside.	−1·575		8	0·1696	x_{12}
		Inside.		−1·714	14	0·1419	
Bronze 19	Sheepshanks	Outside.	+0·213		32	0·0989	x_{14}
		Inside.		+0·119	32	0·0976	
	Henderson	Outside.	+0·370		10	0·1913	x_{16}
		Inside.		−0·430	10	0·1661	
	Dunkin	Outside.	−0·400		10	0·1456	x_{18}
		Inside.		+0·010	10	0·1473	
	Simms	Outside.	+0·288		8	0·1624	x_{20}
		Inside.		+0·288	8	0·0919	
	W. Simms, jun.	Outside.	−0·425		4	0·1458	x_{22}
		Inside.		−0·400	4	0·0634	
	De la Rue	Outside.	−0·413		8	0·1217	x_{24}
		Inside.		+0·400	8	0·1785	
Bronze 20	Sheepshanks	Outside.	+0·514		28	0·0996	x_{26}
		Inside.		+0·107	28	0·1043	
	Henderson	Outside.	+1·040		10	0·2246	x_{28}
		Inside.		−0·040	10	0·2192	
	Simms	Outside.	+0·550		8	0·1365	x_{30}
		Inside.		+0·363	8	0·0908	
	W. Simms, jun.	Outside.	+0·750		4	0·0773	x_{32}
		Inside.		+0·900	4	0·0924	
	De la Rue	Outside.	+0·670		6	0·0895	x_{34}
		Inside.		+0·600	6	0·1947	
Bronze 2	Sheepshanks	Outside.	−0·495		38	0·0862	x_{36}
		Inside.		−0·187	30	0·0892	
	Henderson	Outside.	−0·745		20	0·1444	x_{38}
		Inside.		−0·805	20	0·1281	
	W. Simms, jun.	Outside.	−1·530		4	0·1336	x_{40}
		Inside.		−2·275	4	0·0904	
	De la Rue	Outside.	−0·600		4	0·0411	x_{42}
		Inside.		−0·880	4	0·3095	
Bronze 7	Sheepshanks	Outside.	+0·506		32	0·1133	x_{44}
		Inside.		+1·750	32	0·0881	
	W. Simms, jun.	Outside.	−0·500		4	0·0985	x_{46}
		Inside.		−0·620	4	0·0965	

TABLE (continued).

Excess above Bronze 7.

Name of Bar.	Observer.	Position of Bronze 7.	Excess when Bronze 7 is outside.	Excess when Bronze 7 is inside.	Number of observations.	Probable Error of Mean.	Symbol for Excess, independent of position but dependent on observer.
			d.	d.		d.	
Bronze 10	SHEEPSHANKS.....	Outside.	−2·711		28	0·1008	$x_3 - x_{44}$
		Inside.		−2·142	24	0·0973	
	HENDERSON	Outside.	−1·210		10	0·1454	x_{48}
		Inside.		−1·360	10	0·1744	
	DUNKIN	Outside.	−2·390		10	0·1653	x_{50}
		Inside.		−1·630	10	0·2988	
	W. SIMMS, jun. ...	Outside.	−0·350		4	0·0506	$x_{10} - x_{46}$
		Inside.		−1·550	4	0·1728	

It must be remarked that the "probable error" in the Table is the probable chance-error in the mean of observations of the bar under given circumstances, derived in the usual way from the discordances of the observations under those circumstances, and not applying in any degree to a constant error of the observer or a constant error depending on position.

The results were now treated in the following way:—

From examination of Mr. SHEEPSHANKS's probable errors, it appeared that the probable error of a single "observation" by Mr. SHEEPSHANKS is about $0^d{\cdot}55$. This quantity will be called E.

On comparing the probable errors of the other observers with those of Mr. SHEEPSHANKS, it appeared that their proportions are nearly as follows:—

HENDERSON	: SHEEPSHANKS	:: 9 : 8
DUNKIN	: SHEEPSHANKS	:: 1 : 1
SIMMS	: SHEEPSHANKS	:: 3 : 4
W. SIMMS, jun.	: SHEEPSHANKS	:: 5 : 8
DE LA RUE	: SHEEPSHANKS	:: 6 : 7

It appeared from these that no sensible error would be committed by assuming the observations of all the observers to be equally good, except those of W. SIMMS, jun., one of whose observations is equivalent to two of any other observer. (This estimate agrees entirely with the high opinion of Mr. W. SIMMS's accuracy which Mr. SHEEPSHANKS had often expressed to me long before this time.) In other words, the probable error of one of Mr. W. SIMMS's observations will be $\frac{E}{\sqrt{2}}$, that for any other observer being E. And, to make the probable error, of the equation formed for the mean of n observations, equal to E, the equation must be multiplied, for observers in general, by $\sqrt{n}$: but for Mr. W. SIMMS, jun. it must be multiplied by $\sqrt{2n}$.

Now let x_3 be Mr. SHEEPSHANKS's value of the excess of Bronze 10 above Bronze 28; y the increase in this value produced by the circumstance of Bronze 28 being Outside

(and consequently $-y$ the increase when Bronze 28 is Inside), y being independent of the Observer and of the Bar.

Then the equations formed from the means of Mr. SHEEPSHANKS's two sets of comparisons of Bronze 10 with Bronze 28 are,—

$$x_3+y=-1\cdot382$$
$$x_4-y=-1\cdot131,$$

these equations being subject to the *probable* errors $\frac{E}{\sqrt{78}}$, $\frac{E}{\sqrt{64}}$.

Or

$$\sqrt{78}.x_3+\sqrt{78}.y=-\sqrt{78}\times1\cdot382$$
$$\sqrt{64}.x_4-\sqrt{64}.y=-\sqrt{64}\times1\cdot131,$$

where each equation is subject to the *probable* error E. In this shape, the equations are fit for use in determining the most probable values of x_3, x_4, &c., and y.

But we shall afterwards have need to determine the probable errors of the values of x_3, x_4, &c., y, so found. For this purpose, it will be most convenient to introduce here the consideration of the *actual* error in each equation. Let then e_1, e_2 be the *actual* errors in Mr. SHEEPSHANKS's first and second mean result. Then the following equations are strictly correct:—

$$\sqrt{78}.x_3+\sqrt{78}.y=-\sqrt{78}\times1\cdot382+\sqrt{78}.e_1$$
$$\sqrt{64}.x_4-\sqrt{64}.y=-\sqrt{64}\times1\cdot131+\sqrt{64}.e_2$$

and similar equations for the other observers and the other bars. The first and last pairs of equations for the comparison of Bronze 10 with Bronze 7 will be—

$$\sqrt{28}.x_3-\sqrt{28}.x_{44}+\sqrt{28}.y=-\sqrt{28}\times2\cdot711+\sqrt{28}\times e_{31}$$
$$\sqrt{24}.x_3-\sqrt{24}.x_{44}-\sqrt{24}.y=-\sqrt{24}\times2\cdot142+\sqrt{24}\times e_{32}$$
$$\sqrt{8}\ .x_{10}-\sqrt{8}\ .x_{46}+\sqrt{8}\ .y=-\sqrt{8}\ \times0\cdot350+\sqrt{8}\ \times e_{33}$$
$$\sqrt{8}\ .x_{10}-\sqrt{8}\ .x_{46}-\sqrt{8}\ .y=-\sqrt{8}\ \times1\cdot550+\sqrt{8}\ \times e_{34}.$$

And from all this assemblage of equations, new equations equal in number to the number of unknown quantities are to be formed according to the rules of the Calculus of Probabilities.

The equations are for the most part very simple, those only which relate to x_3, x_{10}, x_{44}, x_{46}, and y, presenting any complexity. I shall only give details of the following, as specimens of the form and the treatment of the whole.

By solution of the third and fourth equations, x_4 is found $=-1\cdot78+\frac{1}{2}e_3+\frac{1}{2}e_4$. This expression is strictly accurate, but it cannot be used immediately because e_3 and e_4 (the *actual* errors) are not known. But the *probable* values of e_3 and e_4 are $\frac{E}{\sqrt{20}}$, $\frac{E}{\sqrt{20}}$: the *probable* values of $\frac{1}{2}e_3$ and $\frac{1}{2}e_4$ are therefore $\frac{E}{\sqrt{80}}$ and $\frac{E}{\sqrt{80}}$; and therefore (by the theorems of Probabilities) the *probable* value of the combination of these is

$$\sqrt{\left\{\frac{E^2}{80}+\frac{E^2}{80}\right\}}=\frac{E}{\sqrt{40}}.$$

Hence the value of x_4 is $-1{\cdot}78$ with a probable error $\frac{E}{\sqrt{40}}$ or $0^d{\cdot}088$.

The expressions for x_7, x_{10}, x_{40}, x_{43}, found from solution of the equations, contain six terms each, multiplying e_1, e_2, e_{13}, e_{14}, e_{31}, e_{32}. The expression for y contains fifty-four such terms. In each of these cases, the *probable* value of each term is to be set down separately (not combined with the others by numerical addition or subtraction), the square of each value is to be formed, the sum of the squares is to be taken, and the square root of that sum will be the *probable error* of the quantity under treatment. In this manner, the Results included in the following Table are formed:—

Abstract of results.

y	Apparent increase in measure of referred bar, when the referee is outside ..... d. $-0{\cdot}018$	Probable Error... d. $0{\cdot}018$.	
Excesses of the Lengths of Bars above Length of Bronze 28.			
	Name of Bar.	Observer.	Excess.	Probable Error.
			d.	d.
x_3	Bronze 10	Sheepshanks.....	$-1{\cdot}28$	$0{\cdot}042$
x_4		Henderson	$-1{\cdot}78$	$\cdot088$
x_6		Dunkin	$-1{\cdot}41$	$\cdot101$
x_8		Simms	$-1{\cdot}39$	$\cdot079$
x_{10}		W. Simms, jun.	$-1{\cdot}83$	$\cdot074$
x_{12}		De la Rue.....	$-1{\cdot}66$	$\cdot117$
x_{14}	Bronze 19	Sheepshanks.....	$+0{\cdot}17$	$\cdot069$
x_{16}		Henderson	$-0{\cdot}03$	$\cdot123$
x_{18}		Dunkin	$-0{\cdot}20$	$\cdot123$
x_{20}		Simms	$+0{\cdot}29$	$\cdot097$
x_{22}		W. Simms, jun.	$-0{\cdot}41$	$\cdot138$
x_{24}		De la Rue.....	$-0{\cdot}01$	$\cdot138$
x_{26}	Bronze 20	Sheepshanks.....	$+0{\cdot}31$	$\cdot074$
x_{28}		Henderson	$+0{\cdot}50$	$\cdot123$
x_{30}		Simms	$+0{\cdot}46$	$\cdot097$
x_{32}		W. Simms, jun.	$+0{\cdot}83$	$\cdot138$
x_{34}		De la Rue.....	$+0{\cdot}63$	$\cdot159$
x_{36}	Bronze 2	Sheepshanks.....	$-0{\cdot}36$	$\cdot067$
x_{38}		Henderson	$-0{\cdot}78$	$\cdot087$
x_{40}		W. Simms, jun.	$-1{\cdot}90$	$\cdot138$
x_{42}		De la Rue.....	$-0{\cdot}74$	$\cdot195$
x_{44}	Bronze 7	Sheepshanks.....	$+1{\cdot}15$	$\cdot055$
x_{46}		W. Simms, jun.	$-0{\cdot}72$	$\cdot104$
Excess of the Length of Bar above Length of Bronze 7.				
x_{48}	Bronze 10	Henderson	$-1{\cdot}29$	$\cdot123$
x_{50}		Dunkin	$-2{\cdot}01$	$\cdot123$
Excess of the Length of Bar above Length of Bronze 28 through the intermediation of Bronze 10.				
x_4-x_{48}	Bronze 7	Henderson	$-0{\cdot}49$	$\cdot150$
x_6-x_{50}		Dunkin	$+0{\cdot}60$	$\cdot159$

It seems difficult here to resist the conclusion that there is a very sensible personal equation. The difference between the results of different observers for the same bar does in every instance greatly exceed any amount of discordance which could be inferred as probable from the combination of probable errors. It is to be particularly remarked that (so far as I can trace in the observations) all the observations of any one bar Inside were taken by all the observers without disturbing the bar, and then all the observations of the same bar Outside were taken by all the observers without disturbing the bar, or *vice versâ*.

On comparing the value found for y with its probable error, it does not appear certain that there is any difference between results with Bar Inside and Bar Outside.

The best way of combining the results of the different observers, with due attention to the circumstance that the chance-error of each observer's result is far below their constant difference, appears to be, to take the simple arithmetical mean. Thus we obtain—

	d.
Bronze 10 = Bronze 28	−1·56
Bronze 19 = Bronze 28	−0·03
Bronze 20 = Bronze 28	+0·55
Bronze 2 = Bronze 28	−0·95
Bronze 7 = Bronze 28	+0·14

In the year 1853, nearly the whole time from March to September was occupied with experiments on the relative expansions of bars of several metals; Bronze 28 being in all cases the standard of reference. The principal part of the observations was made by Mr. SHEEPSHANKS: some however were made by Mr. W. SIMMS, jun. I am aware that these observations were made with very great care, although I cannot state every particular of the cautions which were employed. The temperature was given by the temperature of the water in the external trough; and I find that, when the temperature had been altered by pouring in hot water, &c., no observations were made until six or more hours after the alteration. In some cases, the trough was wrapped in blankets during the night. The readings of L and R were converted into degrees of FAHRENHEIT by the Table made from the last comparisons of these thermometers. The observations were subsequently reduced by Mr. W. ELLIS at the Royal Observatory.

Each "observation," as before, is the mean of three comparisons; and one revolution $= 0^{in}\cdot003587$.

It will be sufficient to exhibit, as a specimen, the reduction of the observations of one bar; and for that purpose I will take Swedish Iron B.

Excess of Swedish Iron B above Bronze 28.

		r.
Sum of excesses in 24 observations at mean temperature	40°·972	= +17·0053
Sum of excesses in 28 observations at mean temperature	59·313	= + 2·6529
Sum of excesses in 16 observations at mean temperature	63·215	= − 0·5738
Sum of excesses in 16 observations at mean temperature	68·143	= − 3·2412

Let x be the excess of Swedish B above Bronze 28 at the temperature 0°; y the relative expansion for 1°. Then the first line gives the equation

$$24 \times x + 983{\cdot}33 \times y = +17{\cdot}0053,$$

and so for the others. These equations, assumed to be of equal value, were treated by the method of least squares, and the result was

$$x = +2{\cdot}081441$$
$$y = -0{\cdot}03350280.$$

On substituting these values in the four equations, the residual errors are

$$-0{\cdot}0050, \quad +0{\cdot}0127, \quad +0{\cdot}0092, \quad -0{\cdot}0168,$$

which prove that there are no important errors of observation or reduction.

To find the excess of Swedish Iron B at temperature 62°, we must form $x + y \times 62$. Its value is $+0^{r}{\cdot}004267$.

And, as the relative expansion is $-0^{r}{\cdot}0335$ or $-0^{in.}{\cdot}000120$: if to this we apply the absolute expansion of Bronze $+0^{in.}{\cdot}000341$ (as found above), the absolute expansion of Swedish Iron B for 1° of FAHRENHEIT is $+0^{in.}{\cdot}000221$. This applies to the length 36 inches.

In all the other instances, the range of temperature is almost exactly the same as for Swedish Iron B, but the number of observations is greater. For Cast Steel B the number is 148.

The following is an Abstract of Results:—

Name of Bar.	Excess above Bronze 28 at 62° FAHRENHEIT.	Relative expansion of 36 inches for 1° FAHRENHEIT.
	r.	r.
Cast Steel B	−0·006565	−0·03730370
Low Moor B	+0·005113	−0·03356648
Swedish B	+0·004267	−0·03350280
Cast Iron C	−0·018638	−0·03979733
Brass 2	+0·082740	+0·00072975
DONKIN's Copper	+0·011634	−0·00743411

About 1853, October 26, Mr. SHEEPSHANKS introduced the use of the "block-and-stirrup" mounting, and I believe that after this time the system of floating the bars in quicksilver was rarely or never used.

Between 1853, August 20, and December 24, Mr. SHEEPSHANKS made numerous comparisons of several bars, by the use of BAILY's Apparatus (the comparing apparatus of the Royal Astronomical Society). He was however much dissatisfied with the results, and never reduced them to the state of Abstracts. In a letter to myself, dated 1854, Jan. 2, Mr. SHEEPSHANKS says, "There is a clear difference in the comparisons depending on the position of the bars: the inside bar is always the longest. But this difference varies most lawlessly, from 3^{d} to 10^{d}," and then gives the means of comparisons on four days, each mean being derived from 18 observations, as follows:—

	d.	d.	d.	d.
Brass 2 farthest—Kew Bar nearest	−1·93	− 0·97	−2·42	−4·02
Brass 2 nearest—Kew Bar farthest	+9·20	+10·05	+7·35	+4·10

SECTION VII.—*Comparisons of numerous bars from* 1851 *to* 1855; *suspicion of change in Bronze* 28; *removal of the suspicion; close of the operations for defining the Yard by line-measure.*

I have interrupted the chronological order, for the sake of bringing together in the last Section the observations of fundamental character, and in the present Section the mere determinations of the length of numerous bars.

In a former Section I gave the differences found by Mr. SHEEPSHANKS between Bronze 12 and the following bars (among others)—

Bronze 6
8
18
21
22
23
24
25
26
27
29
and also Bronze 28

These bars were subsequently compared with Bronze 28, and they therefore afforded very good means of comparing the two fundamental bars Bronze 12 and Bronze 28. Their excesses above Bronze 28 were as follows:—

	d.		d.	
Bronze	6 +2·90	Bronze	24 +3·12	
	8 +1·73		25 +4·77	
	18 −3·32		26 +4·47	and direct comparisons had
	21 +2·22		27 +1·84	given for Bronze 12+1^{d}·55.
	22 +2·62		29 +3·71	
	23 +3·12			

Combining these differences with the differences of the same bars from Bronze 12, inferring from each the value of Bronze 12 with respect to Bronze 28, and taking the mean of all, with fivefold weight for the direct comparison,

Bronze 12=Bronze 28+1^{d}·84.

This value was used for connecting the inferred lengths (from comparison with Bronze 12) with the inferred lengths (from comparison with Bronze 28): then, in all cases where both bases had been used, a general mean was taken.

Thus by comparison, sometimes with one base, sometimes with two, the following differences from Bronze 28 were determined. For convenience of reference, I have included in the same Table the differences determined in the last Section.

Bronze Bars.

	d.		d.
Bronze 1	+2·08	Bronze 23	+2·21
2	−0·95	24	+3·02
3	−2·28	25 (redivided)	+0·25
4	+4·63	26	+5·05
5	+0·52	27	+1·79
6	+2·58	28	(0·00)
7	+0·14	29	+4·60
8	+1·70	30	+3·24
9	+7·14	31	−0·34
10	−1·56	32	−0·36
11	+1·96	33	+0·71
12	+1·84	34	−0·25
13	+1·35	35	+0·47
14	+3·85	36	−0·75
15	−2·92	37	−0·62
16	+0·56	38	−1·33
17	−1·30	39	+1·21
18	−2·48	40	+0·07
19	−0·03	Baily's A	−3·50
20	+0·55	B	+3·09
21	+1·86	C	+4·05
22	+2·40	D	+3·01

Brass Bars.

(Brass 2 is a fundamental bar, whose value was obtained in a preceding Section. The Kew Bar was compared with Brass 2 and Bronze 28: all the others with Bronze 28 only. The Flat Brass Scale appears to have been compared on Baily's apparatus: all the others, I believe, with Mr. Sheepshanks's apparatus.)

	d.		d.
Brass 1	+0·07	Royal Astronomical Society's Tubular	+11·67
2	+8·27	Danish Tubular	+12·01
Flat Brass Scale	−0·12	Simms's Tubular 4	+13·88
Kew Bar	+1·54	Baily's Tubular 5	+19·42
New Tubular Scale	−10·27		

Copper Bars.

(Donkin's is in the last Section. Maudslays' and May's were compared with Bronze 12.)

	d.
Donkin's Copper	+1·16
Maudslays' Copper	+1·60
May's Alloy	+4·15

Iron and Steel Bars.

(Swedish B, Low Moor B, Cast Iron C, and Cast Steel B, are in the last Section.)

		d.
Swedish	A, by comparison with {Bronze 12, Bronze 28}	−9·50
Swedish	B	+0·43
Swedish	C, by comparison with Low Moor B	+1·43
Low Moor	A, by comparison with {Bronze 28, Low Moor B}	−3·56
Low Moor	B	+0·51
Low Moor	C, by comparison with Low Moor B	+2·16
Low Moor	D, by comparison with Low Moor B	+1·28
Cast Iron	A, by comparison with {Bronze 12, Low Moor B}	−3·08
Cast Iron	B, by comparison with Low Moor B	−4·96
Cast Iron	C	−1·86
Cast Iron	D, by comparison with {Bronze 28, Low Moor B}	−3·52
Cast Steel	A, by comparison with Low Moor B	−1·28
Cast Steel	B	−0·66
Cast Steel	C, by comparison with Low Moor B	−2·70
Cast Steel	D, by comparison with {Swedish B, Low Moor B, Cast Iron C, Cast Steel B}	−3·61

Bars and scales which had formerly been compared with the Imperial Standard.

		d.
Ordnance 1 A, by comparison with	Bronze 12	− 9·12
	Brass 2	− 7·63
	Swedish A	−11·03
	Low Moor A	−10·19
	Low Moor B	− 8·57
Ordnance 2 A, by comparison with	Bronze 12	− 3·87
	Brass 2	+ 1·14
	Swedish A	−12·31
	Low Moor A	− 4·58
	Low Moor B	− 4·74
Kater's Royal Society Standard, by comparison with	Bronze 28	−20·78
	Bronze 12	−15·11
	Brass 2	−14·61
Shuckburgh's Scale, 0—36, by comparison with	Brass 2	+ 8·43
Shuckburgh's Scale, 10—46, by comparison with	Brass 1	+ 4·09
	Brass 2	+ 2·29

Ancient Bars.

		d.
Henry VII.,	by comparison with Brass 2	−1041·34
Elizabeth,	by comparison with Brass 2	− 285·93
Excheq.,	by comparison with Brass 2	− 155·01
Tower,	by comparison with Brass 2	+ 53·27
Anonymous,	by comparison with Brass 2	+ 49·43

Mr. SHEEPSHANKS usually investigated the "probable error" of every result. It has appeared to me unnecessary to record them here. They may be understood to range, in general, a little above or a little below $0^{d}\cdot1$, when line-divisions on firm bars were compared.

It will be convenient, in estimating the import of the various numbers given above, that the reader should remember the following values:—

$1^{d}\cdot00 = 0^{in.}\cdot000036$ nearly = one-millionth part of the yard-measure.

$1^{d}\cdot00$ = Expansion of 36 inches of Bronze for $0^{\circ}\cdot10$ FAHRENHEIT.

= Expansion of 36 inches of Iron for $0^{\circ}\cdot17$ FAHRENHEIT.

For accurate reduction, the following thermometric expansions of 36 inches for 1° FAHRENHEIT are to be used:—

	d.	in.
Bronze	9·507	= 0·000341
Brass	9·580	= 0·000344
Copper	8·764	= 0·000314
Wrought Iron	6·153	= 0·000220
Cast Iron	5·527	= 0·000198
Cast Steel	5·777	= 0·000207

In the statements of results of comparison, it is always supposed that both bars are at the temperature 62° FAHRENHEIT, unless a temperature is stated.

Between 1855, March 15 and April 3, Mr. SHEEPSHANKS made three careful series of observations for ascertaining the effect of flexure on the measure of a bar. Bronze D was compared with Bronze 28, the circumstances of Bronze D being varied as follows:—1st. Bronze D was supported on block-and-stirrup, the points of support being those indicated by theory: 2nd. Bronze D was supported in the middle, a trifling preponderance being given to one end in order to prevent oscillation: 3rd. Bronze D was supported by block-and-stirrup at the extreme ends. The position of Bronze 28 was not varied. The results for the excess of Bronze D above Bronze 28 were—

	d.
1st series, by 96 observations	+3·01
2nd series, by 72 observations	+3·03
3rd series, by 72 observations	+3·06

It is sufficiently clear that the effect of flexure is insensible.

I proceed now to allude to a discordance which was a source of great anxiety to Mr. SHEEPSHANKS.

In April 1855, Mr. SHEEPSHANKS was engaged in measuring the bar Cast Steel D. By comparisons with four iron bars (as is stated in the Table above), whose results agreed very closely, the excess of Cast Steel D above Bronze 28 was found to be $-3^{d}\cdot61$. But a direct comparison of Cast Steel D with Bronze 28, immediately preceding, had given $-0^{d}\cdot46$. This comparison was made at the temperature $45^{\circ}\cdot54$, or $16^{\circ}\cdot46$ below the standard temperature. A trifling error of expansion might account for part of the discordance; and ordinary errors of observation might account for part. But in the opinion of Mr. SHEEPSHANKS, though the whole discordance scarcely exceeded the effect of the thermometric expansion of Bronze 28 for $0^{\circ}\cdot3$ FAHRENHEIT, it was impossible so to explain away the whole or a large part of it; and he was convinced that Bronze 28 had sensibly shortened. And so deeply and so painfully was this impression fixed in his mind, that he actually contemplated the rejection of all the results which had cost so many years of labour, and the commencing the work *de novo*.

The first conjecture on the possible cause of the apparent change was, that Bronze 28 while covered with gold-beater's skin and cement (as in the earlier comparisons) might have been so far constrained by that covering that it could not shrink down to its natural length; but that in the last comparisons with Cast Steel D, when that covering had been removed, it had contracted itself. In order to test this, Mr. SHEEPSHANKS immediately made measures of Bronze 12, Bronze 29, Bronze A, and Bronze B, whose coverings had not been removed; then he had the covering removed, and repeated the measures. The following is a comparison of these measures:—

Excess above Bronze 28.

			d.
Bronze 12, before cleaning,	1855 April	26	+4·50
after cleaning,	May	1	+4·77
Bronze 29, before cleaning,	May	5	+3·70
after cleaning,	May	8	+4·28
Bronze A, before cleaning,	May	9	−2·36
after cleaning,	May	11	−2·46
	June	5 to 9	−2·51
Bronze B, before cleaning,	May	10	+4·15
after cleaning,	May	12	+4·51
	May	31	+3·57
	June	1 and 2	+4·59

It seems sufficiently evident that nothing depends on the covering.

Numerous observations were then made, partly by Mr. SHEEPSHANKS himself and partly by a young assistant, on twenty-seven bars (including the four just mentioned) which had been previously well compared with Bronze 28. The result is exhibited in the following Table:—

Excess above Bronze 28.

Bar.	Old measure.	New measure.	Bar.	Old measure.	New measure.
	d.	d.		d.	d.
Bronze 12	+1·84	+4·50	Bronze 14	+3·85	+3·69
		+4·77			+2·98
Bronze 39	+1·21	+0·45			+3·46
Bronze 15	−2·92	−1·63	Bronze 23	+2·21	+2·50
		+2·58			+1·76
Bronze 11	+1·96	+2·66			+2·87
Bronze 30	+3·24	+3·01	Bronze 21	+1·86	+4·51
		+3·70			+4·80
Bronze 29	+4·60	+4·28	Bronze 9	+7·64	+8·06
Bronze 1	+2·08	+4·05	Bronze 22	+2·40	+3·40
		−2·36	Bronze 3	−2·28	−1·56
Bronze A	−3·50	−2·46			+0·04
		−2·51	Bronze 6	+2·38	+3·89
		+4·15	Bronze 26	+5·05	+4·75
Bronze B	+3·09	+4·51	Bronze 27	+1·79	+1·31
		+3·57			+0·91
		+4·59	Bronze 25	+0·25	+0·10
Bronze 16	+0·56	−0·65			−1·11
Bronze 18	−2·48	−1·10			+2·87
		+0·17	Bronze 13	+1·35	−0·04
Bronze 5	+0·52	−0·20			+2·25
		+0·58			+1·90
Bronze 4	+4·63	+4·83			+4·96
Bronze 8	+1·70	+2·75	Bronze 24	+3·02	+2·88
		−0·07			+4·98
Bronze 17	−1·30	−0·92			+3·96
		−1·28			
		−1·84			

The order in which I have placed the bars is the same as the chronological order of the first of the New Observations on each bar. On examining them it will be found,—First, that there is no evidence whatever of a general preponderance of excess from the New Measures above excess from the Old Measures; the signs + and − being intermixed, in the differences, in all possible ways; and the mean of the whole being less than 0ᵈ·50. Secondly, that the only instance which fairly supports the conclusion deduced from Cast Steel D is the first of all, namely, Bronze 12. Cast Steel D was compared on April 13, 14, and 16; Bronze 12 on April 26, 27, 28, and May 1; Bronze 39 (the next) on April 30. The conclusion, I think, is inevitable, that Bronze 28 really was shortened at the beginning of April; that it recovered its exact length before April 30; but that this recovery took place with some fluctuations, so that on May 1 it was subject nearly to the same error as before. Bronze 21, observed on June 26, exhibits a similar discordance. What circumstances can have produced these changes, or how far the later fluctuations are apparent rather than real, I am wholly unable to conjecture.

I know not whether Mr. SHEEPSHANKS intended to continue this series of comparisons. Those which are given are clearly sufficient to establish the general trustworthiness of all the measures; and Mr. SHEEPSHANKS, as they advanced, had expressed himself in a great measure satisfied.

I have now to give the melancholy termination of this part of the work.

To 1855, July 14 inclusive, the whole of Mr. SHEEPSHANKS's observations, so far as I can discover, are transferred from the Memorandum Book to the Reduction Papers, and the means are taken. The final series of observations are the last two sets of comparisons of Bronze 17 with Bronze 28. After the observations of July 14, the following note is placed in the Memorandum Book:—

"1855, July 25. I went to Reading on the 14th, and felt my head so deranged that I stayed there till Friday 20th, when I came up to arrange my apparatus. I did only a little in the latter affair, but I left pretty full instructions for W. S. [WILLIAM SIMMS, jun.], who has, I believe, carried them out with his usual skill. I told him that the microscopes had not the same scale, and that D would require to be set a trifle lower. He preferred taking a bit off A, and so get them nearer to the same length and to have the focus of distinct vision in the same horizontal line. But it will alter the scale of A."

Then follow a few observations (one page only, with a few subtractions on the next page) which I am not able to interpret with certainty. They are not dated.

Mr. SHEEPSHANKS had returned to London, on some day in the week commencing with Monday, July 23, and had, as I conceive, made some trials of the apparatus, which were registered (for the moment only) in the unintelligible entries to which I have alluded. I understand that he was engaged on the apparatus on Saturday, July 28; but several persons had remarked that he was evidently in great distress. On the afternoon of that day he went to his residence at Reading, carrying with him a large travelling-bag full of papers relating to the standards. In the afternoon of Sunday, July 29, while sitting in family society, he fell speechless from his chair, struck with apoplexy, and expired on August 4.

Thus died—almost in the scene of his labours, and with his thoughts still intent on them—a man whose equal, in talent and perseverance, in disinterestedness, in love of justice and truth, I have scarcely known. He had however brought to a satisfactory termination the great division of the Standard-work which best suited his taste, having well overcome the last of the difficulties which had presented themselves, and leaving the work in such a state that not a single additional comparison of line-measures was required. All that was necessary was, to collect and arrange the papers, to complete some few means and abstracts which he had been unable to finish, and to draw up such an account as I have attempted here. The formation of thermometers was left in an imperfect condition, but (under pressing occupations) I have not felt myself called upon to finish them. The essential object was, to construct accurate copies of the National Standard, which rests upon an arbitrary basis; this work could be done by no persons except those who had daily access to the National Standard. The basis of thermometers is natural, and they can be constructed by one physicist or workman as well as by another. And at the present time, it is very much easier to procure in commerce thermometers of well-established accuracy than it was when Mr. SHEEPSHANKS commenced his labours.

Section VIII.—*Formation of End-measure Bars.*

The form given to the end-bars depended in some degree upon the process adopted for comparing end-measure bars with line-measure bars, which originated, I believe, in a suggestion from myself. Suppose that we have *two* end-bars, each nearly equal in length (as measured between its extreme ends) to the length of the line-bars (as measured from line to line). Suppose also that each of the end-bars has a well sunk to the middle of its depth, with a transverse line cut upon a pin (as in the line-measures); the line being not necessarily at the middle of the length of the bar, though it is essential that its distance from the centre of the bar's length be nearly the same in both end-bars. As, generally, one segment of the bar, from the transverse line to the end, will be longer than the other segment, let the longer segment be denoted by + and the shorter by −. And suppose that the two bars are placed end to end, the + end of one being opposed to the − end of the other, as shown in Figure 11. The distance between their transverse lines will evidently be the sum of the + segment of the first bar and the − segment of the second bar; and the conditions that I have stated, as to the similar division of the lengths of the two bars, will make that sum so nearly equal to the length of the line-measure, that the difference may be measured by the micrometer-apparatus. Then interchanging the positions of the end-bars, so that their other ends shall be in contact, the distance under observation will be the sum of the − segment of the first bar and the + segment of the second bar. Adding together the two measured distances, we have a comparison of the sum of the two end-bars with double the length of the line-bar.

In order to make this determination available, it is necessary further to determine the difference between the whole lengths of the two end-bars. A felicitous suggestion by Mr. W. Simms, jun. gave the means of doing this. Mr. W. Simms's proposal was, to apply the process above described to *three* end-bars, determining the measure of the sum of each pair of end-bars in reference to double the length of the line-bar; then there are given three equations to find three unknown quantities. The length of each of the three end-bars is thus found without difficulty.

The general principle of determining the lengths of the end-bars by the contact of their ends (without the use of supplementary pieces, such as had been employed in all previous determinations) being adopted, it was important to decide on the form of those ends. In the opinion of the Committee, the use of plane ends could not be sanctioned. One difficulty attending them is, the uncertainty as to the removal of particles of dust, &c. from every part, and the certainty that if any such particles remain they will disturb the measure of the bar; while at the same time the act of pressing two bars together, either with or without a slight movement, does not tend to remove the particles. Another difficulty is that, even though we were assured that the ends are perfectly clean, yet we cannot be assured that the bars, when laid in a horizontal position end-to-end, will touch in the whole breadth of their ends; inasmuch as, if the position of one of the bars is slightly erroneous in azimuth, or slightly erroneous in inclination to the horizon,

no force which it is practicable to apply in the mounting of the bars would tend to correct the error. The only way which seems practicable is, to place the bars in the vertical position, one bar standing freely upon the end of the other; but this requires a rather inconvenient arrangement of the microscopes.

These errors and inconveniences are entirely avoided by giving to the ends the form of a spherical surface whose centre is the centre of the division-line at the middle of the bar's length. There is a tendency to thrust away particles of dust from the point of contact; and a small error in the elevation, or inclination, or azimuthal position, of either bar, produces no error on the distance between the centres of the division-lines of the two bars.

Accordingly, about the beginning of 1854, Mr. SHEEPSHANKS arranged an apparatus for grinding the ends of end-bars. The bar was grasped by its middle, in a cramp which turned in jimbals or universal-joint; and a plane surface carrying a grinding powder was placed at the proper distance. The operation requires no explanation.

It was then necessary to decide on the substance which should be used for the hard ends of the bars. The properties required were, hardness, toughness, and susceptibility of polish. After several trials, it was found best to use a very slightly conical plug, of either agate or hardened steel (the reason for subsequently abandoning the hardened steel will be given hereafter), and to fix the plug in the corresponding very slightly conical hole at the end of the bar, by fitting it in when the bar was somewhat heated (below the boiling-point of water) and allowing the metal of the bar to shrink upon it. Then the metal and the plug were ground at the same time, so as to make a flush joint. It was found best to give any final reduction of length by a polishing-touch of the hand.

Mr. SHEEPSHANKS had commenced the preparation of some end-bars; but this process, in which a bar if too much shortened is injured beyond the possibility of restoration except by inserting a new plug, is very tedious. This was more particularly felt, when the grinding and polishing were effected in Mr. SIMMS's workshop, and the optical test was applied at Somerset House. No efficient progress was therefore made, up to the time of Mr. SHEEPSHANKS's decease. On then revising the condition of the work, and after having consulted with Professor MILLER, I requested Mr. SIMMS to undertake the whole operation of testing and adjusting the bars in his own workshops, and finally of making the definitive comparisons with Bronze 28 on Mr. SHEEPSHANKS's apparatus at Somerset House.

The bars first tried were those prepared by Mr. SHEEPSHANKS, some having agate ends, some having ends of hardened steel. It happened that an end of agate and an end of hardened steel were left in contact for about a day, when it was found that the distance between their division-lines had increased about $\frac{1}{1000}$th of an inch. Upon examining the end of the steel plug, it was found that the proper point of contact and a disk about $\frac{1}{10}$th of an inch in diameter was clean; but an annulus surrounding it, of the breadth perhaps of $\frac{1}{16}$th of an inch, was thickly rusted; and that the rust had acted like a wedge, forcing the bars asunder in opposition to a spring-pressure of about 3 lbs. It

would seem that the steel and agate formed a galvanic combination, like zinc and copper; and that the air, or its moisture, then acted energetically upon the steel. The steel plugs were then withdrawn, and agate plugs were substituted, and no further difficulty was experienced.

In the definitive comparisons at Somerset House, the bars were always supported on lever-frames, in the manner shown in figures 11 and 12.

The following is an abstract of the comparisons:—

Bronze End-bars called No. 45, No. 69, and No. 74 compared with Bronze 28. Observations by Mr. W. SIMMS, jun. and Mr. JAMES SIMMS. One division=$0^{in.}\cdot0000359$.

$\overset{+}{69}$ with $\overset{-}{74}$, by 48 observations, exceed Bronze 28 by $-13{\cdot}66^{d.}$

$\overset{+}{74}$ with $\overset{-}{69}$, by 48 observations, exceed Bronze 28 by $+17{\cdot}67$

Therefore No. 74+No. 69 exceed 2×Bronze 28 by $+4^{d.}{\cdot}01$.

$\overset{+}{69}$ with $\overset{-}{45}$, by 48 observations, exceed Bronze 28 by $-\ 1{\cdot}43$

$\overset{+}{45}$ with $\overset{-}{69}$, by 48 observations, exceed Bronze 28 by $+\ 2{\cdot}79$

Therefore No. 69+No. 45 exceed 2×Bronze 28 by $+1^{d.}{\cdot}36$.

$\overset{+}{45}$ with $\overset{-}{74}$, by 48 observations, exceed Bronze 28 by $-14{\cdot}28$

$\overset{+}{74}$ with $\overset{-}{45}$, by 48 observations, exceed Bronze 28 by $+13{\cdot}37$

Therefore No. 45+No. 74 exceed 2×Bronze 28 by $-0^{d.}{\cdot}91$.

Solving the three equations,

Bronze End-bar No. 45=Line-measure Bronze 28$-1{\cdot}78^{d.}$
Bronze End-bar No. 69=Line-measure Bronze 28$+3{\cdot}14$
Bronze End-bar No. 74=Line-measure Bronze 28$+0{\cdot}87$

End-bars called Steel 72, Steel 73, and Iron 70 compared with Line-measure Steel B.

$\overset{+}{73}$ with $\overset{-}{72}$, by 48 observations, exceed Steel B by $+5{\cdot}31^{d.}$

$\overset{+}{72}$ with $\overset{-}{73}$, by 48 observations, exceed Steel B by $-3{\cdot}92$

Therefore Steel 73+Steel 72 exceed 2×Steel B by $+1^{d.}{\cdot}39$.

In these comparisons, the temperature is unimportant.

$\overset{+}{70}$ with $\overset{-}{72}$, by 48 observations, exceed Steel B by $-12{\cdot}71$

$\overset{+}{72}$ with $\overset{-}{70}$, by 48 observations, exceed Steel B by $+\ 9{\cdot}10$

Therefore Iron 70+Steel 72 exceed 2×Steel B by $-3^{d.}{\cdot}61$.

These comparisons were made at mean temperature $50^{\circ}{\cdot}48$. But the expansion of Iron (36 inches) for 1° FAHRENHEIT=$6^{d.}{\cdot}153$; that of Steel=$5^{d.}{\cdot}777$: the relative expansion of Iron therefore =$+0^{d.}{\cdot}376$; or for $11^{\circ}{\cdot}52$ FAHRENHEIT=$+4^{d.}{\cdot}35$. Hence at

temperature 62° FAHRENHEIT,

Iron 70+Steel 72 exceed 2×Steel B by $+0^{d}.74$.

Steel 73 with Iron 70, by 48 observations, exceed Steel B by $+13^{d}.97$

Iron 70 with Steel 73, by 48 observations, exceed Steel B by $-16^{d}.73$

Therefore Steel 73+Iron 70 exceed 2×Steel B by $-2^{d}.76$.

These comparisons were made at mean temperature 51°.69. The correction for expansion is $+3^{d}.89$. Hence at 62° FAHRENHEIT,

Steel 73+Iron 70 exceed 2×Steel B by $+1^{d}.13$.

Solving the three equations,

Iron End-bar No. 70 = Line-measure Steel B + 0·24

Steel End-bar No. 72 = Line-measure Steel B + 0·50

Steel End-bar No. 73 = Line-measure Steel B + 0·89

But Line-measure Steel B = Bronze $28-0^{d}.66$.

Hence, finally,

Iron End-bar No. 70 = Line-measure Bronze 28 − 0·42

Iron End-bar No. 72 = Line-measure Bronze 28 − 0·16

Iron End-bar No. 73 = Line-measure Bronze 28 + 0·23

For obtaining the length of another Iron End-bar No. 71, the End-bars Iron No. 70 and Iron 71 were compared with the Iron line-measure Low Moor B:

Iron 70 with Iron 71, by 48 observations, exceed Low Moor B by $-13^{d}.48$

Iron 71 with Iron 70, by 48 observations, exceed Low Moor B by $+11^{d}.95$

Therefore Iron 70+Iron 71 exceed 2×Low Moor B by $-1^{d}.53$.

But Low Moor B = Bronze $28+0^{d}.51$: and Iron 70 = Bronze $28-0^{d}.42$. Hence Iron 71 = Bronze $28+1^{d}.02+0^{d}.42-1^{d}.53$. Or

Iron End-bar No. 71 = Line-measure Bronze $28-0^{d}.09$.

These are all the End-bars whose lengths were independently determined.

SECTION IX.—*Closing proceedings of Official Character; Extracts from the Final Report of the Commission; Extracts from the Act of Parliament legalizing the New Standard; standard temperatures for the compared bars; disposal of the bars.*

The very elaborate comparison of Bars described in Section VI., and the careful comparison of other Bars in Section VII., gave the Committee ample choice of Bars for National Standards. At the suggestion of Mr. SHEEPSHANKS, the selection was in some measure guided, not only by the near approach in length to the presumed length of the Lost Imperial Standard, but also by the clearness of the division and the general good workmanship.

It will be remarked that no one of the Bars compared agrees *precisely* in length with

the Imperial Standard. But this disagreement is a mere exhibition of figures, in regard to a matter to which the accuracy of those figures is totally inapplicable. The Old Imperial Standard was so badly defined that its length was truly indeterminate, to an extent infinitely beyond the disagreement to which I allude. There is also a small range of uncertainty in the modern determinations, but very far below that in the references to the Old Standard. With all these considerations before them, the Committee felt themselves justified in considering Bronze 19 and Bronze 28 as exact copies of the Old Standard, and in recommending the adoption of Bronze 19 as the New Parliamentary Standard.

It was now found convenient to distinguish the Bars by a new order of numeration, commencing with No. 1 for the Parliamentary Standard. It was also thought convenient to distinguish the lengths of the Bars, not by stating their linear difference from Bronze 19 or Bronze 28 at the common temperature 62°·00 FAHRENHEIT, but by stating the temperature at which the length of each bar is the same as the length of Bronze 19 at 62° FAHRENHEIT. Some of these arrangements were not completely carried out until after the death of Mr. SHEEPSHANKS.

The Final Report of the Committee, addressed to the Lords Commissioners of Her Majesty's Treasury, was signed on 1854, March 28. It will be better to give here a few Extracts from the Report than to state in detail the later proceedings of the Committee.

Extracts from the "Report of the Commissioners appointed to superintend the Construction of New Parliamentary Standards of Length and Weight."

"2. By the authority of the Lords Commissioners of the Treasury, and by the courtesy of scientific societies, the following standard length-scales were placed at our service:—

Length-scales which had been compared with the lost Imperial Yard Standard.

Two 3-feet Iron Bars belonging to the Department of the Ordnance Survey.

SHUCKBURGH'S Scale, in the possession of the Royal Society.

The Brass Scale of the Royal Society, No. 46.

The Brass Tubular Scale of the Royal Astronomical Society.

"3. We have also had access to other length-scales, which were not available for absolute restoration of the values of the lost standards, but which we have thought it desirable to examine, as illustrating the construction of the lost standards. Such are,—

The ruined Imperial Yard Standard (in the custody of the Right Honourable the Speaker of the House of Commons).

TROUGHTON'S 5-feet scale (presented by W. SIMMS, Esq., and now preserved at the Royal Observatory, Greenwich).

RAMSDEN'S 20-feet bar, and ROY'S 3-feet bar (purchased by the Committee, and now deposited at the Royal Observatory, Greenwich).

Construction of the Standard of Length.

"4. In regard to the linear measure to be represented by the Standard of Length, we see no reason to depart from the decision in the Report of 1841, December 21, Article 19, to adopt the length of One Yard.

"5. The first important question which was left open to us by Article 20 of the same Report, was whether the measure of length should be defined by the whole length of a bar, or by the distance between two points or lines marked on it; and to this question we gave our early and careful consideration.

"6. The most celebrated of the standards constructed in modern times is the new Prussian Standard, made by the late astronomer, BESSEL. In this standard, the whole length of the bar is adopted as the measure of length. The reasons which induced BESSEL to substitute a standard of this construction (or standard *à bouts*) for the standard in which the measure is defined by the distance between points or lines upon its surface (or standard *à traits*), which was the principle of construction of the former Prussian Standard, are explained in page 7 of the Introduction to BESSEL's Account of the construction of the new Prussian Standard*. The first of these reasons is, that if a flexible bar be supported on two points, the extreme length of the bar from the centre of one end to the centre of the other end is not sensibly altered by its flexure, but the distance between two points or lines upon the upper surface may be considerably altered. BESSEL has himself, however, remarked, that this objection to line-measure is removed if the lines be engraved on surfaces which are depressed to the middle of the thickness of the bar (a principle long since employed in English standard bars); and, moreover, the tendency to alter the apparent length, whether at the surface or at the middle of the thickness, may be destroyed by proper adjustment of the points of support, and perhaps still more surely by supporting the bar at numerous points by lever-frames which will ensure equal supporting forces at all the points (a construction of which the theory has been treated by one of the members of the Committee), or in special experiments, by floating the bar in quicksilver. A second reason assigned by BESSEL is, that the principle of end-measure is more convenient than the line-measure for the production (that is, practically, the comparison) of copies of the standard. It appears probable that this remark is well founded, as regards the construction of secondary standards for commercial purposes, but it is doubtful whether it applies to secondary standards for scientific purposes, and it can scarcely apply to primary standards, in regard to which the consideration of convenience (in the few comparisons that may be made at intervals of many years) has no weight, when contrasted with the consideration of conservation of the standard length. Another reason is, that it is more convenient for use; but the members of the Committee, who have witnessed the operation of measuring a geodetic base by means of bars carrying the line-measure, have been led to form a high estimate of the convenience of the line-principle in that instance. It would appear, too, that in Captain KATER's measurement of

* Darstellung der Untersuchungen und Maassregeln, welche in den Jahren 1835 bis 1838 durch die Einheit des Preussischen Längenmasses veranlasst worden sind, von F. W. BESSEL, Berlin 1839.

the pendulum, and Mr. BAILY's comparisons of the mètre *à traits* with the yard, no inconvenience from that cause was found by either observer. It is moreover to be remarked, that the whole of the British geodetic bases have been measured by bars constructed on the line-principle; and a standard which is intended to apply advantageously to them must be constructed on the same principle. The end-measure has never, so far as we know, been applied to any scientific determination in England.

"7. The construction of the ends of a standard formed on the end-principle (which must be made of materials harder than the rest of the bar, so mounted as to admit of a slight adjustment), is, in BESSEL's construction, complicated and unsatisfactory; that of the ends of a standard on the line-principle is simple. The tendency of an end-standard to alter its length by wearing is in only one direction; that of a line-standard, so far as change can be conjectured, is matter of chance; it will however be practically invariable.

"8. A single comparison by end-measure is perhaps more accurate than a single comparison by line-measure. On the other hand, it seems doubtful whether this accuracy may not be impaired by the immersion in fluid which has been thought necessary for preventing unequal expansion. There has also been recognized in the line-measures which have been made in the course of these observations, a personal equation different with different observers; but this was only discovered after most of the observations were made. We have no evidence whether such equation exists in comparisons by end-measure, as the whole of the comparisons for the Prussian Standard were made by BESSEL himself. Still it is scarcely doubted that in this respect the system of end-measures has an advantage. On the other hand, there is no doubt that, by repeating the measures, and by varying the observers, the errors in line-measures arising from recognized causes will be eliminated, and unexceptionable accuracy will be given to the results.

"9. After consideration of all these points, as far as they were known at the time; remarking, in particular, the freedom of a line-standard from any tendency to constant wear, the applicability of a line-standard to our geodetic measures, the facility of comparing all other measures of a similar construction with a scale carefully subdivided, the circumstance that there exist no means of reproducing the old standard length but by reference to existing line-measures, and the historical fact that the British standard for many years past has been a line-standard; we decided on adopting a line-standard, or measure *à traits*, for the Parliamentary Standard of Length.

"10. The point which next engaged our attention was the material of which the new Standard of Length should be formed. The qualities required in the material are invariability and durability. Already we had reason to suspect that bars, which have received their form by undergoing while cold the violent mechanical operations of hammering, rolling, or drawing, will not preserve an invariable length; and these suspicions were raised to certainty by Mr. BAILY's examination of the standard scale which *primâ facie* appeared to be the most trustworthy representative of the old Imperial

Standard, namely, the Royal Astronomical Society's drawn tubular scale. Remarking then, among the fusible metals and alloys, that copper is soft, and is not easily cast; that cast brass is considered unsound until it has been hammered; that gun-metal is easily cast, and may be made (by due proportion of its component metals) pretty hard and elastic, and is free from liability to corrosion; and that cast steel is perfectly manageable; remarking, also, that platinum, although it possesses the advantageous properties of being very durable, and little affected by change of temperature, is inconveniently heavy and soft; but that iron, whether cast or forged, is manageable, and probably invariable: we considered that our choice of materials was practically limited to the following metals and alloys:—

Gun-metal (in proportions of copper, tin, and zinc to be determined by experiment).
Cast Steel.
Cast Iron.
Forged Iron.

"11. A series of experiments was made by Mr. BAILY on gun-metal formed of different proportions of the simple metals. It was found at length that an alloy formed in the following proportions,—

Copper	16
Tin	$2\frac{1}{2}$
Zinc, ...	1

appeared to possess every desirable property. It is hard; it is highly elastic, recovering its form (as to sense) perfectly after every strain, until the weight placed on it is sufficient to break it, when it breaks without bending; it is also strong.

"12. In the gun-metal above described, and in cast steel, and in cast or wrought iron, we were satisfied that we possessed materials for an *invariable* measure. In considering the question of *durability*, we could not fail to remark that bars of iron or steel are liable to rust; and that there is great danger of injury or even total destruction from this cause if the bars are inaccessible for many years or for centuries, or if, although accessible, they are very rarely examined. We at length decided on adopting, for the material of the Standards of Length, gun-metal composed in the proportions mentioned above.

"13. The form adopted for the Standard of Length and for all its copies is that of a solid square bar, 38 inches long and 1 inch square in transverse section. Near to each end a cylindrical hole is sunk (the distance between the centres of the two holes being 36 inches) to the depth of 0·5 inch. At the bottom of this hole is inserted in a smaller hole a gold plug or pin about 0·1 in diameter, and upon the surface of this pin there are cut three fine lines at intervals of about 0·01 inch transverse to the axis of the bar, and two lines at nearly the same interval parallel to the axis of the bar. The measure of length is given by the interval between the middle transversal line at one end and the middle transversal line at the other end; the part of each line which is employed being the point midway between the longitudinal lines. The other transversal lines were used, in the operations of comparison, only for assigning the scales of the micrometers.

"14. It was intended that the permanent rest of each bar, after the termination of the comparisons, should be upon a system of eight roller-cylinders, permitting the free thermal expansion of the bars, these rollers being carried by lever-frames, so arranged that the pressure received by each roller is necessarily one-eighth part of the whole weight of the bar, and the distances between the rollers being nearly those which are defined by the theory of the flexure of bars as proper to ensure that the measure between two lines (even if cut upon the surface of the bar) shall not be sensibly altered by the flexure of the bar. During the comparisons, however, the bars were floated in quicksilver.

"15. Under the direction of Mr. SHEEPSHANKS, and with our approval, the following apparatus was prepared for the comparison of the bars. A massive foundation of masonry was established in the cellar beneath the apartments of the Royal Astronomical Society, in the block of buildings east of the entrance-gateway of Somerset House. Upon one part of this foundation were supported stone piers and transoms, which carried the comparing microscopes. Upon another part there travelled a large carriage, bearing the trough of water which contained the trough of quicksilver in which the bars floated. It is not necessary here to describe the places of the thermometers, the contrivances for illuminating the divisions, &c. We desire, however, confidently to express our opinion, that no apparatus possessing equal firmness and delicacy with this, or ensuring equal immunity from irregular thermal expansion, had ever before been used for the comparison of standards *à traits*.

"16. The selection of authorities for the reproduction of the Imperial Standard was a matter of considerable difficulty. A careful collation of all the direct comparisons of different scales with the Imperial Standard, and of cross comparisons between the different scales, presented anomalies which it was impossible to explain. The scale which at a late date had been compared with greater care than any of the others, namely, the scale of the Royal Astronomical Society, was proved by Mr. BAILY to have undergone considerable change. This discovery necessarily took away all authority from other scales of a similar construction. The results obtained from SHUCKBURGH'S scale, according as KATER'S or BAILY'S measures are followed, are exceedingly discordant. Indeed, BAILY himself rejected the result of his own comparisons through SHUCKBURGH'S scale, as being quite incompatible with the result of direct comparisons with the Imperial Standard. After an elaborate examination of all these measures, Mr. SHEEPSHANKS arrived at these conclusions:—First, that no scales ought to be used for restoration of the value of the Imperial Standard, except those which had been compared *directly* with the Imperial Standard. Secondly, that among these the only admissible scales are, the Royal Society's brass scale No. 46, and the two 3-feet iron bars of the Ordnance Department. After examination of the reasons adduced by Mr. SHEEPSHANKS, we assented to these conclusions.

"17. The definition of a length, as represented by a standard bar, requires a statement of the temperature at which the standard bar is to be used. It appears, upon examining

the amount of the expansion of a metallic bar for a given change of temperature, that the effect of a change of 0°·01 or 0°·02 FAHRENHEIT is sensible in the measure of the length of a bar; and it was therefore necessary to employ methods of ascertaining the temperature, which should be certain to that degree of accuracy. No thermometer in England, at the time of commencing these experiments, could be trusted to such a nicety. Mr. SHEEPSHANKS, therefore, commenced his labours by forming a system of original thermometers; determining independently the freezing-point, the boiling-point, and the volume of every section of the tube of each. This process occupied a considerable time. To these new thermometers the thermometers employed in the former comparisons of the Imperial Standard with the Royal Society's scale and the Ordnance bars were referred, and the comparisons were reduced anew; and the relations between the lengths of the adopted authorities and the Imperial Standard were ascertained with all the accuracy which the comparative rudeness of construction of the Imperial Standard permitted. The thermometers to be used in the details of the new comparisons were also referred to the new thermometers; the thermometrical expansions of the various bars employed for the new standards, or of similar bars, for temperatures not differing materially from 62° FAHRENHEIT, were ascertained; and the whole of the thermometrical determinations were placed on a satisfactory basis.

"18. Very numerous comparisons were made by Mr. SHEEPSHANKS of the adopted authorities with bars which he called "Brass 2" and "Split-plug A" (the division or *trait* in the latter being not a cut on a flat face of metal, but the line produced by forcing two small surfaces of metal into contact, and then cutting by a transversal section through both pieces of metal; the section of the contact-plane then exhibits a delicate line). There were, however, discordances in the latter comparisons which prevented him from giving full confidence to the results. At length a better mode of illuminating the lines was devised, and then the whole of the preceding work was rejected, and new observations were begun. By means of these, the lengths of bars called Bronze 12 and Bronze 28 were determined in relation to the lost Imperial Standard, and the latter was used as a basis of reference for other bars.

"19. The Report of 1841 recommended (Article 5) that, besides the legal standard, there should be prepared four copies, to be deposited in places to be afterwards determined. Mr. SHEEPSHANKS therefore proceeded to make a series of comparisons of Bronze 28 with numerous bars, in order to select from them five bars which, in respect of the distinctness of their engraving, their floating evenly in quicksilver, and their near approach to the length of the lost Imperial Standard, might seem well adapted to be taken as Parliamentary Standard and Parliamentary Copies. When the observations had been carried to an extent which, it was supposed, would be perfectly satisfactory, it appeared that the results of comparison by different observers were sensibly different. An extensive series of new observations by numerous observers was at once commenced. The discussion of these observations showed clearly, as had been suspected before, that there is a difference among the results of different observers far exceeding the uncertainty

inferred by the theory of probable errors from the observations of each observer taken separately. Thus, in the comparisons of Bronze 10 with Bronze 28, there is a discordance between the results obtained by Mr. SHEEPSHANKS from 142 sets of observations, and by Mr. WILLIAM SIMMS, junior, from 24 sets of observations, amounting (in ten-millionth parts of an inch) to 198; whereas the probable errors of the two determinations, deduced in the usual way from the observations of each observer separately, are only 15 and 26. In other words, each observer has a personal equation in the microscopic observation of the engraved lines. The existence of discordances of similar character between the astronomical observations of different observers has been long since recognized. The origin of this personal peculiarity is very obscure, but the fact seems to be beyond doubt. There appeared to be no proper method of treating the results but to take the means of the final results of the different observers, giving equal weight to each.

"20. The expansions of these bars corresponding to a given change of temperature had been sufficiently determined in the course of the experiments; and it was then judged expedient, instead of stating the difference in the length of the selected bars at the same temperature, to infer the difference of temperature which would cause all to represent the same length, by the application of which it would be possible to assign the specific temperature at which each bar represents precisely the length of one yard. Thus it was found that the length of one yard, as given by the lost Imperial Standard, is represented with no sensible uncertainty, except in the measures of the Imperial Standard itself, by the following bars, at the temperatures placed opposite to them:—

Bronze 19 or No. 1, at $62^{\circ}\cdot00$ FAHRENHEIT.
Bronze 20 or No. 2, at 61·94 FAHRENHEIT.
Bronze 2 or No. 3, at 62·10 FAHRENHEIT.
Bronze 7 or No. 4, at 61·98 FAHRENHEIT.
Bronze 10 or No. 5, at 62·16 FAHRENHEIT.
Bronze 28 or No. 6, at 62·00 FAHRENHEIT.

"21. The degrees of temperature for the use of these standards are defined as proportional to the corresponding apparent increase of volume of quicksilver in the thermometer-tube; the degree 32° representing the freezing-point of water; and the degree 212° representing the temperature of steam under LAPLACE's standard atmospheric pressure, or the atmospheric pressure corresponding to the following number of inches in the barometric reading reduced to 32° FAHRENHEIT:—

$$29\cdot9218+0\cdot0766\times\text{cosine}(2.\text{ latitude})+0\cdot00000179\times\text{height in feet above the sea};$$

and the degree 62° denoting the temperature which produces in the quicksilver an apparent expansion equal to $\frac{30}{180}$ of the expansion between 32° and 212°; and so in proportion for other degrees.

"23. We propose that the bar No. 1 be adopted by the Legislature as the PARLIAMENTARY STANDARD OF ONE YARD; that Nos. 2, 3, 4, 5 be adopted as Parliamentary Copies; and that No. 6 be retained by some officer of the Government for the comparison of

other bars, or for other scientific purposes in which reference to the Standard may soon be required. We will advert shortly to the proposed places of deposit of the Standards Nos. 1, 2, 3, 4, 5.

Places of Deposit of the Standards.

"32. It appears from the copy of the Bill of 1758, to which reference has been made, that the custody of the Standards by the Clerk of the House of Commons, which continued to the time of the destruction of the Houses of Parliament, was a matter of accident. It is recited in that Bill that the bar and the metallic weight, which it is proposed to adopt as legal standards (and which are described with sufficient accuracy to ensure their identity with the late Imperial Standards), together with their duplicates, are in the custody of the Clerk of the House of Commons; and it proposes to enact that "the said standard Troy pound and the said standard yard shall on or before the 9th September 1765 be deposited and for ever remain in the Court of the Receipt of the Exchequer, and be there safely kept under the seals of the Chancellor of the Exchequer and of the Chief Baron, and the Seal of Office of the Chamberlains of the Exchequer, which seals shall not at any time be opened but by the order and in the personal presence of the Chancellor of the Exchequer and the Chief Baron for the time being, and then for the purposes only of guarding the same from all damage or injury, or of inspecting, examining, and comparing the copies and models hereinafter described, &c." The Bill afterwards provides for the appointment of four commissioners, with clerks, workmen, &c., and for the establishment of a proper office; and that "as soon as the said office shall be fixed and established, the said copies of the said standard yard and the said standard Troy pound, and the said models of parts and multiples, &c., shall be delivered to the said commissioners by the Clerk of the House of Commons, &c., and that one set shall be kept in the office under the seal of the Chief Baron of the Exchequer and the seals of the commissioners, and the other set shall be used for the purpose of sizing and adjusting, &c." It appears that this subject had engaged the attention of the House of Commons in 1758 and 1759, and that the Bill above mentioned was introduced by Lord Carysfort in 1760, and had passed through committee; but in the pressure of business attending the death of George II., and the succession of George III., and the dissolution of Parliament soon following, it was not carried through all its stages. There appears, however, to be no doubt that the intention of the Committee of the House of Commons may be sufficiently gathered from the language of the Bill as cited above.

"33. Remarking this probable intention of the Legislature in the year 1760, and remarking also the evident propriety of placing the National Standards under the care of the Executive, we recommend that the Parliamentary Standards of One Yard and One Pound be deposited in the office of the Exchequer, there to be preserved under such regulations as to Parliament may appear fitting.

"34. In selecting places for the preservation of the authentic Copies of the Parliamentary Standards, we have been guided by a consideration of the general fitness of

the offices named; by an appreciation of the careful and accurate habits of the persons employed in them; by the limited accessibility to standards preserved in such offices, which it may be presumed will be made available for their legitimate purposes only; and by the occasional utility of accurate standards in verification of the operations to which these offices are devoted. We have in some measure departed from the recommendation contained in the last clause of Article 11 of the Report of 1841, December 21, in proposing that one set be transferred to the Royal Society of London. In this proposal we have been actuated by the considerations that the character of the Royal Society in every respect fits it for recognition as a depositary of scientific standards; that the first attempt in Britain to obtain an accurate standard of length, and the only attempt to secure a durable standard of weight, were made by that body; that it possesses at present a very valuable collection of standards; and that acknowledgment is due to it for its very important contribution to the means of restoring the value of the lost Standards. We adhere to the recommendation in Article 6 of the Report of 1841, that one set of copies should be imbedded in the masonry of a public building. In the distribution of the different copies, we have been guided by trifling peculiarities in the copies themselves.

"35. After careful consideration, we recommend,—

That the copy of Length Standard, No. 2, and the copy of Weight Standard, PC, No. 1, be deposited in the Royal Mint:

That the copy of Length Standard, No. 3, and the copy of Weight Standard, PC, No. 2, be transferred to the Royal Society:

That the copy of Length Standard, No. 5, and the copy of Weight Standard, PC, No. 3, be deposited in the Royal Observatory of Greenwich:

That the copy of Length Standard, No. 4, and the copy of Weight Standard, PC, No. 4, be immured in the cill of the recess on the east side of the Lower Waiting Hall in the New Palace at Westminster.

"36. We have to report, that, under the sanction of the Lords Commissioners of the Treasury, as conveyed by letter of Sir CHARLES E. TREVELYAN, dated 1853, June 23, and with the permission of the Commissioners of Public Works for immuring one set of standards in the masonry of the New Palace, as conveyed by letter of T. W. PHILIPPS, Esq., dated 1853, March 21, we have deposited the Standards and the several copies above named in the places specified above.

On the question of referring the Values of the Measure and Weight represented by the Standards to natural elements.

"40. After due consideration of this question, referring to the reasons explained in Chapter II. of the Report of 1841, December 21, we adhere to the recommendation contained in that chapter, and embodied in Articles 1 and 2 of the same Report, that no reference be made to natural elements for the values represented by the standards.

"41. We consider the ascertaining of the earth's dimensions and of the length of the seconds' pendulum in terms of the Standard of Length, and of the weight of a certain

volume of water in terms of the Standard of Weight, as philosophical determinations of the highest importance, to the prosecution of which we trust that Her Majesty's Government will always give their most liberal assistance; but we do not urge them on the Government at present as connected with the conservation of standards.

Derived Standards à bouts.

"44. We have already alluded (Article 6 above) to the value of length-standards *à bouts* for the ordinary uses of commerce; and from the commencement of our labours we have proposed to ourselves the formation of a standard *à bouts* equivalent in length to the Parliamentary Standard *à traits*, as necessary for the completeness of our undertaking. After consideration of the methods of verifying and using such a standard, we recommend that the metal be steel; and that the ends be hardened steel, or hard stone as quartz or sapphire, or some sufficiently hard and incorrodible material; and that the end surfaces be curved, forming portions of one sphere whose centre is the centre of the bar.

"45. At our request Mr. SHEEPSHANKS has made preparations for forming such standards, to be compared with a steel Standard *à traits* whose length has been verified in the usual way by the Standard No. 6, to which reference is made in Article 20 above. We are not yet able to report the completion of these standards."

The Lords Commissioners of the Treasury having adopted generally the recommendations of the Report, as applying to the Standard of Length and its Copies, a Bill was prepared and brought into Parliament, which received the Royal Assent on 1855, July 30. The Act is intituled "Anno decimo octavo et decimo nono Victoriæ Reginæ, Cap. LXXII. An Act for legalizing and preserving the restored Standards of Weights and Measures. [30th July 1855]."

The Preamble recites the provisions in the Act 5 Geo. IV. Cap. 74 for legalization and contingent restoration of the Standards; states the destruction of the Standards in the Fire of the Houses of Parliament; and then proceeds as follows (I have omitted various points relating to the Standard of Weight):—

"And whereas by the Researches of scientific Men Doubts were thrown on the Accuracy of the Methods provided by the said Act for the Restoration of the said Standards: And whereas there exist Bars which had been accurately compared with the said Standard Yard so destroyed as aforesaid, which afforded sufficient Means for restoring such original Standards: And whereas scientific Men acting for that Purpose under the Direction of the Commissioners of Her Majesty's Treasury have constructed a Standard of Length equivalent to the Imperial Standard Yard so destroyed, and Four accurate Copies of the Standard so constructed: And whereas the Form adopted for the Standard of Length and for all the Copies thereof is that of a solid square Bar Thirty-eight Inches long and One Inch square in transverse Section, the Bar being of Bronze or Gun Metal; near to each End a cylindrical Hole is sunk (the Distance between the Centres of the Two Holes being Thirty-six Inches) to the Depth of Half an Inch; at the Bottom of this Hole is inserted in a smaller Hole a Gold Plug or Pin about One

Tenth of an Inch in Diameter, and upon the Surface of this Pin there are cut Three fine lines at Intervals of about the One Hundredth Part of an Inch transverse to the Axis of the Bar, and Two Lines at nearly the same Interval parallel to the Axis of the Bar; the Measure of Length is given by the Interval between the Middle transversal Line at one End and the Middle transversal Line at the other End, the Part of each Line which is employed being the Point Midway between the longitudinal Lines; and the said Points are herein referred to as the Centres of the said Gold Plugs or Pins: And whereas the Standard of Length so constructed as aforesaid, the Bronze Bar, being marked "Copper 16 oz., Tin 2½, Zinc 1. Mr. Baily's Metal. No. 1. Standard Yard at 62°·00 Fahrenheit. Cast in 1845. Troughton and Simms, London," and the said Standard of Weight marked P.S. 1844, 1 lb., have respectively been deposited in the Office of the Exchequer at *Westminster*, and One of the said Copies of the said Standard of Length, the Bronze Bar, being marked "Copper 16 oz., Tin 2½, Zinc 1. Mr. Baily's Metal. No. 2. Standard Yard at 61°·94 Fahrenheit. Cast in 1845. Troughton and Simms, London," and One of the said Copies of the Standard of Weight marked No. 1, P.C. 1844, 1 lb., have been deposited at the Royal Mint; and One other of the said Copies of the Standard of Length, the Bronze Bar, being marked "Copper 16 oz., Tin 2½, Zinc 1. Mr. Baily's Metal. No. 3. Standard Yard at 62°·10 Fahrenheit. Cast in 1845. Troughton and Simms, London," and One other of the said Copies of the Standard of Weight marked No. 2. P.C. 1844, 1 lb., have been delivered to the Royal Society of *London*; and One other of the said Copies of the Standard of Length, the Bronze Bar, being marked "Copper 16 oz., Tin 2½, Zinc 1. Mr. Baily's Metal. No. 5. Standard Yard at 62°·16 Fahrenheit. Cast in 1845. Troughton and Simms, London," and One other of the said Copies of the Standard of Weight marked No. 3, P.C. 1844, 1 lb., have been deposited in the Royal Observatory of *Greenwich*; and the other of the said Copies of the Standard of Length, the Bronze Bar, being marked "Copper 16 oz., Tin 2½, Zinc 1. Mr. Baily's Metal. No. 4. Standard Yard at 61°·98 Fahrenheit. Cast in 1845. Troughton and Simms, London," and the other of the said Copies of the Standard of Weight marked No. 4. P.C. 1844, 1 lb., have been immured in the Cill of the Recess on the East Side of the Lower Waiting Hall in the New Palace at *Westminster*: And whereas it is expedient to legalize the Standards so constructed and to provide for the Preservation thereof: Be it therefore enacted by the Queen's most Excellent Majesty, by and with the Advice and Consent of the Lords Spiritual and Temporal, and Commons, in this present Parliament assembled, and by the Authority of the same, as follows:

"I. So much of the said Act of the Fifth Year of King *George* the Fourth as relates to the Restoration of the Imperial Standard Yard and of the Standard Troy Pound respectively, in case of Loss, Destruction, Defacement, or other Injury, shall be repealed.

"II. The straight Line or Distance between the Centres of the Two Gold Plugs or Pins in the Bronze Bar deposited in the Office of the Exchequer as aforesaid shall be the genuine Standard of that Measure of Length called a Yard, and the said straight

Line or Distance between the Centres of the said Gold Plugs or Pins in the said Bronze Bar (the Bronze being at the Temperature of Sixty-two Degrees by FAHRENHEIT'S Thermometer) shall be and be deemed to be the Imperial Standard Yard.

"VII. If at any Time hereafter the said Imperial Standard Yard and Standard Pound Avoirdupois respectively, or either of them, be lost, or in any Manner destroyed, defaced, or otherwise injured, the Commissioners of Her Majesty's Treasury may cause the same to be restored by Reference to or Adoption of any of the Copies so deposited as aforesaid, or such of them as may remain available for that Purpose."

I have now only to give the new nomenclature of the Bars and the temperatures at which they represent the British Yard; and a statement of their distribution up to the present time (1857, April).

Old Name of Bar.	New Number.	Standard Temperature.	Distribution of Bars.
Bronze 19	1	62·00	NATIONAL STANDARD, at the Exchequer, in the custody of the Right Honourable the Comptroller of the Exchequer.
Bronze 20	2	61·94	Parliamentary Copy, at the Royal Mint.
Bronze 2	3	62·10	Parliamentary Copy, in charge of the Royal Society.
Bronze 7	4	61·98	Parliamentary Copy, immured in the New Palace, Westminster.
Bronze 10	5	62·16	Parliamentary Copy, at the Royal Observatory, Greenwich.
Bronze 28	6	62·00	Retained at the Royal Observatory, as accessible representative of the National Standard.
Bronze 1	7	61·78	To the East India Company.
Bronze 8	8	61·82	To the Government of Russia.
Bronze 9	9	61·25	To the Government of Prussia.
Bronze 4	10	61·51	To the Government of Bavaria.
Bronze 11	11	61·79	To the Government of the United States of America.
Bronze 12	12	61·80	At the Royal Observatory, Greenwich, still disposable.
Bronze 13	13	61·86	To the Government of Austria.
Bronze 14	14	61·59	To the Municipality of Edinburgh.
Bronze 15	15	62·31	To the Municipality of Dublin.
Bronze 16	16	61·94	To the Colony of Canada.
Bronze 17	17	62·14	To the Colony of the Cape of Good Hope.
Bronze 18	18	62·26	To the Colony of New South Wales.
Bronze 5	19	61·95	To the Government of Portugal.
Bronze 3	20	62·23	To the Government of Spain.
Bronze 21	21	61·80	To the Government of France.
Bronze 22	22	61·75	To the Government of Belgium.
Bronze 23	23	61·77	To the Government of Holland.
Bronze 24	24	61·68	To the Government of Hanover.
Bronze 25	25	61·97	To the Government of Saxony.
Bronze 26	26	61·47	To the Government of Sweden.
Bronze 27	27	61·81	To the Government of Denmark.
Bronze 6	28	61·73	To the Government of Switzerland.
Bronze 29	29	61·51	To the Ordnance Map Office.
Bronze 30	30	61·66	At the Royal Observatory, Greenwich, still disposable.
Bronze 31	31	62·04	To William Simms, Esq.
Bronze 32	32	62·04	To the Government of Sardinia.
Bronze 33	33	61·92	To the Government of the Papal States.
Bronze 34	34	62·03	To the Government of Naples and Sicily.
Bronze 35	35	61·95	To the Government of Turkey.
Bronze 36	36	62·08	To the Government of Brazil.
Bronze 37	37	62·07	To the Colony of Tasmania.

Table (continued).

Old Name of Bar.	New Number.	Standard Temperature.	Distribution of Bars.
Bronze 38	38	62°·14	To the Government of Buenos Ayres.
Bronze 39	39	61·87	To the Government of Chili.
Bronze 40	40	61·99	To the Colony of Victoria.
Baily's A	41	62·37	At the Royal Observatory, Greenwich, still disposable.
Baily's B	42	61·67	
Baily's C	43	61·57	
Baily's D	44	61·68	
Brass 1	46	61·99	At the Royal Observatory, Greenwich, still disposable.
Brass 2	47	61·14	
Flat Brass Scale	48	62·00	
Kew Bar	49	61·84	To the Kew Observatory.
New Tubular Scale	50	63·09	At the Royal Observatory, Greenwich, still disposable.
Donkin's Copper	51	61·87	At the Royal Observatory, Greenwich, still disposable.
Maudslays' Copper	52	61·82	
May's Alloy	53	61·53	
Swedish A	54	63·55	To the Government of Bavaria.
Swedish B	55	61·93	To the Ordnance Map Office.
Swedish C	56	61·77	To the Government of Russia.
Low Moor A	57	62·58	To the Government of the United States of America.
Low Moor B	58	61·92	At the Royal Observatory, Greenwich, still disposable.
Low Moor C	59	61·65	To the East India Company.
Low Moor D	60	61·79	To the Government of Prussia.
Cast Iron A	61	62·56	At the Royal Observatory, Greenwich, still disposable.
Cast Iron B	62	62·90	
Cast Iron C	63	62·34	
Cast Iron D	64	62·63	To the Government of Austria.
Cast Steel A	65	62·22	At the Royal Observatory, Greenwich, still disposable.
Cast Steel B	66	62·11	
Cast Steel C	67	62·47	
Cast Steel D	68	62·62	To the Académie des Sciences of Paris.
End-bars.			
Bronze	69	61·67	At the Royal Observatory, Greenwich, still disposable.
Swedish A	70	62·07	
Low Moor A	71	62·01	
Cast Steel	72	62·03	To Joseph Whitworth, Esq.
Cast Steel	73	61·96	At the Royal Observatory, Greenwich, still disposable.
Bronze	74	61·91	
Bronze	45	62·19	
Old Scales.			
Baily's Tubular 5	.....	59·97	These have never been the property of the British Government.
Simms's Tubular 4	.....	60·55	
R. Ast. Soc. Tubular	.....	60·78	
Danish Tubular	.....	60·74	

In addition to these there are,—No. 75, an undivided bar, marked by mistake: Nos. 76, 77, 78, one-foot end-bars; Nos. 79, 80, six-inch end bars: and Nos. 81, 82, three-inch end-bars: intended for secondary comparison at the Exchequer or other offices, but at present lodged at the Royal Observatory.

The Documents of every kind, relating to the Standard Commissions and the preparation of Standards, are preserved at the Royal Observatory, Greenwich.

1857, April 8. G. B. Airy.

ADDENDUM.

Since the preparation of this paper, the following bars have been lodged in the Exchequer Office of Weights and Measures, for use:—

Line-Bar	No. 30.
End-Bars	No. 70.
	No. 45.
	No. 76.
	No. 79.
	No. 81.

1858, *February* 19. G. B. Airy.

Note on the Tubular Scale of the Royal Astronomical Society.

The information which the Committee possessed as to the discordance in the relative lengths of the intervals A*a*, B*b*, C*c*, D*d*, on this scale, as ascertained at different times (to which allusion is made in page 645), was derived from Mr. Baily's letters, and from an Abstract entered by Mr. Baily himself in his manuscript book of observations. On careful examination of those observations, in the course of reading the proof-sheets of this Account, it was found that Mr. Baily had given all the signs erroneously, for the observations of 1844, April 21. The signs as now printed in page 645 are correct. On examining into the effect of this change, it will be seen that the discordance is really far less than it was supposed to be by Mr. Baily; and that the Tubular Scale merits greater credit than it obtained with the Committee and Mr. Sheepshanks.

1858, *March* 3. G. B. Airy.

XXXI. *Memoir on the Resultant of a System of two Equations.*

By ARTHUR CAYLEY, *Esq.*, *F.R.S.*

Received December 18, 1856,—Read January 8, 1857.

THE Resultant of two equations such as

$$(a, b, \ldots)(x, y)^n = 0,$$
$$(p, q, \ldots)(x, y)^m = 0,$$

is, it is well known, a function homogeneous in regard to the coefficients of each equation separately, viz. of the degree n in regard to the coefficients $(a, b, \ldots)$ of the first equation, and of the degree m in regard to the coefficients $(p, q, \ldots)$ of the second equation; and it is natural to develope the resultant in the form $k\mathrm{AP} + k'\mathrm{A'P'} +$ &c., where A, A', &c. are the combinations (powers and products) of the degree n in the coefficients $(a, b, \ldots)$, P, P', &c. are the combinations of the degree m in the coefficients $(p, q, \ldots)$, and k, k', &c. are mere numerical coefficients. The object of the present memoir is to show how this may be conveniently effected, either by the method of symmetric functions, or from the known expression of the Resultant in the form of a determinant, and to exhibit the developed expressions for the resultant of two equations, the degrees of which do not exceed 4. With respect to the first method, the formula in its best form, or nearly so, is given in the 'Algebra' of MEYER HIRSCH, and the application of it is very easy when the necessary tables are calculated: as to this, see my "Memoir on the Symmetric Functions of the Roots of an Equation*." But when the expression for the Resultant of two equations is to be calculated without the assistance of such tables, it is I think by far the most simple process to develope the determinant according to the second of the two methods.

Consider first the method of symmetric functions, and to fix the ideas, let the two equations be

$$(a, b, c, d)(x, y)^3 = 0,$$
$$(p, q, r\ \ )(x, y)^3 = 0.$$

Then writing

$$(a, b, c, d)(1, z)^3 = a(1-\alpha z)(1-\beta z)(1-\gamma z),$$

so that

$$-\frac{b}{a} = \alpha+\beta+\gamma \quad = (1),$$

$$+\frac{c}{a} = \alpha\beta+\alpha\gamma+\beta\gamma = (1^2),$$

$$-\frac{d}{a} = \alpha\beta\gamma \quad = (1^3),$$

* Philosophical Transactions, 1857, pp. 489–497.

the Resultant is

$$(p, q, r\chi(\alpha, 1)^3.(p, q, r\chi(\beta, 1)^3.(p, q, r\chi(\gamma, 1)^3,$$

which is equal to

$$r^3+qr^2(\alpha+\beta+\gamma)+pr^2(\alpha^2+\beta^2+\gamma^2)+pqr(\alpha^2\beta+\alpha\beta^2+\beta^2\gamma+\beta\gamma^2+\gamma\alpha^2+\gamma^2\alpha)+ \&c.$$

Or adopting the notation for symmetric functions used in the memoir above referred to, this is

$$\begin{cases} r^3 \\ +qr^2\ (1) \\ +pr^2\ (2) \\ +q^2r\ (1^2) \\ +pqr\ (21) \\ +q^3\ (1^3) \\ +p^2r\ (2^2) \\ +pq^2\ (21^2) \\ +p^2q\ (2^21) \\ +p^3\ (2^3) \end{cases},$$

the law of which is best seen by dividing by r^3 and then writing

$$\frac{q}{r}=[1], \quad \frac{p}{r}=[2],$$

and similarly,

$$\frac{q^2}{r^2}=[1^2], \quad \frac{pq}{r^2}=[21], \ \&c.;$$

the expression would then become

$$1+[1](1)+[2](2)+[1^2](1^2)+[21](21)+[1^3](1^3)+[2^2](2^2)+[21^2](21^2)+[2^21](2^21)+[2^3](2^3),$$

where the terms within the [ ] and ( ) are simply all the partitions of the numbers 1, 2, 3, 4, 5, 6, the greatest part being 2, and the greatest number of parts being 3. And in like manner in the general case we have all the partitions of the numbers 1, 2, 3...mn, the greatest part being n, and the greatest number of parts being m.

The symmetric functions (1), (2), (1^2), &c. are given in the Tables (b) of the Memoir on Symmetric Functions, but it is necessary to remark that in the Tables the first coefficient a is put equal to unity, and consequently that there is a power of the coefficient a to be restored as a factor: this is at once effected by the condition of homogeneity. And it is not by any means necessary to write down (as for clearness of explanation has been done) the preceding expression for the Resultant; any portion of it may be taken out directly from one of the Tables (b). For instance, the bracketed portion

$$\begin{cases} +pqr\ (21), \\ +q^3\ (1^3), \end{cases}$$

which corresponds to the partitions of the number 3, is to be taken out of the Table III(b). as follows: a portion of this Table (consisting as it happens of consecutive lines and

columns, but this is not in general the case) is

=	d	be
(1^2)	$+3$	-1
(1^3)	-1	

if in this we omit the sign $=$, and in the outside line write for homogeneity ad instead of d, and in the outside column first substituting q, p for 1, 2, then write for homogeneity pqr instead of pq, we have

	ad	be
pqr	$+3$	-1
q^2	-1	

viz. $pqr\times(+3ad-1be)+q^2(-1ad)$, for the value of the portion in question; this is equivalent to

	pqr	q^2
ad	$+3$	-1
be	-1	

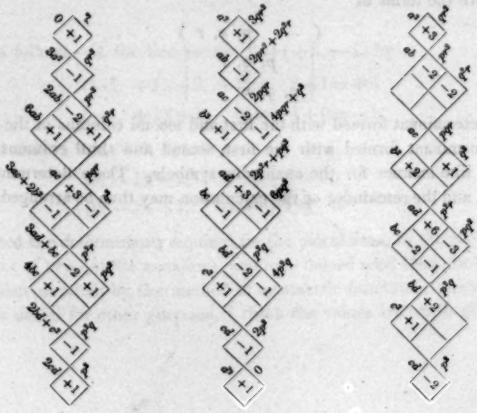
, or as it may be more conveniently written,

in which form it constitutes a part of the expression given in the sequel for the Resultant of the two functions in question; and similarly the remainder of the expression is at once derived from the Tables (b) I. to VI.

As a specimen of a mode of verification, it may be remarked that the Resultant qua invariant ought, when operated upon by the sum of the two operations,

$$3a\partial_b+2b\partial_c+c\partial_d \text{ and } 2p\partial_q+q\partial_r,$$

to give a result zero. The results of the two operations are originally obtained in the forms in the first and second columns, and the first column, and the second column, with all the signs reversed, are respectively equal to the third column, and consequently the sum of the first and second columns vanishes, as it ought to do.



Next to explain the second method, viz. the calculation of the resultant from the expression in the form of a determinant.

Taking the same example, as before, the resultant is

$$\begin{vmatrix} & a, & b, & c, & d \\ a, & b, & c, & d, & \\ & & p, & q, & r \\ & p, & q, & r & \\ p, & q, & r, & & \end{vmatrix}$$

which may be developed in the form

$$\left.\begin{matrix} +12.345 \\ -13.245 \end{matrix}\right\}$$
$$\left.\begin{matrix} +14.235 \\ +23.145 \end{matrix}\right\}$$
$$\left.\begin{matrix} -15.234 \\ -24.135 \end{matrix}\right\}$$
$$\left.\begin{matrix} +25.134 \\ +34.125 \end{matrix}\right\}$$
$$\left.\begin{matrix} -35.124 \\ +45.123 \end{matrix}\right\}$$

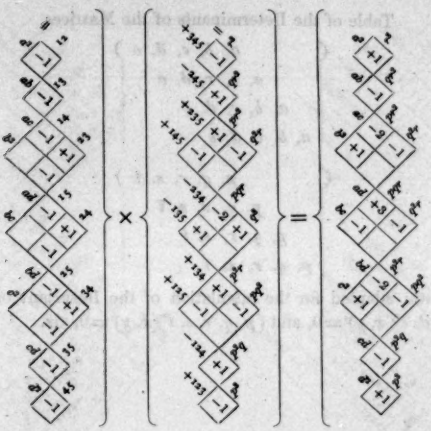
where 12, 13, &c. are the terms of

$$\begin{pmatrix} & a, & b, & c, & d \\ a, & b, & c, & d & \end{pmatrix}$$

and 123, &c. are the terms of

$$\begin{pmatrix} & & p, & q, & r \\ & p, & q, & r & \\ p, & q, & r & & \end{pmatrix}$$

viz. 12 is the determinant formed with the first and second columns of the upper matrix, 123 is the determinant formed with the first, second and third columns of the lower matrix, and in like manner for the analogous symbols. These determinants must be first calculated, and the remainder of the calculation may then be arranged as follows:—



where it is to be observed that the figures in the squares of the third column are obtained from those in the corresponding squares of the first and second columns by the ordinary rule for the multiplication of determinants,—taking care to multiply the dexter lines (*i. e.* lines in the direction $\diagdown$) of the first square by the sinister lines (*i. e.* lines in the direction $\diagup$) of the second square in order to obtain the sinister lines of the third square. Thus, for instance, the figures in the square

are obtained as follows, viz. the first sinister line $(+3, -1)$ by

$$(-1, +1)(-2, +1) = 2+1 = +3$$
$$(-1, +1)(+1, \ 0) = -1+0 = -1,$$

and the second sinister line $(-1, 0)$ by

$$(0, -1)(-2, +1) = 0-1 = -1$$
$$(0, -1)(+1, \ 0) = 0+0 = \ 0.$$

I have calculated the determinants required for the calculation, by the preceding process, of the Resultant of two quartic equations, and have indeed used them for the verification of the expression as found by the method of symmetric functions; as the determinants in question are useful for other purposes, I think the values are worth preserving.

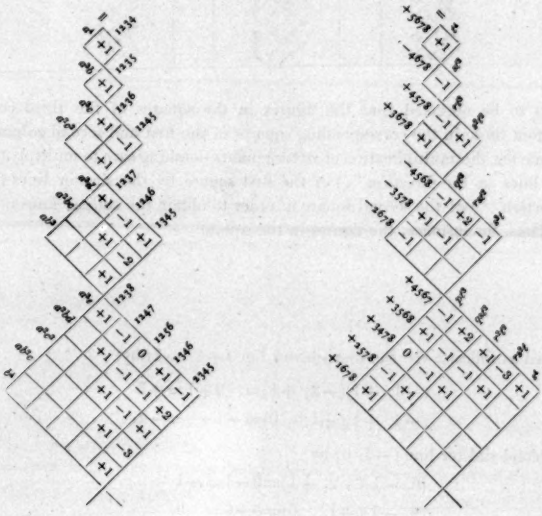
Table of the Determinants of the Matrices,

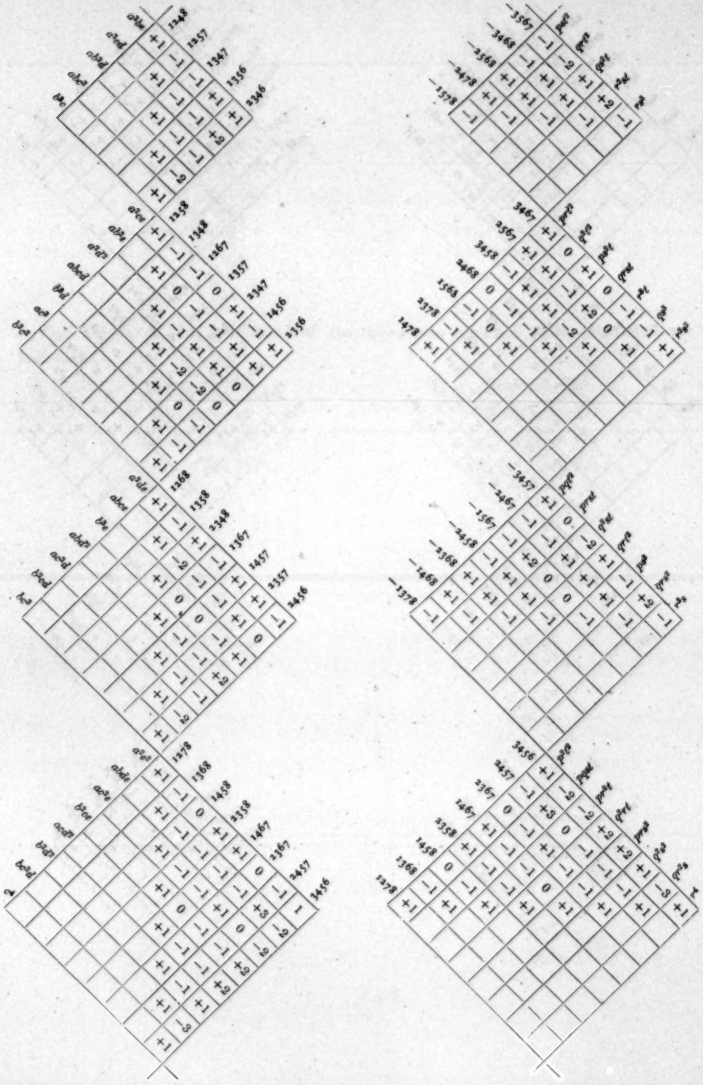
$$\begin{pmatrix} a, & b, & c, & d, & e \\ & a, & b, & c, & d, & e \\ & & a, & b, & c, & d, & e \\ & & & a, & b, & c, & d, & e \end{pmatrix}$$

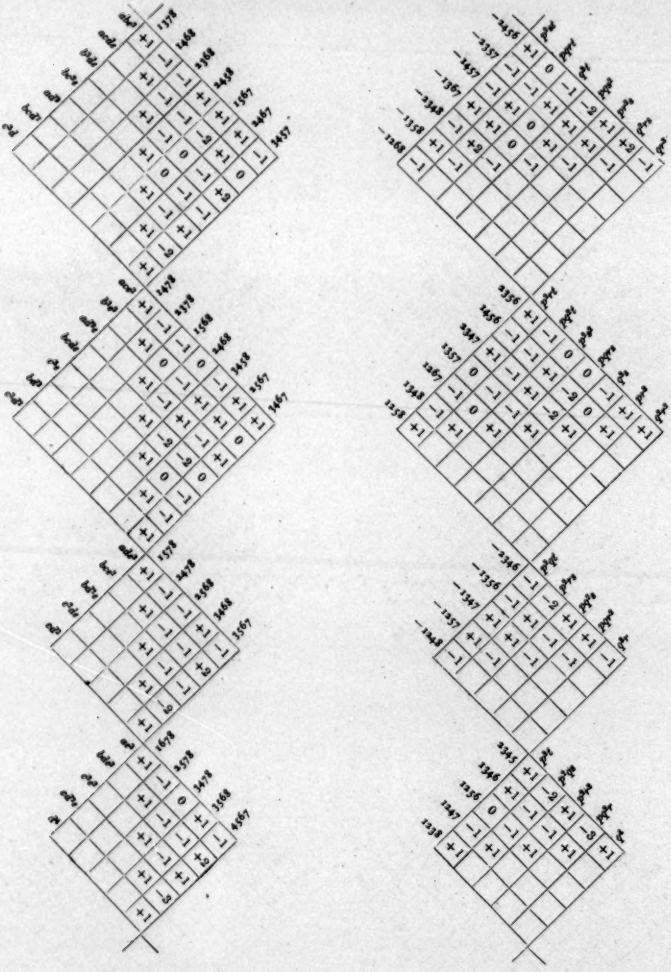
and

$$\begin{pmatrix} p, & q, & r, & s, & t \\ & p, & q, & r, & s, & t \\ & & p, & q, & r, & s, & t \\ & & & p, & q, & r, & s, & t \end{pmatrix}$$

arranged in the form adapted for the calculation of the Resultant of the two quartic equations $(a, b, c, d, e)(x, y)^4=0$, and $(p, q, r, s, t)(x, y)^4=0$, viz.







The Tables of the Resultants of two equations which I have calculated are as follows:—

Table (2, 2).

Resultant of
$(a, b, c\chi(x, y)^2$,
$(p, q, r\chi(x, y)^2$.

Table (3, 2).

Resultant of
$(a, b, c\ d\chi(x, y)^3$,
$(p, q, r\ \chi(x, y)^2$.

Table (4, 2).

Resultant of
$(a, b, c, d, e)(x, y)^4$,
$(p, q, r)(x, y)^2$.

Table (3, 3).

Resultant of
$(a, b, c, d)(x, y)^3$,
$(p, q, r, s)(x, y)^3$.

Table (4, 3).

Resultant of
$(a, b, c, d, e)(x, y)^4$,
$(p, q, r, s)(x, y)^3$.

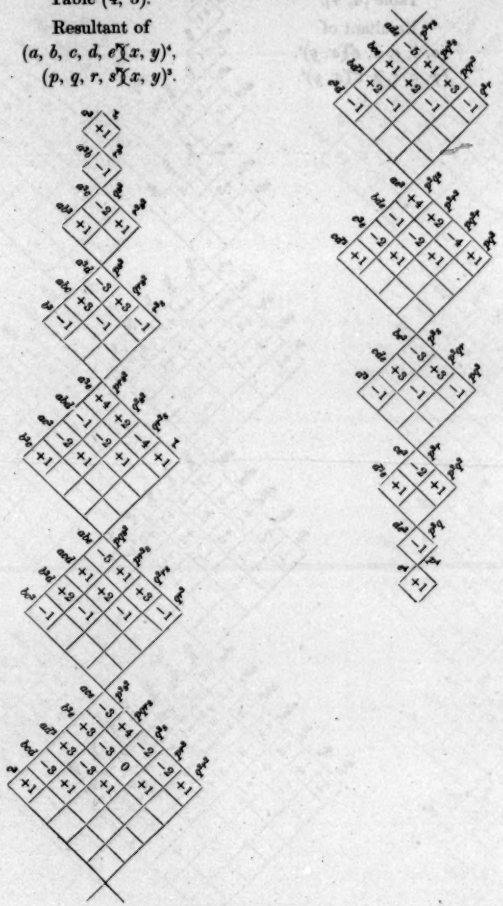


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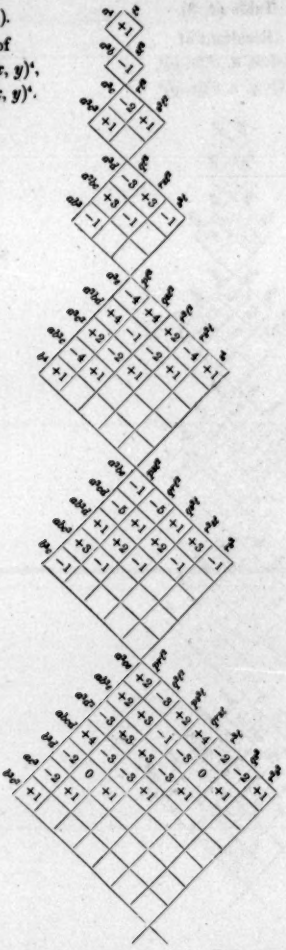
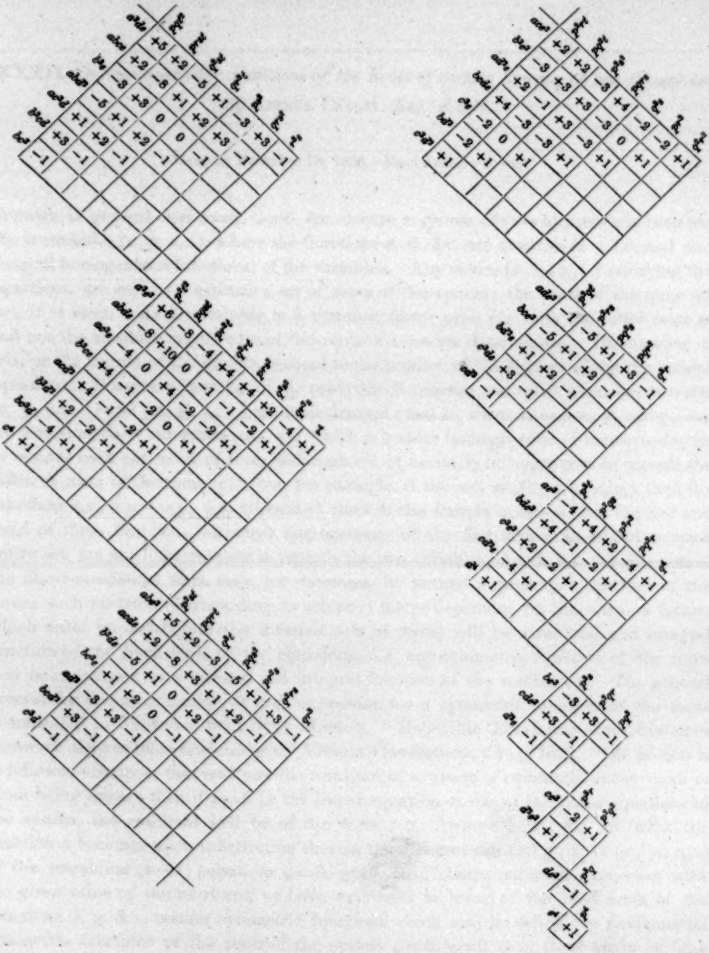
Table (4, 4).

Resultant of

$(a, b, c, d, e)(x, y)^4$,

$(p, q, r, s, t)(x, y)^4$.





XXXII. *On the Symmetric Functions of the Roots of certain Systems of two Equations.*

By ARTHUR CAYLEY, *Esq.*, *F.R.S.*

Received December 18, 1856,—Read January 8, 1857.

SUPPOSE in general that $\phi=0$, $\psi=0$, &c. denote a system of $(n-1)$ equations between the n variables $(x, y, z, ..)$, where the functions ϕ, ψ, &c. are quantics (*i. e.* rational and integral homogeneous functions) of the variables. Any values $(x_1, y_1, z_1, ..)$ satisfying the equations, are said to constitute a set of roots of the system; the roots of the same set are, it is clear, only determinate to a common factor *près*, *i. e.* only the ratios *inter se* and not the absolute magnitudes of the roots of a set are determinate. The number of sets, or the degree of the system, is equal to the product of the degrees of the component equations. Imagine a function of the roots which remains unaltered when any two sets $(x_1, y_1, z_1, \ldots)$ and $(x_2, y_2, z_2, ..)$ are interchanged (that is, when x_1 and x_2, y_1 and y_2, &c. are simultaneously interchanged), and which is besides homogeneous of the same degree as regards each entire set of roots, although not of necessity homogeneous as regards the different roots of the same set; thus, for example, if the sets are (x_1, y_1), (x_2, y_2), then the functions x_1x_2, $x_1y_2+x_2y_1$, y_1y_2 are each of them of the form in question; but the first and third of these functions, although homogeneous of the first degree in regard to each entire set, are not homogeneous as regards the two variables of each set. A function of the above-mentioned form may, for shortness, be termed a symmetric function of the roots; such function (disregarding an arbitrary factor depending on the common factors which enter implicitly into the different sets of roots) will be a rational and integral function of the coefficients of the equations, *i. e.* any symmetric function of the roots may be considered as a rational and integral function of the coefficients. The general process for the investigation of such expression for a symmetric function of the roots is indicated in Professor SCHLÄFLI's Memoir, "Ueber die Resultante eines Systemes mehrerer algebraischer Gleichungen," Vienna Transactions, t. iv. (1852). The process is as follows:—Suppose that we know the resultant of a system of equations, one or more of them being linear; then if $\phi=0$ be the linear equation or one of the linear equations of the system, the resultant will be of the form $\phi_1\phi_2..$, where ϕ_1, ϕ_2, &c. are what the function ϕ becomes upon substituting therein the different sets $(x_1, y_1, z_1..)$, $(x_2, y_2, z_2..)$ of the remaining $(n-1)$ equations $\psi=0$, $\chi=0$, &c.; comparing such expression with the given value of the resultant, we have expressed in terms of the coefficients of the functions ψ, χ, &c., certain symmetric functions which may be called the fundamental symmetric functions of the roots of the system $\psi=0$, $\chi=0$, &c.; these are in fact the symmetric functions of the first degree in respect to each set of roots. By the aid of these fundamental symmetric functions, the other symmetric functions of the roots of the system $\psi=0$, $\chi=0$, &c. may be expressed in terms of the coefficients, and then com-

bining with these equations a non-linear equation $\Phi=0$, the resultant of the system $\Phi=0$, $\psi=0$, $\chi=0$, &c. will be what the function $\Phi_1\Phi_2\ldots$ becomes, upon substituting therein for the different symmetric functions of the roots of the system $\psi=0$, $\chi=0$, &c. the expressions for these functions in terms of the coefficients. We thus pass from the resultant of a system $\phi=0$, $\psi=0$, $\chi=0$, &c., to that of a system $\Phi=0$, $\psi=0$, $\chi=0$, &c., in which the linear function ϕ is replaced by the non-linear function Φ. By what has preceded, the symmetric functions of the roots of a system of $(n-1)$ equations depend on the resultant of the system obtained by combining the $(n-1)$ equations with an arbitrary linear equation; and moreover, the resultant of any system of n equations depends ultimately upon the resultant of a system of the same number of equations, all except one being linear; but in this case the linear equations determine the ratios of the variables or (disregarding a common factor) the values of the variables, and by substituting these values in the remaining equation we have the resultant of the system. The process leads, therefore, to the expressions for the symmetric functions of the roots of any system of $(n-1)$ equations, and also to the expression for the resultant of any system of n equations. Professor SCHLÄFLI discusses in the general case the problem of showing how the expressions for the fundamental symmetric functions lead to those of the other symmetric functions, but it is not necessary to speak further of this portion of his investigations. The object of the present Memoir is to apply the process to two particular cases, viz. I propose to obtain thereby the expressions for the simplest symmetric functions (after the fundamental ones) of the following systems of two ternary equations; that is, first, a linear equation and a quadric equation; and secondly, a linear equation and a cubic equation.

First, consider the two equations

$$(a, b, c, f, g, h)(x, y, z)^2=0,$$
$$(\alpha, \beta, \gamma)(x, y, z)=0,$$

and join to these the arbitrary linear equation

$$(\xi, \eta, \zeta)(x, y, z)=0,$$

then the two linear equations give

$$x : y : z=\beta\zeta-\gamma\eta : \gamma\xi-\alpha\zeta : \alpha\eta-\beta\xi;$$

and substituting in the quadratic equation, we have for the resultant of the three equations,

$$(a, b, c, f, g, h)(\beta\zeta-\gamma\eta, \gamma\xi-\alpha\zeta, \alpha\eta-\beta\xi)^2=0,$$

which may be represented by

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(\xi, \eta, \zeta)^2=0,$$

where the coefficients are given by means of the Table.

	a	b	c	f	g	h	
a=		$+\gamma^2$	$+\beta^2$	$-2\beta\gamma$			(ξ^2)
b=	$+\gamma^2$		$+\alpha^2$		$-2\gamma\alpha$		(η^2)
c=	$+\beta^2$	$+\alpha^2$				$-2\alpha\beta$	(ζ^2)
f=	$-\beta\gamma$			$-\alpha^2$	$+\alpha\beta$	$+\gamma\alpha$	$2(\eta\zeta)$
g=		$-\gamma\alpha$		$+\alpha\beta$	$-\beta^2$	$+\beta\gamma$	$2(\zeta\xi)$
h=			$-\alpha\beta$	$+\gamma\alpha$	$+\beta\gamma$	$-\gamma^2$	$2(\xi\eta)$

viz. $\mathrm{a}=b\gamma^2+c\beta^2-2f\beta\gamma$, &c.

But if the roots of the given system are

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2),$$

then the resultant of the three equations will be

$$(x_1, y_1, z_1)(\xi, \eta, \zeta).(x_2, y_2, z_2)(\xi, \eta, \zeta)=0;$$

and comparing the two expressions, we have

$$\begin{aligned} a &= x_1x_2, \\ b &= y_1y_2, \\ c &= z_1z_2, \\ 2f &= y_1z_2+y_2z_1, \\ 2g &= z_1x_2+z_2x_1, \\ 2h &= x_1y_2+x_2y_1, \end{aligned}$$

which are the expressions for the six fundamental symmetric functions, or symmetric functions of the first degree in each set, of the roots of the given system.

By forming the powers and products of the second order a^2, ab, &c., we obtain linear relations between the symmetric functions of the second degree in respect to each set of roots. The number of equations is precisely equal to that of the symmetric functions of the form in question, and the solution of the linear equations gives—

$$\begin{aligned} a^2 &= x_1^2x_2^2, \\ b^2 &= y_1^2y_2^2, \\ c^2 &= z_1^2z_2^2, \\ bc &= y_1z_1y_2z_2, \\ ca &= z_1x_1z_2x_2, \\ ab &= x_1y_1x_2y_2, \\ 4f^2-2bc &= y_1^2z_2^2+y_2^2z_1^2, \\ 4g^2-2ca &= z_1^2x_2^2+z_2^2x_1^2, \\ 4h^2-2ab &= x_1^2y_2^2+x_2^2y_1^2, \\ 2af &= x_1y_1z_2x_2+z_1x_1x_2y_2, \\ 2bg &= y_1z_1x_2y_2+x_1y_1y_2z_2, \\ 2ch &= z_1x_1y_2z_2+y_1z_1z_2x_2, \\ 4gh-2af &= x_1^2y_2z_2+x_2^2y_1z_1, \\ 4hf-2bg &= y_1^2z_2x_2+y_2^2z_1x_1, \\ 4fg-2ch &= z_1^2x_2y_2+z_2^2x_1y_1, \\ 2bf &= y_1^2y_2z_2+y_2^2y_1z_1, \\ 2cg &= z_1^2z_2x_2+z_2^2z_1x_1, \\ 2ah &= x_1^2x_2y_2+x_2^2x_1y_1, \end{aligned}$$

$$2cf = x_1^2y_2z_3 + x_2^2y_1z_1,$$
$$2ag = x_1^2z_2x_3 + x_2^2z_1x_1,$$
$$2bh = y_1^2x_2y_3 + y_2^2x_1y_1.$$

Proceeding next to the powers and products of the third order a^3, a^2b, &c., the total number of linear relations between the symmetric functions of the third degree in respect to each set of roots exceeds by unity the number of the symmetric functions of the form in question; in fact the expressions for abc, af^2, bg^2, ch^2, fgh contain, not five, but only four symmetric functions of the roots; for we have

$$abc = x_1y_1z_1 . x_2y_2z_2$$
$$4af^2 = (x_1y_1^2x_2z_2^2 + x_2y_2^2x_1z_1^2) + 2x_1y_1z_1x_2y_2z_2,$$
$$4bg^2 = (y_1z_1^2y_2x_2^2 + y_2z_2^2y_1x_1^2) + 2x_1y_1z_1x_2y_2z_2,$$
$$4ch^2 = (z_1x_1^2z_2y_2^2 + z_2x_2^2z_1y_1^2) + 2x_1y_1z_1x_2y_2z_2,$$
$$\left.\begin{aligned} 8fgh = {} & (x_1y_1^2x_2z_2^2 + x_2y_2^2x_1z_1^2) \\ & + (y_1z_1^2y_2x_2^2 + y_2z_2^2y_1x_1^2) \\ & + (z_1x_1^2z_2y_2^2 + z_2x_2^2z_1y_1^2) \end{aligned}\right\} + 2x_1y_1z_1x_2y_2z_2,$$

and consequently the quantities a, b, c, f. g, h, are not independent, but are connected by the equation

$$abc - af^2 - bg^2 - ch^2 + 2fgh = 0,$$

an equation, which is in fact verified by the foregoing values of a, &c. in terms of the coefficients of the given system.

The expressions for the symmetric functions of the third degree considered as functions of a, b, c, f, g, h, are consequently not absolutely determinate, but they may be modified by the addition of the term $\lambda(abc - af^2 - bg^2 - ch^2 + 2fgh)$, where λ is an indeterminate numerical coefficient.

The simplest expressions are those obtained by disregarding the preceding equation for fgh, and the entire system then becomes—

$$a^3 = x_1^3x_2^3,$$
$$b^3 = y_1^3y_2^3,$$
$$c^3 = z_1^3z_2^3,$$
$$b^3c = y_1^3z_1y_2^3z_2,$$
$$c^3a = z_1^3x_1z_2^3x_2,$$
$$a^3b = x_1^3y_1x_2^3y_2,$$
$$bc^3 = y_1z_1^3y_2z_2^3,$$
$$ca^3 = z_1x_1^3z_2x_2^3,$$
$$ab^3 = x_1y_1^3x_2y_2^3,$$
$$abc = x_1y_1z_1x_2y_2z_2,$$

$$
\begin{aligned}
2a^2f &= x_1^2y_1z_2x_2^2 + x_2^2y_2z_1x_1^2,\\
2b^2g &= y_1^2z_1x_2y_2^2 + y_2^2z_2x_1y_1^2,\\
2c^2h &= z_1^2x_1y_2z_2^2 + z_2^2x_2y_1z_1^2,
\end{aligned}
$$

$$
\begin{aligned}
2a^2g &= x_1^2z_2x_2^2 + x_2^2z_1x_1^2,\\
2b^2h &= y_1^2x_2y_2^2 + y_2^2x_1y_1^2,\\
2c^2f &= z_1^2y_2z_2^2 + z_2^2y_1z_1^2,
\end{aligned}
$$

$$
\begin{aligned}
2a^2h &= x_1^2x_2^2x_3 + x_2^2x_1^2x_1,\\
2b^2f &= y_1^2x_2^2y_3 + y_2^2x_1^2y_1,\\
2c^2g &= z_1^2y_2^2z_3 + z_2^2y_1^2z_1,
\end{aligned}
$$

$$
\begin{aligned}
2bcf &= y_1^2z_1y_2z_2^2 + y_2^2z_2y_1z_1^2,\\
2cag &= z_1^2x_1z_2x_2^2 + z_2^2x_2z_1x_1^2,\\
2abh &= x_1^2y_1x_2y_2^2 + x_2^2y_2x_1y_1^2,
\end{aligned}
$$

$$
\begin{aligned}
2bcg &= y_1z_1^2x_2y_2z_2 + y_2z_2^2x_1y_1z_1,\\
2cah &= z_1x_1^2x_2y_2z_2 + z_2x_2^2x_1y_1z_1,\\
2abf &= x_1y_1^2x_2y_2z_2 + x_2y_2^2x_1y_1z_1,
\end{aligned}
$$

$$
\begin{aligned}
2bch &= y_1^2z_1x_2y_2z_2 + y_2^2z_2x_1y_1z_1,\\
2caf &= z_1^2x_1x_2y_2z_2 + z_2^2x_2x_1y_1z_1,\\
2abg &= x_1^2y_1x_2y_2z_2 + x_2^2y_2x_1y_1z_1,
\end{aligned}
$$

$$
\begin{aligned}
4af^2 - 2abc &= x_1y_1^2z_2^2x_2 + x_2y_2^2z_1^2x_1,\\
4bg^2 - 2abc &= y_1z_1^2x_2^2y_2 + y_2z_2^2x_1^2y_1,\\
4ch^2 - 2abc &= z_1x_1^2y_2^2z_2 + z_2x_2^2y_1^2z_1,
\end{aligned}
$$

$$
\begin{aligned}
4bf^2 - 2b^2c &= y_1^2y_2z_2^2 + y_2^2y_1z_1^2,\\
4cg^2 - 2c^2a &= z_1^2z_2x_2^2 + z_2^2z_1x_1^2,\\
4ah^2 - 2a^2b &= x_1^2x_2y_2^2 + x_2^2x_1y_1^2,
\end{aligned}
$$

$$
\begin{aligned}
4cf^2 - 2bc^2 &= z_1^2y_2^2z_3 + z_2^2y_1^2z_1,\\
4ag^2 - 2ca^2 &= x_1^2z_2^2x_3 + x_2^2z_1^2x_1,\\
4bh^2 - 2ab^2 &= y_1^2x_2^2y_3 + y_2^2x_1^2y_1,
\end{aligned}
$$

$$
\begin{aligned}
4agh - 2a^2f &= x_1^2x_2y_2z_2 + x_2^2x_1y_1z_1,\\
4bhf - 2b^2g &= y_1^2x_2y_2z_2 + y_2^2x_1y_1z_1,\\
4cfg - 2c^2h &= z_1^2x_2y_2z_2 + z_2^2x_1y_1z_1,
\end{aligned}
$$

$$\begin{aligned}
4bgh - 2abf &= y_1^2 z_1 x_2^2 y_3 + y_2^2 z_2 x_3^2 y_1, \\
4chf - 2bcg &= z_1^2 x_1 y_2^2 z_3 + z_2^2 x_2 y_3^2 z_1, \\
4afg - 2cah &= x_1^2 y_1 z_2^2 x_3 + x_2^2 y_2 z_3^2 x_1,
\end{aligned}$$

$$\begin{aligned}
4cgh - 2acf &= y_1 z_1^2 x_2^2 z_3 + y_2 z_2^2 x_3^2 z_1, \\
4ahf - 2bag &= z_1 x_1^2 y_2^2 x_3 + z_2 x_2^2 y_3^2 x_1, \\
4bfg - 2cbh &= x_1 y_1^2 z_2^2 y_3 + x_2 y_2^2 z_3^2 y_1,
\end{aligned}$$

$$\begin{aligned}
8f^3g - 4chf - 2bcg &= z_1^2 x_2 y_3^2 + z_2^2 x_1 y_1^2, \\
8g^3h - 4afg - 2cah &= x_1^2 y_2 z_3^2 + x_2^2 y_1 z_1^2, \\
8h^3f - 4bgh - 2abf &= y_1^2 z_2 x_3^2 + y_2^2 z_1 x_1^2,
\end{aligned}$$

$$\begin{aligned}
8fg^3 - 4chg - 2acf &= z_1^2 x_2^2 y_3 + z_2^2 x_1^2 y_1, \\
8gh^3 - 4afh - 2bag &= x_1^2 y_2^2 z_3 + x_2^2 y_3^2 z_1, \\
8hf^3 - 4bgf - 2cbh &= y_1^2 z_2^2 x_3 + y_2^2 z_3^2 x_1,
\end{aligned}$$

$$\begin{aligned}
8f^3 - 6bcf &= y_1^2 z_2^2 + y_2^2 z_1^2, \\
8g^3 - 6cag &= z_1^2 x_2^2 + z_2^2 x_1^2, \\
8h^3 - 6abh &= x_1^2 y_2^2 + x_2^2 y_1^2.
\end{aligned}$$

Secondly, consider the system of equations

$$\begin{aligned}
(a, b, c, f, g, h, i, j, k, l)(x, y, z)^3 &= 0, \\
(\alpha, \beta, \gamma)(x, y, z) &= 0,
\end{aligned}$$

where the cubic function written at full length is

$$ax^3 + by^3 + cz^3 + 3fy^2z + 3gz^2x + 3hx^2y + 3iyz^2 + 3jzx^2 + 3kxy^2 + 6lxyz$$

Joining to the system the linear equation

$$(\xi, \eta, \zeta)(x, y, z) = 0,$$

the linear equations give

$$x : y : z = \beta\zeta_0 - \gamma\eta : \gamma\xi - \alpha\zeta_0 : \alpha\eta - \beta\xi,$$

and the resultant is

$$(a, b, c, f, g, h, i, j, k, l)(\beta\zeta - \gamma\eta, \gamma\xi - \alpha\zeta, \alpha\eta - \beta\xi)^3 = 0,$$

which may be represented by

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{i}, \mathrm{j}, \mathrm{k}, \mathrm{l})(\xi, \eta, \zeta)^3 = 0,$$

where the coefficients a, b, &c. are given by means of the Table :—

	a	b	c	f	g	h	i	j	k	l	
$a=$		$+\gamma^3$	$-\beta^3$	$-3\beta\gamma^2$			$+3\beta^2\gamma$				ξ^3
$b=$	$-\gamma^3$		$+\alpha^3$		$-3\gamma\alpha^2$			$+3\gamma^2\alpha$			η^3
$c=$	$+\beta^3$	$-\alpha^3$				$-3\alpha\beta^2$			$+3\alpha^2\beta$		ζ^3
$f=$	$+\beta\gamma^2$				$+\alpha^2\beta$	$-\gamma^2\alpha$	$-\alpha^3$	$-2\alpha\beta\gamma$		$+2\gamma\alpha^3$	$3\eta^2\zeta$
$g=$		$+\gamma\alpha^2$		$-\alpha^2\beta$		$+\beta^2\gamma$		$-\beta^3$	$-2\alpha\beta\gamma$	$+2\alpha\beta^3$	$3\zeta^2\xi$
$h=$			$+\alpha\beta^2$	$+\gamma^2\alpha$	$-\beta^2\gamma$		$-2\alpha\beta\gamma$		$-\gamma^3$	$+2\beta\gamma^2$	$3\xi^2\eta$
$i=$	$-\beta^2\gamma$			$+\alpha^3$		$+2\alpha\beta\gamma$		$+\alpha\beta^2$	$-\gamma\alpha^2$	$-2\alpha^2\beta$	$3\eta\zeta^2$
$j=$		$-\gamma^2\alpha$		$+2\alpha\beta\gamma$	$+\beta^3$		$-\alpha\beta^2$		$+\beta\gamma^2$	$-2\beta^2\gamma$	$3\zeta\xi^2$
$k=$			$-\alpha^2\beta$		$+2\alpha\beta\gamma$	$+\gamma^3$	$+\gamma\alpha^2$	$-\beta\gamma^2$		$-2\gamma^2\alpha$	$3\xi^2\eta$
$l=$				$-\gamma\alpha^2$	$-\alpha\beta^2$	$-\beta\gamma^2$	$+\alpha^2\beta$	$+\beta^2\gamma$	$+\gamma^2\alpha$		$6\xi\eta\zeta$

viz. $$\mathrm{a}=bf^3-cb^3-3f\beta\gamma^3+3i\beta^2\gamma, \text{ \&c.};$$

but if the roots of the given system are

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3),$$

then the resultant of the three equations may also be represented by

$$(x_1, y_1, z_1)(\xi, \eta, \zeta).(x_2, y_2, z_2)(\xi, \eta, \zeta).(x_3, y_3, z_3)(\xi, \eta, \zeta);$$

and comparing with the former expression, we find—

$$\mathrm{a}=x_1x_2x_3,$$
$$\mathrm{b}=y_1y_2y_3,$$
$$\mathrm{c}=z_1z_2z_3,$$

$$3\mathrm{f}=y_1y_2z_3+y_2y_3z_1+y_3y_1z_2,$$
$$3\mathrm{g}=z_1z_2x_3+z_2z_3x_1+z_3z_1x_2,$$
$$3\mathrm{h}=x_1x_2y_3+x_2x_3y_1+x_3x_1y_2,$$

$$3\mathrm{i}=y_1z_2z_3+y_2z_3z_1+y_3z_1z_2,$$
$$3\mathrm{j}=z_1x_2x_3+z_2x_3x_1+z_3x_1x_2,$$
$$3\mathrm{k}=x_1y_2y_3+x_2y_3y_1+x_3y_1y_2,$$

$$6\mathrm{l}=x_1y_2z_3+x_2y_3z_1+x_3y_1z_2+x_1y_3z_2+x_2y_1z_3+x_3y_2z_1.$$

But there is in the present case a relation independent of the quantities a, &c., viz. we have $(\alpha, \beta, \gamma)(x_1, y_1, z_1)=0$, $(\alpha, \beta, \gamma)(x_2, y_2, z_2)=0$, $(\alpha, \beta, \gamma)(x_3, y_3, z_3)=0$, and thence eliminating the coefficients (α, β, γ), we find

$$\nabla=x_1y_2z_3+x_2y_3z_1+x_3y_1z_2-x_1y_3z_2-x_2y_1z_3-x_3y_2z_1=0.$$

By forming the powers and products of the second degree a^2, ab, &c., we obtain 55 equations between the symmetric functions of the second degree in each set of roots. But we have $\nabla^2=0=$a symmetric function of the roots, and thus the entire number of linear relations is 56, and this is in fact the number of the symmetric functions of the second degree in each set. I use for shortness the sign S to denote the sum of the

distinct terms obtained by permuting the different sets of roots, so that the equations for the fundamental symmetric functions are—

$$\begin{aligned}
\mathrm{a} &= x_1x_2x_3,\\
\mathrm{b} &= y_1y_2y_3,\\
\mathrm{c} &= z_1z_2z_3,\\
3\mathrm{f} &= \mathrm{S}\,y_1y_2z_3,\\
3\mathrm{g} &= \mathrm{S}\,z_1z_2x_3,\\
3\mathrm{h} &= \mathrm{S}\,x_1x_2y_3,\\
3\mathrm{i} &= \mathrm{S}\,y_1z_2z_3,\\
3\mathrm{j} &= \mathrm{S}\,z_1x_2x_3,\\
3\mathrm{k} &= \mathrm{S}\,x_1y_2y_3,\\
6\mathrm{l} &= \mathrm{S}\,x_1y_2z_3;
\end{aligned}$$

then the complete system of expressions for the symmetric functions of the second order is as follows, viz.—

$$\begin{aligned}
\mathrm{a}^2 &= x_1^2x_2^2x_3^2,\\
\mathrm{b}^2 &= y_1^2y_2^2y_3^2,\\
\mathrm{c}^2 &= z_1^2z_2^2z_3^2,\\
\mathrm{bc} &= y_1z_1y_2z_2y_3z_3,\\
\mathrm{ca} &= z_1x_1z_2x_2z_3x_3,\\
\mathrm{ab} &= x_1y_1x_2y_2x_3y_3,\\
3\mathrm{af} &= \mathrm{S}\,x_1y_1x_2y_2z_3x_3,\\
3\mathrm{bg} &= \mathrm{S}\,y_1z_1y_2z_2x_3y_3,\\
3\mathrm{ch} &= \mathrm{S}\,z_1x_1z_2x_2y_3z_3,\\
3\mathrm{bf} &= \mathrm{S}\,y_1^2y_2^2y_3z_3,\\
3\mathrm{cg} &= \mathrm{S}\,z_1^2z_2^2z_3x_3,\\
3\mathrm{ah} &= \mathrm{S}\,x_1^2x_2^2x_3y_3,\\
3\mathrm{cf} &= \mathrm{S}\,y_1z_1y_2z_2z_3^2,\\
3\mathrm{ag} &= \mathrm{S}\,z_1x_1z_2x_2x_3^2,\\
3\mathrm{bh} &= \mathrm{S}\,x_1y_1x_2y_2y_3^2,\\
3\mathrm{ai} &= \mathrm{S}\,x_1y_1z_2x_3z_3x_3,\\
3\mathrm{bj} &= \mathrm{S}\,y_1z_1x_2y_2x_3y_3,\\
3\mathrm{ck} &= \mathrm{S}\,z_1x_1y_2z_2y_3z_3,
\end{aligned}$$

$$3bi = S\, y_1^2 y_2 z_2 y_3 z_3,$$
$$3cj = S\, z_1^2 z_2 x_2 z_3 x_3,$$
$$3ak = S\, x_1^2 x_2 y_2 x_3 y_3,$$
$$3ci = S\, y_1 z_1 z_2^2 z_3^2,$$
$$3aj = S\, z_1 x_1 x_2^2 x_3^2,$$
$$3bk = S\, x_1 y_1 y_2^2 y_3^2,$$
$$6al = S\, x_1^2 x_2 y_2 z_2 x_3,$$
$$6bl = S\, y_1^2 y_2 z_2 x_2 y_3,$$
$$6cl = S\, z_1^2 z_2 x_2 y_2 z_3,$$
$$9f^2 - 6bi = S\, y_1^2 y_2^2 z_3^2,$$
$$9g^2 - 6cj = S\, z_1^2 z_2^2 x_3^2,$$
$$9h^2 - 6ak = S\, x_1^2 x_2^2 y_3^2,$$
$$9i^2 - 6cf = S\, y_1^2 z_2^2 z_3^2,$$
$$9j^2 - 6ag = S\, x_1^2 x_2^2 x_3^2,$$
$$9k^2 - 6bh = S\, x_1^2 y_2^2 y_3^2,$$
$$9fg - 3ck = S\, x_1 y_1 z_2 z_3^2,$$
$$9gh - 3ai = S\, y_1 z_1 z_2 x_2 x_3^2,$$
$$9hf - 3bj = S\, z_1 x_2 x_2 y_2 y_3^2,$$
$$9jk - 3af = S\, x_1^2 x_2 y_2 y_3 z_3,$$
$$9ki - 3bg = S\, y_1^2 y_2 z_2 z_3 x_3,$$
$$9ij - 3ch = S\, z_1^2 z_2 x_2 x_3 y_3,$$
$$9fi - 3bc = S\, y_1^2 y_2 z_2 z_3^2,$$
$$9gj - 3ca = S\, z_1^2 z_2 x_2 x_3^2,$$
$$9hk - 3ab = S\, x_1^2 x_2 y_2 y_3^2,$$
$$3(fj + gk + hi - l^2) = S\, x_1 y_1 z_2 x_2 y_3 z_3,$$
$$3(2fj - gk - hi + l^2) = S\, x_1 y_1 x_2 y_2 z_3^2,$$
$$3(2gk - hi - fj + l^2) = S\, y_1 z_1 y_2 z_2 x_3^2,$$
$$3(2hi - fj - gk + l^2) = S\, z_1 x_1 z_2 x_2 y_3^2,$$
$$3(6fl - 3ki - bg) = S\, x_1 y_1 y_2^2 z_3^2,$$
$$3(6gl - 3ij - ch) = S\, y_1 z_1 z_2^2 x_3^2,$$
$$3(6hl - 3jk - af) = S\, z_1 x_1 x_2^2 y_3^2,$$

$$3(6il - 3fg - ck) = S\, x_1x_1y_2^2z_3^2,$$
$$3(6jl - 3gh - ai) = S\, x_1y_1z_2^2x_3^2,$$
$$3(6kl - 3hf - bj) = S\, y_1z_1x_2^2y_3^2,$$

$$6(-fj - gk - hi + 4l^3) = S\, x_1^2y_2^2z_3^2.$$

As an instance of the application of the formulæ, let it be required to eliminate the variables from the three equations,

$$(a, b, c, f, g, h, i, j, k, l)(x, y, z)^3 = 0,$$
$$(a', b', c', f', g', h')(x, y, z)^3 = 0,$$
$$(\alpha, \beta, \gamma)(x, y, z) = 0.$$

This may be done in two different ways; first, representing the roots of the linear equation and the quadric equation by (x_1, y_1, z_1), (x_2, y_2, z_2), the resultant will be

$$(a, \ldots)(x_1, y_1, z_1)^3 . (a, \ldots)(x_2, y_2, z_2)^3,$$

which is equal to

$$a^3 x_1^2x_2^2 + \&c.,$$

where the symmetric functions $x_1^2x_2^2$, &c. are given by the formula $a'^3 = x_1^2x_2^2$, &c., where the coefficients of the quadratic equation being (a', b', c', f', g', h') I have written a' instead of a. Next, if the roots of the linear equation and the cubic equation are represented by (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), then the resultant will be

$$(a', \ldots)(x_1, y_1, z_1)^3 . (a', \ldots)(x_2, y_2, z_2)^2 (a', \ldots)(x_3, y_3, z_3)^2,$$

which is equal to

$$a'^3\, x_1^2x_2^2x_3^2 + \&c.,$$

the symmetric functions $x_1^2x_2^2x_3^2$, &c. being given by the formulæ $a^2 = x_1^2x_2^2x_3^2$, &c. The expression for the Resultant is in each case of the right degree, viz. of the degrees 6, 3, 2, in the coefficients of the linear, the quadric, and the cubic equations respectively: the two expressions, therefore, can only differ by a numerical factor, which might be determined without difficulty. The third expression for the resultant, viz.

$$(\alpha, \beta, \gamma)(x_1, y_1, z_1) . (\alpha, \beta, \gamma)(x_2, y_2, z_2) \ldots (\alpha, \beta, \gamma)(x_6, y_6, z_6),$$

(where $(x_1, y_1, z_1), \ldots (x_6, y_6, z_6)$ are the roots of the cubic and quadratic equations) compared with the foregoing value, leads to expressions for the fundamental symmetric functions of the cubic and quadratic equations, and thence to expressions for the other symmetric functions of these two equations; but it would be difficult to obtain the actually developed values even of the fundamental symmetric functions. I hope to return to the subject, and consider in a general point of view the question of the formation of the expressions for the other symmetric functions by means of the expressions for the fundamental symmetric functions.

XXXIII. *A Memoir on the Conditions for the Existence of given Systems of Equalities among the Roots of an Equation.* By ARTHUR CAYLEY, *Esq.*, *F.R.S.*

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IT is well known that there is a symmetric function of the roots of an equation, viz. the product of the squares of the differences of the roots, which vanishes when any two roots are put equal to each other, and that consequently such function expressed in terms of the coefficients and equated to zero, gives the condition for the existence of a pair of equal roots. And it was remarked long ago by Professor SYLVESTER, in some of his earlier papers in the 'Philosophical Magazine,' that the like method could be applied to finding the conditions for the existence of other systems of equalities among the roots, viz. that it was possible to form symmetric functions, each of them a sum of terms containing the product of a certain number of the differences of the roots, and such that the entire function might vanish for the particular system of equalities in question; and that such functions expressed in terms of the coefficients and equated to zero would give the required conditions. The object of the present memoir is to extend this theory and render it exhaustive, by showing how to form a series of types of all the different functions which vanish for one or more systems of equalities among the roots; and in particular to obtain by the method distinctive conditions for all the different systems of equalities between the roots of a quartic or a quintic equation, viz. for each system conditions which are satisfied for the particular system, and are not satisfied for any other systems, except, of course, the more special systems included in the particular system. The question of finding the conditions for any particular system of equalities is essentially an indeterminate one, for given any set of functions which vanish, a function syzygetically connected with these will also vanish; the discussion of the nature of the syzygetic relations between the different functions which vanish for any particular system of equalities, and of the order of the system composed of the several conditions for the particular system of equalities, does not enter into the plan of the present memoir. I have referred here to the indeterminateness of the question for the sake of the remark that I have availed myself thereof, to express by means of invariants or covariants the different systems of conditions obtained in the sequel of the memoir; the expressions of the different invariants and covariants referred to are given in my 'Second Memoir upon Quantics,' Philosophical Transactions, vol. cxlvi. (1856).

1. Suppose, to fix the ideas, that the equation is one of the fifth order, and call the roots $\alpha, \beta, \gamma, \delta, \varepsilon$. Write $12 = \Sigma\varphi(\alpha-\beta)^l$, $12.13 = \Sigma\varphi(\alpha-\beta)^l(\alpha-\gamma)^m$, $12.34 = \Sigma\varphi(\alpha-\beta)^l(\gamma-\delta)^n$, &c., where φ is an arbitrary function and l, m, &c. are positive integers. It is hardly necessary to remark that similar types, such as 12, 13, 45, &c., or as 12.13 and 23.25, &c., denote identically the same sums. Two types, such as 12.13 and 14.15.23.24.25.34.35.45,

may be said to be complementary to each other. A particular product $(\alpha-\beta)(\gamma-\delta)$ does or does not enter as a term (or factor of a term) in one of the above-mentioned sums, according as the type 12.34 of the product, or some similar type, does or does not form part of the type of the sum; for instance, the product $(\alpha-\beta)(\gamma-\delta)$ is a term (or factor of a term) of each of the sums 12.34, 13.45.24, &c., but not of the sums 12.13.14.15, &c.

2. If, now, we establish any equalities between the roots, *e. g.* $\alpha=\beta$, $\gamma=\delta$, the effect will be to reduce certain of the sums to zero, and it is easy to find in what cases this happens. The sum will vanish if each term contains one or both of the factors $\alpha-\beta$, $\gamma-\delta$, *i. e.* if there is no term the complementary of which contains the product $(\alpha-\beta)(\gamma-\delta)$, or what is the same thing, whenever the complementary type does not contain as part of it, a type such as 12.34. Thus for the sum 14.15.24.25.34.35.45, the complementary type is 12.13.23, which does not contain any type such as 12.34, *i. e.* the sum 14.15.24.25.34.35.45 vanishes for $\alpha=\beta$, $\gamma=\delta$. It is of course clear that it also vanishes for $\alpha=\beta=\epsilon$, $\gamma=\delta$ or $\alpha=\beta=\gamma=\delta$, &c., which are included in $\alpha=\beta$, $\gamma=\delta$. But the like reasoning shows, and it is important to notice, that the sum in question does not vanish for $\alpha=\beta=\gamma$: and of course it does not vanish for $\alpha=\beta$. Hence the vanishing of the sum 14.15.24.25.34.35.45 is characteristic of the system $\alpha=\beta$, $\gamma=\delta$. A system of roots α, β, γ, δ, ϵ may be denoted by 11111; but if $\alpha=\beta$, then the system may be denoted by 2111, or if $\alpha=\beta$, $\gamma=\delta$, by 221, and so on. We may then say that the sum 14.15.24.25.34.35.45 does not vanish for 2111, vanishes for 221, does not vanish for 311, vanishes for 32, 41, 5.

3. For the purpose of obtaining the entire system of results it is only necessary to form Tables, such as the annexed Tables, the meaning of which is sufficiently explained by what precedes: the mark (×) set against a type denotes that the sum represented by the complementary type vanishes, the mark (o) that the complementary type does not vanish, for the system of roots denoted by the symbol at the top or bottom of the column; the complementary type is given in the same horizontal line with the original type. It will be noticed that the right-hand columns do not extend to the foot of the Table; the reason of this of course is, to avoid a repetition of the same type. Some of the types at the foot of the Tables are complementary to themselves, but I have, notwithstanding this, given the complementary type in the form under which it naturally presents itself.

4. The Tables are—

Table for the equal Roots of a Quartic.

211	22	31	4
×	×	×	×
o	×	×	×
o	×	×	×
o	o	×	×
o	×	×	×
o	o	×	×
o	×	o	×
211	22	31	4

	211	22	31	4
12.13.14.23.24.34	o	o	o	o
12 | 13.14.23.24.34	o	o	o	×
12.13 | 14.23.24.34	o	o	o	×
12.34 | 13.14.23.24	o	o	×	×
12.13.14 | 23.24.34				
12.13.24 | 14.23.34	211	22	31	4
12.13.23 | 14.24.34				

Table for the equal Roots of a Quintic.

2111	221	311	32	41	5	
×	×	×	×	×	×	12.13.14.15.23.24.25.34.35.45
o	×	×	×	×	×	13.14.15.23.24.25.34.35.45
o	×	×	×	×	×	12.13.14.15.23.24.25.34.35.45
o	o	×	×	×	×	12.34.13.14.15.23.24.25.35.45
o	×	×	×	×	×	12.13.14.15.23.24.25.34.35.45
o	o	×	×	×	×	12.13.45.14.15.23.24.25.34.35
o	o	×	×	×	×	12.13.24.14.15.23.25.34.35.45
o	×	o	×	×	×	12.13.23.14.15.24.25.34.35.45
o	×	×	×	×	×	12.13.14.15.23.24.25.34.35.45
o	o	o	×	×	×	12.13.14.23.15.24.25.34.35.45
o	o	×	×	×	×	12.13.14.25.15.23.24.34.35.45
o	o	×	×	×	×	12.13.24.34.14.15.23.25.35.45
o	o	×	×	×	×	12.14.23.35.13.15.24.25.34.45
o	o	o	o	×	×	12.13.23.45.14.15.24.25.34.35
o	o	o	×	×	×	12.13.14.15.23.24.25.34.35.45
o	o	o	×	×	×	12.13.14.23.24.15.25.34.35.45
o	o	o	×	×	×	12.13.14.23.25.15.24.34.35.45
o	o	o	×	×	×	12.13.15.24.34.14.23.25.35.45
o	o	o	o	×	×	12.13.14.23.45.15.24.25.34.35
o	o	×	×	×	×	12.15.23.34.45.13.14.24.25.35
2111	221	311	32	41	5	

2111	221	311	32	41	5
o	o	o	o	o	o
o	o	o	o	o	×
o	o	o	o	o	×
o	o	o	o	×	×
o	o	o	o	×	×
o	o	o	o	×	×
o	o	o	×	o	×
o	o	o	×	×	×
o	o	o	×	×	×
o	o	o	×	×	×
o	o	o	×	×	×
o	o	×	×	×	×
2111	221	311	32	41	5

The two Tables enable the discussion of the theory of the equal roots of a quartic or quintic equation: first for the quartic:—

5. In order that a quartic may have a pair of equal roots, or what is the same thing, that the system of roots may be of the form 211, the type to be considered is

$$12.13.14.23.24.34;$$

this of course gives as the function to be equated to zero, the discriminant of the quartic.

6. In order that there may be two pairs of equal roots, or that the system may be of the form 22, the simplest type to be considered is

$$14.24.34;$$

this gives the function

$$\Sigma(\alpha-\delta)(\beta-\delta)(\gamma-\delta)(x-\alpha y)^2(x-\beta y)^2(x-\gamma y)^2,$$

which being a covariant of the degree 3 in the coefficients and the degree 6 in the variables, can only be the cubicovariant of the quartic.

7. In order that the quartic may have three equal roots, or that the system of roots may be of the form 31, we may consider the type

$$13.14.23.24,$$

and we obtain thence the two functions

$$\Sigma(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta),$$
$$\Sigma(\alpha-\gamma)^2(\alpha-\delta)(\beta-\gamma)(\beta-\delta)^2,$$

which being respectively invariants of the degrees 2 and 3, are of course the quadrinvariant and the cubinvariant of the quartic. If we had considered the apparently more simple type

$$12.34,$$

this gives the function

$$\Sigma(\alpha-\beta)^3(\gamma-\delta)^3,$$

which is the quadrinvariant, but the cubinvariant is not included under the type in question.

8. Finally, if the roots are all equal, or the system of roots is of the form 4, then the simplest type is

$$12;$$

and this gives the function

$$\Sigma(\alpha-\beta)^2(x-\gamma y)^2(x-\delta y)^2,$$

a covariant of the degree 2 in the coefficients and the degree 4 in the variables; this is of course the Hessian of the quartic.

Considering next the case of the quintic:—

9. In order that a quintic may have a pair of equal roots, or what is the same thing, that the system of roots may be of the form 2111, the type to be considered is

$$12.13.14.15.23.24.25.34.35.45;$$

this of course gives as the function to be equated to zero, the discriminant of the quintic.

10. In order that the quintic may have two pairs of equal roots, or that the system of roots may be 221, the simplest type to be considered is

$$14.15.24.25.34.35.45;$$

a type which gives the function

$$\Sigma(\alpha-\delta)(\alpha-\varepsilon)(\beta-\delta)(\beta-\varepsilon)(\gamma-\delta)(\gamma-\varepsilon)(\delta-\varepsilon)^2(x-\alpha y)^2(x-\beta y)^2(x-\gamma y)^2.$$

This is a covariant of the degree 5 in the coefficients and of the degree 9 in the variables; but it appears from the memoir above referred to, that there is not any irreducible covariant of the form in question; such covariant must be a sum of the products (No. 13)(No. 20), (No. 13)(No. 14)^2, (No. 15)(No. 16) (the numbers refer to the Covariant Tables given in the memoir), each multiplied by a merely numerical coefficient. These numerical coefficients may be determined by the consideration that there being two pairs of equal roots, we may by a linear transformation make these roots 0, 0, ∞, ∞, or what is the same thing, we may write $a=b=e=f=0$, the covariant must then vanish identically. The coefficients are thus found to be 1, -4, 50, and we have for a covariant vanishing in the case of two pairs of equal roots,

$$\begin{aligned} &1\ (\text{No. } 13)(\text{No. } 20)\\ -\ &4\ (\text{No. } 13)(\text{No. } 14)^2\\ +\ &50\ (\text{No. } 15)(\text{No. } 16). \end{aligned}$$

In fact, writing $a=b=e=f=0$, and rejecting, where it occurs, a factor x^2y^2, the several covariants become functions of cx, dy; and putting, for shortness, x, y instead of cx, dy, the equation to be verified is

$$\begin{aligned} &1.10(x+y)(6x^4+8x^3y+28x^2y^2+8xy^3+6y^4)\\ -\ &4.10(x+y)(3x^3+2xy+3y^2)^2\\ +\ &50(6x^2+8xy+6y^2)(x^3+x^2y+xy^2+y^3)=0. \end{aligned}$$

And dividing out by $(x+y)$ and reducing, the equation is at once seen to be identically true.

11. In order that the quintic may have three equal roots, or that the system of roots may be of the form 311, the simplest type to be considered is

$$12.13.23.45;$$

this gives the function

$$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(\delta-\varepsilon)^4,$$

which being an invariant, and being of the fourth degree in the coefficients, must be the quartinvariant of the quintic. The same type gives also the function

$$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(\delta-\varepsilon)^2(x-\delta y)^2(x-\varepsilon y)^2,$$

which is a covariant of the degree 4 in the coefficients and the degree 4 in the variables; and it must vanish when $a=b=c=0$, this can only be the covariant

$$3(\text{No. } 20)-2(\text{No. } 14)^2,$$

which it is clear vanishes as required.

12. In order that the quintic may have three equal roots and two equal roots, or that the system of roots may be of the form 32, the simplest type to be considered is

$$12.13.14.15,$$

which gives the function

$$\Sigma(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)(x-\beta y)^2(x-\gamma y)^2(x-\delta y)^2(x-\varepsilon y)^2,$$

a covariant of the degree 4 in the coefficients, and the degree 12 in the variables; and it must vanish when $a=b=c=0$, $e=f=0$; this can only be the covariant

$$3(\text{No. } 13)^2(\text{No. } 14)-25(\text{No. } 15)^2,$$

which it is clear vanishes as required.

13. In order that the quintic may have four equal roots, or that the system may be of the form 41, the simplest type to be considered is

$$12.34,$$

which gives the function

$$\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(x-\varepsilon y)^2,$$

a covariant of the degree 2 in the coefficients, and of the same degree in the variables; this can only be the covariant (No. 14).

14. Finally, in order that all the roots may be equal, or that the system of roots may be of the form 5, the type to be considered is

$$12;$$

and this gives the function

$$\Sigma(\alpha-\beta)^2(x-\gamma y)^2(x-\delta y)^2(x-\varepsilon y)^2,$$

a covariant of the degree 2 in the coefficients, and the degree 6 in the variables, and this can only be the Hessian (No. 15).

It will be observed that all the preceding conditions are distinctive; for instance, the covariant which vanishes when the system of roots is of the form 311, does not vanish when the system is of the form 221, or of any other form not included in the form 311.

XXXIV. *Tables of the Sturmian Functions for Equations of the Second, Third, Fourth, and Fifth Degrees.* By Arthur Cayley, *Esq.*, *F.R.S.*

Received December 18, 1856,—Read January 8, 1857.

The general expressions for the Sturmian functions in the form of determinants are at once deducible from the researches of Professor Sylvester in his early papers on the subject in the Philosophical Magazine, and in giving these expressions in the Memoir 'Nouvelles Recherches sur les Fonctions de M. Sturm,' *Liouville*, t. xiii. p. 269 (1848), I was wrong in claiming for them any novelty. The expressions in the last-mentioned memoir admit of a modification by which their form is rendered somewhat more elegant; I propose on the present occasion merely to give this modified form of the general expression, and to give the developed expressions of the functions in question for equations of the degrees two, three, four, and five.

Consider in general the equation

$$U=(a, b, \ldots . j, k)(x, 1)^n,$$

and write

$$P=(a, b, \ldots j \quad)(x, 1)^{n-1}$$
$$Q=(b, \quad \ldots j, k)(x, 1)^{n-1},$$

then supposing as usual that the first coefficient a is positive, and taking for shortness n_1, n_2, &c. to represent the binomial coefficients $\frac{n-1}{1}$, $\frac{n-1.n-2}{1.2}$, &c. corresponding to the index $(n-1)$, the Sturmian functions, each with its proper sign, are as follows, viz.

$$U,\ P,\ \begin{vmatrix} P, & Q \\ a, & b \end{vmatrix},\ -\begin{vmatrix} xP, & P, & xQ, & Q \\ a, & ., & b, & . \\ n_1 b, & a, & n_1 c, & b \\ n_2 c, & n_1 b, & n_2 d, & n_1 c \end{vmatrix},$$

$$+\begin{vmatrix} x^2P, & xP, & P, & x^2Q, & xQ, & Q \\ a, & ., & ., & b, & ., & . \\ n_1 b, & a, & ., & n_1 c, & b, & . \\ n_2 c, & n_1 b, & a, & n_2 d, & n_1 c, & b \\ n_3 d, & n_2 c, & n_1 b, & n_3 e, & n_2 d, & n_1 c \\ n_4 e, & n_3 d, & n_2 c, & n_4 f, & n_3 e, & n_2 d \end{vmatrix}, \text{ &c.}$$

where the terms containing the powers of x, which exceed the degrees of the several

functions respectively, vanish identically (as is in fact obvious from the form of the expressions), but these terms may of course be omitted *ab initio*.

The following are the results which I have obtained; it is well known that the last or constant function is in each case equal to the discriminant, and as the expressions for the discriminant of equations of the fourth and fifth degrees are given, Tables No. 12 and No. 26 in my 'Second Memoir upon Quantics*,' I have thought it sufficient to refer to these values without repeating them at length.

Table for the degree 2.

The Sturmian functions for the quadric $(a, b, c)(x, 1)^2$ are

$$\left(\begin{array}{|c|c|c|} a & 2b & c \end{array}\right)(x, 1)^2,$$

$$\left(\begin{array}{|c|c|} a & b \end{array}\right)(x, 1),$$

$$\begin{array}{|c} -1\ ac \\ +1\ b^2 \end{array}$$

Table for the degree 3.

The Sturmian functions for the cubic $(a, b, c, d)(x, 1)^3$ are

$$\left(\begin{array}{|c|c|c|c|} a & 3b & 3c & d \end{array}\right)(x, 1)^3,$$

$$\left(\begin{array}{|c|c|c|} a & 2b & c \end{array}\right)(x, 1)^2,$$

$$\left(\begin{array}{|c|c|} -2\ ac \\ +2\ b^2 & -\ ad \\ & +\ bc \end{array}\right)(x, 1),$$

$$\begin{array}{|c} +1\ a^2d^2 \\ +6\ abcd \\ -4\ ac^3 \\ -4\ bd^3 \\ -3\ b^2c^2 \end{array}$$

* Philosophical Transactions, t. cxlvi. p. 101 (1856).

Table for the degree 4.

The Sturmian functions for the quartic $(a, b, c, d, e)(x, 1)^4$ are

$$\left(\begin{array}{|c|c|c|c|c|}\hline a & 4b & 6c & 4d & e \\ \hline\end{array}\right)(x, 1)^4,$$

$$\left(\begin{array}{|c|c|c|c|}\hline a & 3b & 3c & d \\ \hline\end{array}\right)(x, 1)^3,$$

$$\left(\begin{array}{|c|c|c|}\hline -3ac & -3ad & -ae \\ +3b^2 & +3bc & +bd \\ \hline\end{array}\right)(x, 1)^2,$$

$$3\left(\begin{array}{|c|c|}\hline -1\ a^2ce & +1\ a^2de \\ +3\ a^2d^2 & -4\ abce \\ +1\ ab^2e & -1\ abd^2 \\ -14\ abcd & +3\ ac^2d \\ +9\ ac^3 & +3\ b^2e \\ +8\ b^2d & -2\ b^2cd \\ -6\ b^2c^2 & \\ \hline\end{array}\right)(x, 1),$$

$$-\left(\begin{array}{|c|}\hline +1\ a^3e^2 \\ +\ \&c. \\ \text{Disct. Tab.} \\ \text{No. 12.} \\ \hline\end{array}\right).$$

Table for the degree 5.

The Sturmian functions for the quintic $(a, b, c, d, e, f)(x, 1)^5$ are

$$\left(\begin{array}{|c|c|c|c|c|c|}\hline a & 5b & 10c & 10d & 5e & f \\ \hline\end{array}\right)(x, 1)^5,$$

$$\left(\begin{array}{|c|c|c|c|c|}\hline a & 4b & 6c & 4d & e \\ \hline\end{array}\right)(x, 1)^4,$$

$$\left(\begin{array}{|c|c|c|c|}\hline -4ac & -6ad & -4ae & -af \\ +4b^2 & +6bc & +4bd & +be \\ \hline\end{array}\right)(x, 1)^3,$$

$$2\left(\begin{array}{|c|c|c|}\hline -8\ a^2ce & -2\ a^2cf & +3\ a^2df \\ +18\ a^2d^2 & +12\ a^2de & -11\ abcf \\ +8\ ab^2e & +2\ ab^2f & -3\ abde \\ -76\ abcd & -42\ abce & +8\ ac^2e \\ +48\ ac^3 & -12\ abd^2 & +8\ b^2f \\ +40\ b^2d & +32\ ac^2d & -5\ b^2ce \\ -30\ b^2c^2 & +30\ b^2e & \\ & -20\ b^2cd & \\ \hline\end{array}\right)(x, 1)^2,$$

$$2\Big(\quad \Big)\,\Upsilon(x, 1),$$

$-2\ a^3cf^3$	$+3\ a^2df^3$
$+24\ a^2def$	$-8\ a^2e^2f$
$-32\ a^2e^3$	$-11\ a^2bef^2$
$+2\ a^2b^2f^3$	$+58\ a^2bdef$
$+264\ a^2bde^2$	$+8\ a^2be^3$
$-52\ a^2bcef$	$+104\ a^2c^2ef$
$-96\ a^2bd^2f$	$-156\ a^2cd^2f$
$+64\ a^2c^2df$	$-96\ a^2cde^2$
$+352\ a^2c^2e^2$	$+108\ a^2d^2e$
$-938\ a^2cd^2e$	$+8\ ab^2f^3$
$+432\ a^2d^3$	$-266\ ab^2cef$
$+28\ ab^2ef$	$-8\ ab^2d^2f$
$-970\ ab^2ce^2$	$+35\ ab^2de^2$
$+120\ ab^2d^2e$	$+584\ abc^2df$
$+2480\ abc^2de$	$+120\ abc^2e^2$
$+264\ ab^2cdf$	$-360\ abcd^2e$
$-1440\ abcd^3$	$-288\ ac^4f$
$-192\ abc^3f$	$+160\ ac^3de$
$-960\ ac^4e$	$+120\ b^4ef$
$+640\ ac^3d^2$	$-320\ b^3cdf$
$-160\ b^4df$	$-75\ b^3ce^2$
$+450\ b^4e^2$	$+200\ b^2d^2e$
$-1400\ b^3cde$	$+180\ b^2c^2f$
$+800\ b^3d^3$	$-100\ b^2c^2de$
$+120\ b^3c^2f$	
$+600\ b^2c^2e$	
$-400\ b^2c^2d^2$	

$+1\ a^4f^4$
$+$ &c.
Discr. Tab.
No. 26.

XXXV. *On some of the Products of the Destructive Distillation of Boghead Coal.*—Part II.

By C. Greville Williams, *Lecturer on Chemistry in the Normal College, Swansea.*

Received December 17, 1857,—Read January 14, 1858.

The first part of this paper contained the results of the examination of those hydrocarbons found in the distillate from the Torbane-hill mineral which were remarkable for their resistance to the action of monohydrated nitric and sulphuric acids. It was shown that, when in a state of purity, they possessed the composition, boiling-point, and density in the fluid and gaseous states, of the radicals of the ethylic class of alcohols. The present communication enters on the study of a group of substances as distinguished by the mobility, as the former are by the fixity of their hydrogen atoms. The examination of them has involved more labour than was experienced with their companion hydrocarbons; and the quantities obtained being necessarily small, even when working on a considerable scale, it has been impossible, at present, to do more than determine their constitution. The study of their combinations and decompositions is therefore necessarily deferred.

Up to the present time we have been unacquainted with methods for the proximate analysis of complex mixtures such as that under study. It was necessary, therefore, to devise some plan for the purpose. The presence of benzole and its homologues was almost a matter of certainty, as they appear to be formed in all processes where coaly or bituminous matters are subjected to destructive distillation. In order to separate the hydrocarbons, the following method was adopted. Four ounces of bromine being placed in a large and well-stoppered flask, about eight volumes of water were added. The naphtha boiling in the fourteenth rectification between 71° and 77° Centigrade, was then added in very small portions, the whole being violently agitated after each addition. As the bromine, from its superior density, lay at the bottom of the water and the naphtha at the top, it will be seen that combination could not take place until the flask was shaken, at which time they united with a sharp hissing sound and a rise of temperature. The entire quantity of naphtha was absorbed by the bromine, forming a dense oil, becoming paler and at last colourless as the hydrocarbon increased in quantity. To ensure a definite product more bromine was used, so as to obtain a red oil, the excess being afterwards removed by agitation with mercury. During the process some substance is generated, acting powerfully on the eyes and causing a flow of tears; it is always present in the brominated oil, even when every trace of free bromine is removed. If the brominated oil be kept for some time in contact with water, it will be found that a third layer will make its appearance beneath the oil; this will be described hereafter.

On the crude bromine compound being separated and distilled, all the benzole and radicals will distil away, leaving the brominated oil, the boiling-point of which is much higher. We are not acquainted with any process for separating benzole from the radicals in such a manner as to yield the former pure, but by treatment with fuming nitric acid, in the manner described in Part I., the benzole is transformed into the nitrocompound, from which the propyle may be obtained by distillation on the water-bath.

It was necessary to ascertain whether the nitrocompound obtained by treating with nitric acid the fluid distilling away from the brominated oil, contained any substance (save nitrobenzole) capable of forming alkaloids by reduction. On treatment by Béchamp's process, it yielded a considerable quantity of base having all the characters of aniline, and distilling at 182°.

To confirm this result a portion was dissolved in hydrochloric acid, and to the solution was added about a third of the quantity of bichloride of platinum necessary to precipitate the whole of the base in the state of platinum salt. The precipitate being filtered off, a fresh quantity of the bichloride was added, and so on until three precipitates were obtained. They were washed with a mixture of alcohol and ether previous to drying.

I. ·5197 gramme of the first precipitate gave ·4726 carbonic acid and ·1359 water.

II. ·2980 gramme of the first precipitate gave ·0972 platinum.

III. ·5196 gramme of the second precipitate gave ·4668 carbonic acid and ·1359 water.

IV. ·3408 gramme of the second precipitate gave ·1112 platinum.

V. ·3447 gramme of the third precipitate gave ·1128 platinum.

Or, per cent.—

	Experiment.			Calculation.		
	1st precip.	2nd precip.	3rd precip.			
Carbon	24·5	24·5	.....	C^{12}	72·0	24·0
Hydrogen	2·9	2·9	.....	H^8	8·0	2·7
Nitrogen	.....	.....	.....	N	14·0	4·7
Chlorine	.....	.....	.....	Cl^3	106·5	35·6
Platinum	32·6	32·6	32·7	Pt	99·0	33·0
					299·5	100·0

It is evident, therefore, that the aniline procured in this manner is almost pure, the only admixture being a small portion of toluidine; and it may with safety be concluded that benzole is the only hydrocarbon (distilling away from the brominized oil) which forms a nitrocompound capable of reduction by protacetate of iron.

If the bromine compound obtained from the naphtha boiling between 71° and 77° be distilled, a thermometer being in the tubulature of the retort, no constant boiling-point will be obtained until a temperature of 123° is reached. This arises from the tenacity with which the oil retains benzole and the radicals. It was only after much labour and numerous analyses that this source of difficulty was discovered, and precautions taken to prevent its interfering with the isolation of the hydrocarbons to be presently described. If the heat be very gradually applied to the retort, ebullition will commence at about 80°,

the mercury rapidly rising until 123° is reached, after which the rise becomes very slow. When the propyle and benzole have distilled away, and the point of ebullition of the fluid is elevated to 150° or upwards, rapid evolution of hydrobromic acid takes place, accompanied by a vapour acting powerfully on the eyes. The distillate at this point acquires a fine blue colour. It being plain from these phenomena that the hyperhalyde produced by the action of bromine could not be prepared in a state fit for analysis, the residue in the retort was mixed with a strong alcoholic solution of potash, in order to obtain a halyde capable of being distilled without decomposition. On mixing the two fluids there was a great rise of temperature, and a large quantity of bromide of potassium crystallized out. Sometimes the bromine compound and the alcoholic potash were cohobated together, at others the halyde was prepared by mere digestion of the ingredients and subsequent precipitation of the oil by water; even cohobation over solid hydrate of potash produced the same result. The brominated oil thus prepared was colourless, very dense, and had a powerful odour, somewhat resembling iodide of amyle. Although the hyperhalyde was prepared from a fraction of almost constant boiling-point, and purified carefully by distillation on the oil-bath, all coming over below 130° being rejected, yet the halyde produced from it was not homogeneous, its boiling-point varying from 80° to 150°.

MM. Berthelot and De Luca have shown in their memoir on iodized propylene the great difficulty of obtaining pure products of this class, owing to the simultaneous formation of other substances, both fixed and volatile; and it is evident that if such a difficulty is found with that substance, it must exist to a much greater extent with a homologue having so comparatively high an atomic weight as that yielding the compound under consideration. In order to lessen the sources of error, the bromine compound was fractionated previous to analysis, but, despite every precaution, numerous and perfectly concordant analyses showed the impossibility of obtaining it free from oxygen. This arose from the mode of preparation. The formula of the brominated oil was, originally, $C^{12} H^{11} Br$, but oxidation had proceeded so far as to make the composition almost agree with the expression $C^{12} H^{11} BrO$. The cause of the oxidation of the monobrominated caproylene or hexylene will become apparent from the following experiments.

It has been said that when the hydrocarbons were treated with bromine in presence of water, the fluid separated after a time into three layers, the upper being water containing a small quantity of hydrobromic acid, the middle the brominated oil, and a lower stratum. The latter, on separation by a tap-funnel, proved to be highly acid and perfectly soluble in water. It distilled, with very little evolution of acid fumes, at 127°. A piece of paper moistened with it and held near the fire, became jet-black at the spot touched. Excess of precipitated carbonate of lime was added, and, as soon as effervescence had ceased, the whole was boiled for a few minutes and filtered. The solution evaporated to dryness yielded a white and highly deliquescent mass, a qualitative analysis of which indicated the presence of bromine and calcium. A portion was heated to redness, rapidly cooled in a platinum crucible with a well-fitting lid, and the percentage of calcium estimated.

·2872 gramme of substance gave ·1447 of carbonate of lime.

	Experiment.	Calculation.
Bromine	79·85	Br 80
Calcium	20·15	Ca 20
	100·00	100

The ignited salt, on being moistened with warm water, became extremely hot, and hissed like hot metal when quenched.

The preceding salt being deliquescent, and consequently not convenient for analysis, I digested a fresh portion of the acid upon carbonate of baryta, the latter being in excess; the solution was boiled, filtered and evaporated to dryness on the water-bath. After two hours' desiccation at 100°, it gave on a determination of the barium 43·67 per cent., corresponding to the formula Ba Br, HO; requiring 43·53. By ignition the per-centage of barium was raised to 46·8. Ba Br requires 46·16. During ignition the salt became grey, but regained its original appearance when moistened with water.

A determination of the per-centage of dry hydrobromic acid in the fluid was next made.

I. 2·0624 grammes of acid gave 1·7782 of bromide of silver.

II. 1·1190 gramme of acid gave ·9720 bromide of silver.

Or, per cent.—	I.	II.
	37·2	36·96

The specific gravity of the fluid obtained in this singular manner was 1·320 at 18°. The small quantity of fumes evolved during the distillation points to the existence of a definite hydrate; H Br+15HO would contain 37·5 of hydrobromic acid, a number not far from that found by experiment. The reason for examining thus minutely the lower layer of liquid, was to trace the nature of the reaction resulting in the production of the oxidized fluid. It is plain that water was decomposed by the excess of bromine; the oxidation therefore resembled that produced by the halogens generally.

The examination of the bromine compounds was not pursued further because their products of decomposition appeared more interesting.

The results detailed render it evident that the only method of studying satisfactorily the $C^{n}H^{n}$ group as it existed in the oily fluid would be by means of its derivatives. It is well known that bodies of this class must, after the action of alcoholic potash, be regarded as bearing the same relation to the negative that the hydriodic ethers do to the positive radicals. Like them, the halogen compounds, when decomposed by metals, yield hydrocarbons, having, for four volumes of vapour, double the number of atoms of carbon and hydrogen existing in the iodides, bromides, &c., and these hydrocarbons or radicals, as they are still called, have consequently a higher boiling-point than if the number of carbon and hydrogen atoms remained the same. Thus propylene, $C^{6}H^{6}$, is a gas, while allyle, the negative radical, $C^{12}H^{12}$, is a fluid boiling at 59°. It was therefore probable that the hydrocarbon, $C^{12}H^{13}$, would, by treatment of the bromine compound with sodium, yield a derivative boiling at about 116°.

For the purpose of acting on the bromine compound with sodium, a small glass retort

was used, into the tubulature of which a large pipette was fastened by means of a cork. The neck of the retort was closed. The pipette being removed and the bromine compound introduced, some freshly-cut fragments of sodium were added. The pipette being returned to its place, the whole was allowed to remain until the action of the sodium, at first energetic, became sluggish in consequence of a coating of bromide covering the metal. Gentle heat was then applied, upon which the fluid entered into ebullition, but the vapours condensing in the pipette constantly returned to the retort. During the reaction the sodium acquired a superb blue colour, reminding us of the phenomena observed by M. Bouis when acting on chloride of capryle with the same metal. The addition of the sodium was continued until no more action occurred, upon which the pipette was removed and the fluid distilled over. To ensure absence of bromine, it was once more rectified over sodium, but the metal remained perfectly brilliant until the end of the operation.

The fluid thus procured was not the only product, for the bromide of potassium in the retort was mixed with greasy and carbonaceous matters. The hydrocarbon submitted to a fractional distillation, came over at 71°, instead of 116°, as would have been the case if the bromine had reacted with sodium in an analogous manner to iodized propylene. The following are the analyses of this fluid:—

I. ·1677 gramme yielded ·5251 of carbonic acid and ·2164 water.

II. ·1703 gramme yielded ·5354 carbonic acid and ·2178 water.

	Experiment.		Mean.	Theory $C^6 H^4$.
Carbon	85·4	85·7	85·6	85·7
Hydrogen . . .	14·3	14·2	14·3	14·3
				100·0

The bromine compound is therefore decomposed by sodium in a manner totally different to iodide of allyle. To determine the value of n, I determined the density of the vapour of the fluid. The operation was performed in an apparatus permitting the experiment to be made at any desired pressure. It was originally intended to ascertain the density at several pressures; the experiment was consequently commenced with an extra column of 205·5 millimetres of mercury, the atmospheric pressure being 759·2. The elastic force of the vapour was therefore equal to a column of mercury 964·7 millimetres in height. On consideration, the idea of making more than one experiment with the vapour of this hydrocarbon was abandoned until its formation had been repeated several times.

Corrections made.

Weight of substance	·4240 grm.
Volume of vapour	116·5 cub. cent.
Atmospheric pressure	759·2 mm.
Extra pressure	205·5 mm.
Temperature of vapour	98°·0
Density	3·02

The formula $C^{12}H^{12}$ requires—

12 volumes of carbon vapour . .	$=0{\cdot}8290\times12=9{\cdot}9480$
24 volumes of hydrogen	$=0{\cdot}0692\times24=1{\cdot}6608$
	$\frac{11{\cdot}6088}{4}=2{\cdot}9022$

The excess is partly due to the extra pressure*, but chiefly to the presence of a little $C^{14}H^{14}$.

The fluid procured in this manner is therefore caproylene or hexylene. Its formation by the process detailed is interesting, because it is an instance of the regeneration of a hydrocarbon by means of a complex decomposition similar to that found by MM. Berthelot and De Luca to occur with propylene. The hexylene is not therefore a mere product of decomposition in the ordinary sense of the term, but is identical in kind with the fluid in the naphtha which yielded the bromine compound. This is proved by the identity of boiling-point. The naphtha distilled between 71° and 77°, and the hexylene produced from it boiled at 71°.

Hexylene was discovered by Fremy† among the products of the distillation of hydroleic and metoleic acids.

Bouis found that chloride of capryle treated with sodium in the cold yielded capryle, but that if the reaction took place with heat, caprylene resulted. It appearing probable from this that the products of the action of sodium on the bromine compounds would vary with the temperature, I repeated my experiments with the necessary precautions.

With the view of obtaining the next homologue of hexylene, the fraction boiling between 82° and 88° was selected. About twelve ounces of bromine were saturated with naphtha in the same manner as before, but, during the treatment of the brominated oil with sodium (after the action of alcoholic potash), all rise of temperature was carefully avoided, the apparatus being constantly covered with cold water. The decomposition is of course more gradual than when heat is applied, but it is quite as perfect.

* I have been occupied at intervals during the last two years in ascertaining the amount of influence exercised by pressure on vapour volume, and also in endeavouring to improve the processes generally used for determining the densities of vapours. I hope before long to communicate these results to the Society. In the meantime some values obtained with propyle are annexed, to give an idea of the extent to which the density is increased by pressure.

Vapour-density of Propyle at various pressures.

Theory 2·976=4 volumes.

Temperature.	Pressure.	Density.
100·0	772·1	2·963
97·0	870·6	3·005
97·0	943·8	3·022
98·7	1013·0	3·032

† Ann. de Chim. et de Phys. lxv. 139.

On redistilling the hydrocarbon, fractions were obtained from 82° to 104°. They were analysed as follows:—

I. ·1390 gramme gave ·4366 carbonic acid and ·1785 water.
II. ·1667 gramme gave ·5254 carbonic acid and ·2161 water.
III. ·1532 gramme gave ·4800 carbonic acid and ·1953 water.
IV. ·1862 gramme gave ·5850 carbonic acid and ·2348 water.

Or, per cent.—

	I.	II.	III.	IV.	Mean.	Theory $C^n H^n$.
Carbon	85·7	86·0	85·5	85·7	85·7	85·7
Hydrogen . . .	14·3	14·4	14·2	14·0	14·2	14·3

The first of these analyses was made on a fluid boiling at 85°, the second and third at 90°, and the fourth at 99°. In order to find the value of n, the vapour-density of each of these fractions was determined by GAY-LUSSAC's method.

Boiling-point.	Weight of liquid.	Volume of vapour.	Difference of level.	Barometer.	Temperature of vapour.	Pressing col. of oil in mm. of mercury.	Density.
85°	0·0977	35·75	87·3	755·1	100°	21·7	3·19
90	0·1292	47·50	36·0	732·5	121	20·3	3·22
99	0·1117	43·25	55·0	742·0	146	21·0	3·32

The fraction boiling at 99° has therefore the composition and condensation of œnanthylene or heptylene, the olefiant gas of the œnanthylic alcohol, which requires the following numbers:—

$$14 \text{ volumes of carbon-vapour} = 0{\cdot}8290 \times 14 = 11{\cdot}6060$$
$$28 \text{ volumes of hydrogen} \quad = 0{\cdot}0692 \times 28 = \underline{1{\cdot}9376}$$
$$\frac{13{\cdot}5436}{4} = 3{\cdot}3859$$

The specific gravity of heptylene, in the fluid state, was 0·7184 at 17°.

From these results it is evident that the products of the action of sodium do not vary with the temperature. They also fully confirm the statement, made in another part of this paper, to the effect that the hydrocarbons produced by the processes detailed are identical with those existing in the Boghead naphtha. It will be remembered that the naphtha, from which the bromine compound yielding the heptylene was obtained, boiled between 82° and 88°, and the hydrocarbon produced by the action of sodium on the bromine compound came over between 85° and 99°.

The decomposition which the bromine compound undergoes is by no means simple, large quantities of difficultly combustible charcoal separating, with production of small but distinct traces of fatty acids.

Little is known with regard to the true boiling-points of the $C^n H^n$ series. Those which have been most examined, and consequently appear best adapted for standards with which to compare the boiling-points of the other homologues, do not agree with

each other. The boiling-point of amylene, according to FRANKLAND, is 35°; that of octylene was determined by CAHOURS to be 108°, while BOUIS found it 125°. For convenience of reference the following Table is appended, showing some of the physical properties of the best-known members of the $C^n H^n$ family.

Table of the Physical Properties of some of the $C^n H^n$ series.

	Observer.	Formula.	Boiling-point.	Density.	Density of vapour.	
					Experiment.	Theory.
Amylene.....	FRANKLAND.	$C^{10} H^{10}$	35°	.....	2·386	2·422
Hexylene	FREMY*.	$C^{12} H^{12}$	55	.....	2·875	2·904
Hexylene	C. G. WILLIAMS.	$C^{12} H^{12}$	71	.....	3·020	2·904
Heptylene	C. G. WILLIAMS.	$C^{14} H^{14}$	99	0·718 at 17°	3·320	3·386
Octylene.....	BOUIS†.	$C^{16} H^{16}$	125	0·723 at 17°	3·900	3·875
Octylene.....	CAHOURS‡.	$C^{16} H^{16}$	108	0·708	3·920	3·875
Nonylene	FREMY*.	$C^{18} H^{18}$	110	.....	4·488	4·359

It results from the experiments detailed in the first and second parts of this investigation, that Boghead naphtha consists of a mixture of hydrocarbons belonging to three series, namely,—

The radicals of the alcohols.
Homologues of benzole.
Homologues of olefiant gas.

APPENDIX.

Since the reading of the above, I have seen some experiments recently published by LIMPRICHT on the action of perchloride of phosphorus on œnanthole§. He finds the product to be chloride of œnanthylene, $C^{14} H^{14} Cl^2$. By treatment with sodium he obtains œnanthylene (heptylene), boiling at 95°. By first treating the chloride with alcoholic potash to obtain $C^{14} H^{13} Cl$, and then with sodium, he expected, as I did, to obtain $C^{14} H^{13}$, or rather (for 4 volumes of vapour) $C^{28} H^{26}$. The resulting fluid, however, boiled at 95°, and had all the properties of heptylene. One not very satisfactory analysis agreed better with $C^{14} H^{13}$ than $C^{14} H^{14}$, but he admits that the properties of the hydrocarbon agreed so exactly with œnanthylene, that he could not insist on the accuracy of $C^{14} H^{13}$ as the true formula. My experiments, by proving the regeneration of œnanthylene by the action of sodium on $C^{14} H^{13} Br$, show the true nature of M. LIMPRICHT's result. M. LIMPRICHT states $C^{14} H^{13} Cl$ to be without action on sodium at ordinary temperatures; this difference between the chlorine and bromine compounds is curious and worthy of examination.

* By distillation of hydroleic and metoleic acids.
† By action of sulphuric acid on octylic alcohol.
‡ By distilling pelargonic acid with potash-lime.
§ Chemical Gazette, March 15th 1858.

XXXVI. *On the Comparison of Transcendents, with certain applications to the Theory of Definite Integrals.* By GEORGE BOOLE, *Esq., Professor of Mathematics in the Queen's University. Communicated by Professor* W. F. DONKIN, *F.R.S.*

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1. THE following objects are contemplated in this paper:—

1st. The demonstration of a fundamental theorem for the summation of integrals whose limits are determined by the roots of an algebraic equation.

2ndly. The application of that theorem to the problem of the comparison of algebraic transcendents.

The immediate object of this application will in each case be the finite expression of the value of the sum of a series of integrals, $\Sigma\int X dx$, the differential coefficient, X, being an algebraic function, and the values of x at the limits being determined by the roots of an algebraic equation.

3rdly. The application of the same theorem in a new, and, as is conceived, more remarkable line of investigation, to the comparison of functional transcendents.

The terms 'algebraic' and 'functional' are not here used by way of logical division to indicate classes of transcendents wholly distinct, but the term functional transcendent is simply employed to designate an integral $\int X dx$ in which X involves an arbitrary symbol of functionality.

Under this third head of the comparison of functional transcendents will fall the most important special result of the entire investigation. A case will arise in which, without any limitation of the functional symbol, the several integrals included under the form $\Sigma\int X dx$ will close up, if the expression may be allowed, into a single integral taken between the limits $-\infty$ and ∞. The result is a very remarkable theorem of definite integration, fruitful in important consequences. In its general form, this theorem is, I believe, entirely new. A particular case of it was discovered by me several years ago, and was published without demonstration in the Cambridge and Dublin Mathematical Journal*, and in LIOUVILLE'S Journal de Mathématiques†. A memoir by CAUCHY on integrals taken between the limits 0 and ∞‡, contains also a very limited case of the same theorem. It appears there, however, as an isolated result, quite apart from the doctrine of the comparison of transcendents.

In the concluding sections of this paper I shall apply the results of this part of the investigation to the extension of the theory of multiple definite integrals.

As respects the methods and processes which will be employed in this paper, the only

* Vol. iv. p. 14. † Tom. xiii. ‡ Exercices, vol. i. p. 54.

peculiarity to which it seems necessary to direct attention, is the introduction of a symbol, differing in interpretation only by the addition of one element, from that which CAUCHY has employed in his 'Calculus of Residues.' Of the nature of this connexion I was not aware until my researches were nearly completed; and I should then have abandoned my own symbol and adopted the other, already associated by the labours of its distinguished inventor with so many important discoveries in the higher departments of the integral calculus, if it had not appeared to me that the several elements combined in the interpretation of the former symbol were so allied that no one of them could without a manifest defect of completeness be omitted. It seemed to me also that many of CAUCHY'S own applications of his symbol would gain in simplicity and in generality of expression, by the adoption of the more enlarged interpretation.

2. Beside the above special objects, in the attainment of which whatever claim to originality this paper may possess will consist, I have proposed to myself, as a general object, the simplification of a branch of analysis which possesses some practical and much speculative importance. To this object the introduction of the symbol above referred to, contributes in a very important degree. The necessity of simplification will, I think, be admitted by all who are acquainted with the literature of the subject. As presented in the writings of ABEL and of those who immediately followed in his steps, the doctrine of the comparison of transcendents is repulsive from the complexity of the formulæ in which its general conclusions are embodied. The particular result known as ABEL'S theorem, the only one of its class which has been adopted into English works of education, will at once suggest itself in confirmation of this remark. Perhaps this complexity will not be thought surprising if we consider the nature of the problems involved,—the discovery of finite relations among integrals which derive their very name from the circumstance that individually their finite expression transcends the powers of analysis. On the other hand, and this is a juster ground of inference, the theory upon which such applications rest is far from being difficult or recondite, and, considered *à priori*, should be capable of a simpler analytical development than it has yet received. I hope that I shall be able to show that this anticipation is confirmed by the results of the present inquiry. Simplicity, though it is not to be gained at the expense of that which is the chief object of scientific methods, the discovery of truth, is nevertheless a highly valuable quality. And so far from being inconsistent with generality in the processes and the results of analysis, it is sometimes an indication of the measure of our approach to completeness and unity. I think that this is more especially the case where, as through the labours of ABEL in the present instance, the subject matter of investigation has been clearly defined, and the entire series of methods and results foreshown to be the evolution of some one general principle or idea.

3. It will be proper, before entering upon the special investigation, to give some general account of the doctrine of the comparison of transcendents. In doing this, I cannot but refer to the able report of Mr. LESLIE ELLIS on the Progress of Analysis, published in the Report of the British Association for 1846. It contains a most valuable

summary and criticism of the chief contributions which have been made, both by English and by foreign mathematicians, to the theory of transcendental integrals in all its branches. I had completed my investigations on that particular branch of the theory which relates to the comparison of transcendents before I had the opportunity of studying Mr. ELLIS's report; but I know of but a single point, and that of no real importance, in which I should be disposed to dissent from the general views which he has expressed on this subject.

The fundamental idea upon which the doctrine of the comparison of transcendents rests may be thus stated. We know that where we cannot express in finite terms the values of the roots of an algebraic equation, we can nevertheless finitely determine the values of various symmetrical functions of the roots, *e. g.* their sum, the sum of their squares, the sum of their binary products, &c. Those values will be expressed in terms of the coefficients of the equation, or, speaking more generally, of the independent constants involved in any way in the equation. If we represent the equation in the form

$$\mathrm{E}(x, a_1, a_2, \ldots a_r)=0, \quad \ldots\ldots\ldots\ldots \quad (1.)$$

E being a functional symbol, and $a_1 a_2 \ldots a_r$ the independent constants, and if we represent the roots of the equation by $x_1, x_2, \ldots x_n$, then whatever the interpretation of the functional symbol φ may be, the expression

$$\varphi(x_1)+\varphi(x_2)\ldots+\varphi(x_n) \quad \ldots\ldots\ldots\ldots \quad (2.)$$

will denote a symmetrical function of the roots, whose value, in terms of $a_1, a_2, \ldots a_r$ can always be determined when $\varphi(x_i)$ is a rational function of x_i, and can very often be determined when $\varphi(x_i)$ is an irrational function of x_i. Thus the value of the sum $\Sigma\varphi(x)$, where the different values of x are the roots of an algebraic equation, can often be found when the values of the separate terms of which that sum is composed cannot be found.

Now this suggests to us the question whether it is not possible in certain cases to find the value of the integral-sum $\Sigma\int \mathrm{X}dx$, where we cannot find the value of the separate integrals involved in that sum. What we usually mean by finding the value of the single integral $\int \mathrm{X}dx$, is the expressing of the value of that integral in terms of its superior limit, an arbitrary constant being annexed, and consists, in fact, in determining the function $\phi(x_i)$ in the equation

$$\int^{x_i} \mathrm{X}dx=\phi(x_i)+\mathrm{C}. \quad \ldots\ldots\ldots\ldots \quad (3.)$$

The suggested problem is the finding of the value of the integral-sum

$$\int^{x_1} \mathrm{X}dx+\int^{x_2} \mathrm{X}dx\ldots+\int^{x_n} \mathrm{X}dx, \quad \ldots\ldots\ldots\ldots \quad (4.)$$

where $x_1, x_2, \ldots x_n$ are the roots of an algebraic equation, which we will suppose to be (1). Representing (4.) by $\Sigma\int^{x_i} \mathrm{X}dx$, the solution of the problem will, according to the nature of the analysis employed, assume one of the two following equivalent forms, viz. either the form

$$\Sigma\int^{x_i} \mathrm{X}dx=\varphi(a_1, a_2, \ldots a_r)+\mathrm{C}, \quad \ldots\ldots\ldots\ldots \quad (5.)$$

$a_1, a_2, \ldots a_r$ being the independent constants in the equation (1.) by which the limits are determined, or the form

$$\Sigma\int_{x_1}^{x_2} X dx = \psi(x_1, x_2, \ldots x_n) + C, \qquad (6.)$$

$\psi(x_1, x_2, \ldots x_n)$ being a function, and manifestly a symmetrical function, of the limits $x_1, x_2, \ldots x_n$. Either of these forms may be converted into the other by means of (1.), but the first is the one to which we shall give the preference.

We may remark, that if in the equation (1.) $a_1, a_2, \ldots a_r$ vary, the values of x, determined by the solution of that equation, will vary also. For each set of values of $a_1, a_2, \ldots a_r$, there will exist a simultaneous set of values of x. We may in this way consider the variables x in the several integrals under the sign Σ as always, in the course of their transition from the lower to the upper limits of integration, determined by the roots of the equation

$$E(x, a_1, a_2, \ldots a_r) = 0.$$

According to this more general view, $a_1, a_2 \ldots a_r$ become a set of variables with which x is connected by the above equation, but the variation of x in each integral represents only the variation of one root of the equation. And the determination of the values of x at the upper or at the lower limits of integration by the solution of that equation, particular values being assigned to $a_1, a_2, \ldots a_r$, is only a special case of the determination of the simultaneous values of the variable x.

4. Thus the problem with which we are concerned may be more briefly expressed in this form. *Required the value of the expression* $\Sigma\int X dx$, *the simultaneous values of* x *in the several integrals being determined by an equation of the form*

$$F(x, a_1, a_2, \ldots a_r) = 0, \qquad (7.)$$

in which $a_1, a_2, \ldots a_r$ *are variable quantities, by the assigning of particular values to which in the solution of the equation, the particular values of* x *at the limits of integration are determined.*

The solution of the above problem is to be effected by giving to the expression $\Sigma\int X dx$ the equivalent form $\int \Sigma X dx$, transforming $\Sigma X dx$ into a complete differential with respect to the variables $a_1, a_2, \ldots a_r$, and then effecting the integration. For this reason I shall designate (7.), or, as it may for convenience be written, $E=0$, as the 'transforming equation,' except when it is employed to determine the limits, in which case the designation of 'equation of the limits' will be preferable.

For the solution of the problem, as thus stated, it is usually necessary that the form of the function E in the transforming equation, and the form of the function X under the sign of integration, should have a certain connexion. The connexion implied is the following. The transforming equation must in general be such, that it may be possible by means of it to reduce the differential expression X under the sign of integration to a form $F(x, a_1, a_2, \ldots a_r)$, which, considered as an explicit function of x and of $a_1, a_2, \ldots a_r$, shall be rational with respect to x. This is not a necessity *à priori*. It is a necessity founded in the limitations of the powers of analysis.

5. To illustrate these remarks, let us take as an example the integral

$$\int \frac{dx}{\sqrt{1+x^4}},$$

which is known to express the length of the arc of the lemniscate. The following would be a legitimate example of the kind of problem which it is proposed to investigate, viz. Required the value of the expression

$$\Sigma \int \frac{dx}{\sqrt{1+x^4}},$$

the simultaneous values of x in the successive integrals being given by the roots of the equation

$$1+x^4=(a+x+x^2)^2, \quad \ldots\ldots\ldots \quad (8.)$$

reducible to the form

$$x^3+(a+\tfrac{1}{2})x^2+ax+\frac{a^2-1}{2}=0. \quad \ldots\ldots\ldots \quad (9.)$$

We shall represent these roots by x_1, x_2, x_3.

The solution of the problem would then assume either the form

$$\Sigma \int \frac{dx}{\sqrt{1+x^4}}=\varphi(a)+C, \quad \ldots\ldots\ldots \quad (10.)$$

C being the constant of integration and $\varphi(a)$ the function to be determined, or the form

$$\Sigma \int \frac{dx}{\sqrt{1+x^4}}=\psi(x_1, x_2, x_3)+C, \quad \ldots\ldots\ldots \quad (11.)$$

wherein ψ is to be determined.

It will be observed that the transforming equation (8.) is such as to enable us, in accordance with the requirement of art. 4, to reduce the function under the sign of integration to a form in which, considered as an explicit function of x and a, it is rational with respect to x. For it gives

$$\int \frac{dx}{\sqrt{1+x^4}}=\int \frac{dx}{a+x+x^2}, \quad \ldots\ldots\ldots \quad (12.)$$

the second member of which fulfils the condition in question. It is to be remarked, however, that a and x are still connected by the equation (8.) or (9.), and that we are not permitted to integrate as if a were constant.

In the above problem, only one arbitrary element, a, presents itself in the transforming equation. There might, however, have been two or any greater number of arbitrary elements. Thus the equation might have been

$$1+x^4=(a+bx+x^2)^2,$$

or

$$x^3+\frac{b^2+2a}{2b}x^2+ax+\frac{a^2-1}{2b}=0, \quad \ldots\ldots\ldots \quad (13.)$$

and the solution of the problem would then have assumed one of the two following forms, viz. either the form

$$\Sigma \int \frac{dx}{\sqrt{1+x^4}}=\varphi(a, b)+C, \quad \ldots\ldots\ldots \quad (14.)$$

or the form

$$\Sigma\int\frac{dx}{\sqrt{1+x^4}}=\psi(x_1, x_2, x_3)+C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.)$$

From these solutions the corresponding solutions of the previous and less general problem would be obtained, in the one case by making $b=1$, in the other by imposing upon the limits x_1, x_2, x_3 conditions thereto equivalent.

6. And not only the solution, but the original statement of a problem may be exhibited in the two forms above described. Instead of supposing the limits determined by an equation involving arbitrary quantities in its coefficients, we may suppose them directly connected by symmetrical equations, *i. e.* we may suppose those relations *explicitly* given which are only *implied* in the equation determining the limits, and can only thence be deduced by eliminating the arbitrary elements. Thus (9.) furnishes us with the three following equations:

$$x_1+x_2+x_3=-a-\tfrac{1}{2}.$$
$$x_1x_2+x_2x_3+x_3x_1=a.$$
$$x_1x_2x_3=\frac{1-a^2}{2}.$$

From which, eliminating the arbitrary quantity a, we have

$$\left.\begin{aligned} x_1+x_2+x_3&=x_1x_2+x_2x_3+x_3x_1-\tfrac{1}{2}\\ 2x_1x_2x_3&=1-(x_1x_2+x_2x_3+x_3x_1)^2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.)$$

The problem first considered now assumes the following form. Required the value of the integral expression

$$\Sigma\int\frac{dx}{\sqrt{1+x^4}},$$

when the superior limits x_1, x_2, x_3 are connected by the explicit relations (16.).

The second problem, similarly transformed, would, as there are two arbitrary elements to be eliminated, lead to a single equation between x_1, x_2, x_3, in place of the two equations (16.).

7. We may observe, from the above examples, that when the number of integrals to be added is three, the existence of two arbitrary elements in the equation of the limits involves the existence of one symmetrical equation among the limits, and the existence of one arbitrary constant in the equation of the limits involves the existence of two symmetrical equations among the limits themselves. And thus generally if there be n integrals to be added, the existence of r arbitrary elements in the equation of the limits will involve the existence of $n-r$ symmetrical equations among the limits. The converse of this proposition is obviously true also. If any number r of symmetrical equations among the limits $x_1, x_2, \ldots x_n$ are given, and if we regard $x_1, x_2, \ldots x_n$ as roots of the equation of the nth degree,

$$x^n+p_1x^{n-1}+p_2x^{n-2}\ldots+p_n=0,$$

the symmetrical conditions referred to, will, by the theory of equations, establish among the coefficients $p_1, p_2, \ldots p_n$ a system of relations by means of which we can determine r of

those coefficients as functions of the others deemed arbitrary,—or choosing in some other way $n-r$ arbitrary elements, express all the coefficients by means of those elements.

It is further seen that in the one form of the problem the greater the number of arbitrary quantities in the equation determining the limits,—in the other form the smaller the number of symmetrical equations connecting the limits with each other,—the more general is the statement of the problem itself.

8. The Norwegian and German mathematicians, by whom this branch of analysis has been chiefly cultivated, have almost universally followed ABEL in his mode of stating the general problem, *i. e.* they have regarded the limits as roots of an equation involving a greater or less number of arbitrary elements in its coefficients. Mr. FOX TALBOT, in his interesting papers "On the Comparison of Transcendents," published in the Philosophical Transactions for the years 1836–37, has in the earlier examples of his method expressed by symmetrical equations among the limits the conditions to which the latter are subject: in his later examples he adopts the more succinct notation of ABEL. Indeed, this mode of statement, as it replaces a system of equations by a single equation from which such systems may be considered as derived, is far better suited to general investigations, and will be adopted in this paper.

9. The complete solution of the problem of the comparison of transcendents, as above explained, involves two distinct steps,—1st, regarding the equation

$$E=0$$

of (7.), art. 4, as an equation expressing the dependence of the variable x upon the variables $a_1, a_2, \ldots a_r$, we must seek to convert the differential expression $\Sigma X dx$ into a complete differential relative to $a_1, a_2, \ldots a_r$ as independent variables, and which will therefore virtually be in the form

$$\Sigma X dx = A_1 da_1 + A_2 da_2 \ldots + A_r da_r,$$

each of the differential coefficients $A_1, A_2, \ldots A_r$ being a function of $a_1, a_2, \ldots a_r$; 2ndly, we must integrate this expression.

The introduction of the symbol of operation adverted to in art. 1 will enable us to dispense with the explicit determination of the coefficients $A_1, A_2, \ldots A_r$, and to reduce the corresponding differential expression to one involving only a single variable. I shall now proceed to define the symbol in question, and to investigate its chief properties.

Definition and Properties of the Symbol Θ.

10. It is an evident consequence of TAYLOR'S theorem that we can develope any function of x, $f(x)$, in ascending powers of $x-a$, provided that neither $f(x)$ nor any one of its differential coefficients becomes infinite when $x=a$. To effect this development, we have only to assume $x-a=z$; then $x=a+z$, whence

$$\left.\begin{aligned} f(x) &= f(a+z) \\ &= f(a)+f'(a)z+f''(a)\frac{z^2}{1.2}+\&c. \\ &= f(a)+f'(a)(x-a)+f''(a)\frac{(x-a)^2}{1.2}+\&c. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1.)$$

Now $f(x)$ being as above, let $\mathrm{F}(x)=\frac{f(x)}{(x-a)^m}$ where m is an integer. Then

$$\mathrm{F}(x)=\frac{f(a)}{(x-a)^m}+\frac{f'(a)}{(x-a)^{m-1}}\ldots+\frac{f^{(m-1)}(a)}{\underline{m-1}}\frac{1}{x-a}+\frac{f^{(m)}(a)}{\underline{m}}+\frac{f^{(m+1)}(a)\times(x-a)}{\underline{m+1}}+\&c. \quad . \quad . \quad (2.)$$

Here we obtain, as before, a development of $\mathrm{F}(x)$ in ascending powers of $x-a$, but the development begins with negative powers of that quantity. To this species of development we shall have frequent occasion to refer, and no doubt after this explanation will arise as to what is meant when we speak of the development of any function $\mathrm{F}(x)$ in ascending powers of $x-a$.

These things being premised, let the symbol Θ be thus defined, *viz.* *If* $\phi(\mathrm{x})\mathrm{f}(\mathrm{x})$ *be any function of* x *composed of two factors* $\phi(\mathrm{x})$ *and* $\mathrm{f}(\mathrm{x})$ *whereof* $\phi(\mathrm{x})$ *is rational, let*

$$\Theta[\phi(x)]f(x)$$

denote the result obtained by successively developing the function in ascending powers of each distinct simple factor $\mathrm{x}-\mathrm{a}$ *in the denominator of* $\phi(\mathrm{x})$, *taking in each development the coefficient of* $\frac{1}{\mathrm{x}-\mathrm{a}}$, *adding together the coefficients thus obtained from the several developments, and subtracting from the result the coefficient of* $\frac{1}{\mathrm{x}}$ *in the development of the same function* $\phi(\mathrm{x})\mathrm{f}(\mathrm{x})$ *in descending powers of* x.

It is seen from the above that the interpretation of Θ is relative. It directs us to obtain certain developments, but the nature of these developments, if we except the last of them, depends upon the nature of the function within the brackets [ ]. Thus while in the expression $\Theta[\phi(x)]f(x)$ the operation of the symbol Θ extends over the entire function $\phi(x)f(x)$, the interpretation of Θ, by which the nature of that operation is defined, is derived solely from the factor $\phi(x)$.

Thus to take an example of some generality, let it be required to deduce an expression for

$$\Theta\left[\frac{x^2}{(x-a)(x-b)^2}\right]f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

where $f(x)$ denotes some function of x which does not become infinite when $x=a$ or b.

The distinct simple factors in the denominator of the function within the brackets [ ] are $x-a$ and $x-b$. If we make $x-a=z$, or $x=a+z$, we have to develope

$$\frac{(a+z)^2}{z(a-b+z)^2}f(a+z)$$

in ascending powers of z. The coefficient of $\frac{1}{z}$ in that expansion is

$$\frac{a^2f(a)}{(a-b)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Again, making $x-b=z$ and $x=b+z$, we have to develope the function

$$\frac{(b+z)^2}{z^2(b-a+z)}f(b+z)$$

in ascending powers of z. The development is

$$\frac{1}{x^2}\left\{\frac{b^2f(b)}{b-a}+\frac{d}{db}\frac{b^2f(b)}{b-a}z+\frac{1}{1.2}\frac{d^2}{db^2}\frac{b^2f(b)}{b-a}z^2+\&c.\right\},$$

in which the coefficient of $\frac{1}{z}$ is

$$\frac{d}{db}\cdot\frac{b^2f(b)}{b-a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

Lastly, the coefficient of $\frac{1}{x}$ in the development of the function

$$\frac{x^2f(x)}{(x-a)(x-b)^2}$$

in descending powers of x being represented according to a familiar notation by

$$C_{\frac{1}{x}}\frac{x^2f(x)}{(x-a)(x-b)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

we have, on adding (4.) and (5.), and subtracting (6.) from the sum,

$$\Theta\left[\frac{x^2}{(x-a)(x-b)^2}\right]f(x)=\frac{a^2f(a)}{(a-b)^2}+\frac{d}{db}\frac{b^2f(b)}{b-a}-C_{\frac{1}{x}}\frac{x^2f(x)}{(x-a)(x-b)^2}. \quad . \quad . \quad . \quad (7.)$$

As a particular illustration, let $f(x)=\log\left(c+\frac{1}{x}\right)$, and let us seek the value of the last term in the above expression. Now

$$\frac{x^2}{(x-a)(x-b)^2}=\frac{1}{x}+\frac{a+2b}{x^2}+\&c.$$

on developing in descending powers of x, and

$$\log\left(c+\frac{1}{x}\right)=\log c+\frac{1}{cx}-\frac{1}{c^2x^2}+\&c.$$

Multiplying these together, the coefficient of $\frac{1}{x}$ in the result is $\log c$. Whence

$$\Theta\left[\frac{x^2}{(x-a)(x-b)^2}\right]\log\left(c+\frac{1}{x}\right)=\frac{a^2\log\left(c+\frac{1}{a}\right)}{(a-b)^2}+\frac{d}{db}\frac{b^2\log\left(c+\frac{1}{b}\right)}{b-a}-\log c, \quad . \quad . \quad (8.)$$

in which it only remains to perform the differentiation in the second term.

Formulæ applicable to the determination of the result of the operation Θ in any case, may readily be found by the aid of TAYLOR's theorem. Thus we should have, $f(x)$ not becoming infinite when $x=a$ or $x=b$,

$$\left.\begin{aligned}\Theta\left[\frac{1}{(x-a)^m(x-b)^n}\right]f(x)=&\frac{1}{1.2\ldots(m-1)}\left(\frac{d}{da}\right)^{m-1}\frac{f(a)}{(a-b)^n}\\&+\frac{1}{1.2\ldots(n-1)}\left(\frac{d}{db}\right)^{n-1}\frac{f(b)}{(b-a)^m}-C_{\frac{1}{x}}\frac{f(x)}{(x-a)^m(x-b)^n}\end{aligned}\right\} \quad . \quad . \quad (9.)$$

an expression in which the general law of such formulæ is manifest.

If there be n distinct simple factors in the denominator of the rational fraction within the brackets, the result of the operation Θ will consist of $n+1$ terms, the first n of which

will be determined as above by TAYLOR'S theorem. The $n+1$th term will involve the operation denoted by the symbol $C_{\frac{1}{x}}$, and it is by this term only that the interpretation of Θ differs from that of CAUCHY'S symbol, $\mathcal{E}$. We have in fact

$$\Theta = \mathcal{E} - C_{\frac{1}{x}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

the complete interpretation of Θ involving two *distinct* elements.

It happens in certain problems that one of those elements alone presents itself, the symbol Θ becoming equivalent either to $\mathcal{E}$ or to $-C_{\frac{1}{x}}$. In other problems they both appear, but I am not aware of any problems in which the vanishing of one of the elements is not due to some special circumstance of restriction or limitation affecting the interpretation of the symbol Θ in the case supposed. In a Note to this paper I have endeavoured to illustrate the above remark by employing the symbol Θ in certain problems, in which CAUCHY has made use of the symbol of residues, and to that Note the reader who is interested in the comparison is referred.

11. The properties of the symbol Θ are now to be considered. The two following are the most important of them:—

1st. *The operation Θ is distributive as respects both the function without and the function within the brackets, provided that those functions do not become together infinite for any finite value of* x.

PROOF.—First, as respects the function without the brackets, we have the theorem

$$\Theta[\varphi(x)]\left(f_1(x)+f_2(x)\ldots+f_n(x)\right)=\Theta[\varphi(x)]f_1(x)+\Theta[\varphi(x)]f_2(x)\ldots+\Theta[\varphi(x)]f_n(x); \quad . \quad (1.)$$

for the coefficient of a particular term, as $\frac{1}{x-a}$, $\frac{1}{x}$, &c. in the development of a function is equal to the sum of the coefficients of all the corresponding terms in the development of the several component functions from which the proposed function is formed by addition.

Secondly, as respects the function within the brackets, we have the theorem

$$\Theta[\varphi_1(x)+\varphi_2(x)\ldots+\varphi_n(x)]f(x)=\Theta[\varphi_1(x)]f(x)+\Theta[\varphi_2(x)]f(x)\ldots+\Theta[\varphi_n(x)]f(x). \quad . \quad (2.)$$

Here, it is to be observed that $\varphi_1(x)$, $\varphi_2(x)$, $\ldots\varphi_n(x)$ represent any rational fractions into which the rational fraction $\varphi(x)$ within the brackets [ ] is resolved. Thus the theorem might be written in the form

$$\Theta[\varphi(x)]f(x)=\Theta[\varphi_1(x)]f(x)+\Theta[\varphi_2(x)]f(x)\ldots+\Theta[\varphi_n(x)]f(x), \quad . \quad . \quad . \quad (3.)$$

wherein

$$\varphi(x)=\varphi_1(x)+\varphi_2(x)\ldots+\varphi_n(x).$$

To prove this theorem I shall show that it is necessarily true for each of the operations into which Θ is resolvable. Let one of the *distinct* factors of the denominator of $\varphi(x)$ be $x-a$, then one of the component operations in Θ consists in developing in ascending powers of $x-a$, and taking in the development the coefficient of $\frac{1}{x-a}$. Now

whether this operation be performed at once upon the function $\varphi(x)f(x)$ or separately upon the several functions

$$\varphi_1(x)f(x),\ \varphi_2(x)f(x)\ldots\varphi_n(x)f(x)$$

of which that function is composed, and the several results then collected together, is a matter of indifference. If we represent this part of the operation Θ by R, we have therefore

$$R\varphi(x)f(x)=R\varphi_1(x)f(x)+R\varphi_2(x)f(x)\ldots+R\varphi_n(x)f(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Now any term, $R\varphi_i(x)f(x)$, in the second member of the above will either form a part of the corresponding term $\Theta[\varphi_i(x)]f(x)$ in the second member of (3.), or will vanish. The former will obviously be the case if $x-a$ is contained in the denominator of $\varphi_i(x)$; the latter will be the case if $x-a$ is not included in the denominator of $\varphi_i(x)$, for then the function $\varphi_i(x)f(x)$ not becoming infinite when $x=a$, is developable in a series of the form $A+B(x-a)+C(x-a)^2\ldots$ Art. 10, and in this series the coefficient of $\frac{1}{x-a}$ is 0. Hence, all the terms in (4.) are contained in (3.); neither are there any terms resulting from the component part of the operation Θ denoted by R which are not contained in the second member of (4.).

Hence, if we cause R to stand in succession for each of the component operations of Θ, and add the several equations thus obtained and typically represented by (4.) together, we shall obtain the theorem (3.).

As an example, we have

$$\Theta\left[\frac{2a}{x^2-a^2}\right]f(x)=\Theta\left[\frac{1}{x-a}\right]f(x)+\Theta\left[\frac{-1}{x+a}\right]f(x),$$

which is easily verified, since the first member gives

$$f(a)-f(-a)-C_{\frac{1}{x}}\frac{2af(x)}{x^2-a^2},$$

and the second member

$$f(a)-f(-a)-C_{\frac{1}{x}}\frac{f(x)}{x-a}+C_{\frac{1}{x}}\frac{f(x)}{x+a},$$

an equivalent result.

2ndly. *If* f(x) *be a rational and entire function of* x, *we have always*

$$\Theta[\varphi(x)]f(x)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

PROOF.—As $f(x)$ must be of the form ΣAx^i, i being an integer, we have

$$\Theta[\varphi(x)]f(x)=\Theta[\varphi(x)]\Sigma Ax^i$$
$$=\Sigma A\Theta[\varphi(x)]x^i$$

by the last proposition. Thus we have to consider a series of terms of the form

$$\Theta[\varphi(x)]x^i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

Again, $\varphi(x)$ being a rational fraction may be resolved into a series of terms which will be of the form ax^m or of the form $\frac{b}{(x-c)^n}$. Hence, availing ourselves of the distributive

property of Θ with respect to the term within the brackets, we see that (6.) is resolvable ultimately into a series of terms falling under the two typical forms

$$\Theta[x^m]x^i, \qquad \Theta\left[\frac{1}{(x-a)^n}\right]x^i.$$

All terms of the first form obviously vanish, since they can have no negative indices. It only remains, therefore, to consider terms of the second form.

First, let i be equal to or greater than n. Then putting $x-a=z$, and $x=a+z$, the coefficient of $\frac{1}{z}$ in the development of the function $\frac{(a+z)^i}{z^n}$, to which $\frac{x^i}{(x-a)^n}$ is reduced, in ascending powers of z will be

$$\frac{i(i-1)..(i-n+2)}{1.2..(n-1)}\,a^{i-n+1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

and the coefficient of $\frac{1}{x}$ in the development of the function $\frac{x^i}{(x-a)^n}$ in descending powers of x will be

$$\frac{n(n+1)..i}{1.2..(i-n+1)}\,a^{i-n+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

These expressions are equal, as may be shown by equating them and clearing of fractions. Hence, in this case,

$$\Theta\left[\frac{1}{(x-a)^n}\right]x^i=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

Secondly, if $i=n-1$, each of the above expressions (7.) and (8.) reducing to unity, the equation (9.) is still true.

Lastly, if i is less than $n-1$, neither will any term containing $\frac{1}{x}$ present itself in the ascending development, nor any term containing $\frac{1}{x}$ in the descending development, so that the equation (9.) remains true in this case also.

Wherefore the theorem is proved generally. See Note A.

General Theorem of Transformation.

12. The foregoing properties of the symbol Θ have an important bearing upon the general theorem for the transformation of integrals under the sign Σ, to the demonstration of which we shall now proceed.

THEOREM.—*If* $E=0$ *be an equation connecting the variable* x *with another set of variables* $a_1, a_2, \ldots a_n$, *the function* E *being rational and entire with respect to* x, *and if* F *be any function of* x *and of* $a_1, a_2, \ldots a_n$ *which is rational with respect to* x, *then, provided that* F *does not become infinite when* $E=0$, *we have*

$$\Sigma F dx=\Theta[F]\frac{\delta E}{E},$$

where δ *indicates complete differentiation with respect to the variables* $a_1, a_2, \ldots a_n$, *and the*

symbol Θ *directs us, according to previous definition, to develope the function* $F\frac{\partial E}{E}$ *in ascending powers of* $x-h_1, x-h_2 \ldots$ *the distinct simple factors of the denominator of* F, *to take in those successive developments the coefficients of* $\frac{1}{x-h_1}$, $\frac{1}{x-h_2}$, *&c., and from the sum of these coefficients to subtract the coefficient of* $\frac{1}{x}$ *in the development of the same function in descending powers of* x.

The object of this theorem will become apparent if it be compared with the statement in Art. 4. It will be observed that E stands for the rational and entire function $E(x, a_1, a_2, \ldots a_r)$, and F for the rational function $F(x, a_1, a_2 \ldots a_r)$ in that article. Thus F is that rational function of x to which the differential coefficient X in the integral $\int X dx$ is supposed to be capable of being reduced by means of the equation of transformation $E=0$.

DEMONSTRATION.—13. First, it will be necessary to prove the following subsidiary proposition:—

PROPOSITION.—*If* $\phi(x)$ *be any rational function of* x, *and if* $E=0$ *be any equation rational and entire with respect to* x, *by which* x *is connected with a new set of variables* $a_1, a_2, \ldots a_r$, *then, provided that* $\phi(x)$ *does not become infinite when* $E=0$, *we have*

$$\Sigma\phi(x) = -\Theta[\phi(x)]\frac{d\log E}{dx}.$$

PROOF.—As $\phi(x)$ is a rational function of x, it is capable of being resolved into a series of terms, each of which is either of the form ax^i, or of the form $\frac{a}{(x-p)^i}$, a being constant, and i an integer.

Consider then, first, the expression

$$\Sigma ax^i, \quad \ldots \quad (1.)$$

the different values of x in the several terms under the sign Σ being roots of the equation $E=0$. Representing these roots by $x_1, x_2, \ldots x_n$, any two or more of which may be equal, we have

$$E = A(x-x_1)(x-x_2)\ldots(x-x_n), \quad \ldots \quad (2.)$$

A being constant. Hence

$$\frac{d}{dx}\log E = \frac{1}{x-x_1}+\frac{1}{x-x_2}\ldots+\frac{1}{x-x_n}. \quad \ldots \quad (3.)$$

Developing the several terms of the second member in descending powers of x, the aggregate coefficient of $\frac{1}{x^{i+1}}$ will be

$$x_1^i + x_2^i \ldots + x_n^i,$$

or

$$\Sigma x^i.$$

Hence

$\Sigma ax^i = a \times$ coefficient of $\frac{1}{x^{i+1}}$ in the development of $\frac{d}{dx}\log E$ in descending powers of x.

$=$ coefficient of $\frac{1}{x}$ in the development of $ax^i\frac{d}{dx}\log E$ in descending powers of x.

Now $\Theta[ax^r]\frac{d}{dx}\log \mathrm{E} = -$coefficient of $\frac{1}{x}$ in the development of $ax^r\frac{d}{dx}\log \mathrm{E}$ by the definition of Θ, since, as the function within the brackets is entire, there will be no ascending developments. Hence

$$\Sigma ax^r = -\Theta[ax^r]\frac{d}{dx}\log \mathrm{E}. \quad \ldots\ldots\ldots\ldots (4.)$$

Consider, secondly, the expression

$$\Sigma\frac{a}{(x-p)^i}.$$

Now expressing (3.) in the form

$$\frac{d}{dx}\log \mathrm{E} = \frac{1}{x-p-(x_1-p)} + \frac{1}{x-p-(x_2-p)} \ldots + \frac{1}{x-p-(x_n-p)},$$

and developing the several terms in the second member in ascending powers of $x-p$, the aggregate coefficient of $(x-p)^{i-1}$ will be

$$-\left\{\frac{1}{(x_1-p)^i} + \frac{1}{(x_2-p)^i} \ldots + \frac{1}{(x_n-p)^i}\right\} = -\Sigma\frac{1}{(x-p)^i}.$$

Hence $\Sigma\frac{1}{(x-p)^i} = -$coefficient of $(x-p)^{i-1}$ in the development of $\frac{d}{dx}\log \mathrm{E}$ in ascending powers of $x-p$,

$= -$coefficient of $\frac{1}{x-p}$ in the development of $\frac{1}{(x-p)^i}\frac{d}{dx}\log \mathrm{E}$ in ascending powers of $x-p$;

$\therefore\ \Sigma\frac{a}{(x-p)^i} = -$coefficient of $\frac{1}{x-p}$ in the development of $\frac{a}{(x-p)^i}\frac{d}{dx}\log \mathrm{E}$ in ascending powers of $x-p$.

But $\Theta\left[\frac{a}{(x-p)^i}\right]\frac{d}{dx}\log \mathrm{E} =$ coefficient of $\frac{1}{x-p}$ in the development of $\frac{a}{(x-p)^i}\frac{d}{dx}\log \mathrm{E}$ in ascending powers of $x-p$.

For 1st, the only distinct simple factor in the denominator of the expression within the brackets is $x-p$; 2ndly, there will be no term of the form $\frac{\mathrm{A}}{x}$ in the development of $\frac{a}{(x-p)^i}\frac{d}{dx}\log \mathrm{E}$ in descending powers of x, the first term of that development being evidently $\frac{na}{x^{i+1}}$. Hence

$$\Sigma\frac{a}{(x-p)^i} = -\Theta\left[\frac{a}{(x-p)^i}\right]\frac{d}{dx}\log \mathrm{E}. \quad \ldots\ldots\ldots\ldots (5.)$$

It appears from the above, that when we decompose the rational fraction $\varphi(x)$ into a series of terms $\varphi_1(x), +\varphi_2(x) \ldots +\varphi_m(x)$, which are individually either of the form ax^i or of the form $\frac{a}{(x-p)^i}$, we have

$$\Sigma\varphi_1(x) + \Sigma\varphi_2(x) \ldots + \Sigma\varphi_m(x) = -\Theta[\varphi_1(x)]\frac{d\log \mathrm{E}}{dx} - \Theta[\varphi_2(x)]\frac{d\log \mathrm{E}}{dx} \ldots - \Theta[\varphi_m(x)]\frac{d\log \mathrm{E}}{dx}$$

$$= -\Theta[\varphi_1(x) + \varphi_2(x) \ldots + \varphi_m(x)]\frac{d\log \mathrm{E}}{dx},$$

by art. 11. Whence

$$\Sigma\varphi(x)=-\Theta[\varphi(x)]\frac{d\log \mathrm{E}}{dx}, \quad \ldots\ldots\ldots \quad (6.)$$

which is the expression of the subsidiary proposition in question.

By this proposition, $\Sigma\varphi(x)$, when the different values of x in the terms under the sign Σ are the roots of an equation $\mathrm{E}=0$, is determined as a function of any independent quantities $a_1, a_2, \ldots a_r$ with which x is connected by that equation. In order that these quantities may be independent, it is, of course, necessary that their number should not exceed the index of the degree of the equation $\mathrm{E}=0$.

14. It may be well before continuing the demonstration to exemplify the theorem just obtained. Suppose it then required to determine the value of the expression $\Sigma\frac{x}{x-a}$, the values of x being the roots of the algebraic equation $px^2+(1-p)x+q=0$. Here we have

$$\Sigma\frac{x}{x-a}=-\Theta\left[\frac{x}{x-a}\right]\frac{2px+1-p}{px^2+(1-p)x+q}.$$

Developing the function

$$-\frac{x}{x-a}\cdot\times\frac{2px+1-p}{px^2+(1-p)x+q}$$

in ascending powers of $x-a$, the coefficient of $\frac{1}{x-a}$ will evidently be

$$-a\frac{2pa+1-p}{pa^2+(1-p)a+q}.$$

Again, developing the same function in descending powers of x, the coefficient of $\frac{1}{x}$ will be -2.

Hence

$$\Sigma\frac{x}{x-a}=-\frac{2pa^2+(1-p)a}{pa^2+(1-p)a+q}+2=\frac{(1-p)a+2q}{pa^2+(1-p)a+q}, \quad \ldots\ldots \quad (7.)$$

which is easily verified by the known theory of equations.

The quantity a may be itself a function of p and q without affecting the truth of the above result. The reasoning by which (6.) is established remains quite unaffected by the consideration whether the function $\varphi(x)$ contains, together with x, any of the independent quantities $a_1, a_2, \ldots a_r$ or not, provided only that if they do enter into its expression they enter determinately, *e. g.* that the same value which is given to any radical, as $\sqrt{1+a^2}$, in one of the functions $\varphi(x)$ shall be retained in all.

15. Now let us resume the expression $\Sigma\mathrm{F}dx$, in which the values of x are subject to the condition

$$\mathrm{E}=0.$$

As by this equation the value of x is made to depend upon the quantities $a_1, a_2, \ldots a_r$, we have, on differentiating with respect to all the variables at once,

$$\frac{d\mathrm{E}}{dx}dx+\frac{d\mathrm{E}}{da_1}da_1+\frac{d\mathrm{E}}{da_2}da_2\cdots+\frac{d\mathrm{E}}{da_r}da_r=0. \quad \ldots\ldots\ldots \quad (8.)$$

Or if we appropriate the symbol δ to express complete differentiation with respect to $a_1, a_2, \ldots a_n$,

$$\frac{d\mathrm{E}}{dx}dx+\delta\mathrm{E}=0.$$

Whence

$$dx=-\frac{\delta\mathrm{E}}{\frac{d\mathrm{E}}{dx}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9.)$$

Wherefore

$$\Sigma\mathrm{F}dx=-\Sigma\frac{\mathrm{F}\delta\mathrm{E}}{\frac{d\mathrm{E}}{dx}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10.)$$

Now F being a rational, and E a rational and entire function of x, the expression $\frac{\mathrm{F}\delta\mathrm{E}}{\frac{d\mathrm{E}}{dx}}$ will be rational with respect to x, whence, by the subsidiary proposition just demonstrated,

$$\Sigma\frac{\mathrm{F}\delta\mathrm{E}}{\frac{d\mathrm{E}}{dx}}=-\Theta\left[\mathrm{F}\frac{\delta\mathrm{E}}{\frac{d\mathrm{E}}{dx}}\right]\frac{d\log\mathrm{E}}{dx}.$$

Therefore

$$\Sigma\mathrm{F}dx=\Theta\left[\mathrm{F}\frac{\delta\mathrm{E}}{\frac{d\mathrm{E}}{dx}}\right]\frac{\frac{d\mathrm{E}}{dx}}{\mathrm{E}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11.)$$

Now the distinctive part of the performance of the operation Θ in the second member, consists in developing the entire function

$$\mathrm{F}\frac{\delta\mathrm{E}}{\frac{d\mathrm{E}}{dx}}\times\frac{\frac{d\mathrm{E}}{dx}}{\mathrm{E}}, \text{ or } \mathrm{F}\frac{\delta\mathrm{E}}{\mathrm{E}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12.)$$

in ascending powers of certain simple factors of the form $x-p$, those simple factors being, 1st, such as are found in the denominator of F; 2ndly, such as are not found in the denominator of F, but are found in the denominator of $\frac{\delta\mathrm{E}}{\frac{d\mathrm{E}}{d}}$. It may be shown that the result of that portion of the operation Θ which depends upon the latter class of factors is 0. For the only factors of the form $x-p$ which produce terms of the form $\frac{a}{x-p}$ in the ascending development of (12.), and which are not found in the denominator of F, must be found in E. Let $x-p$ be any such factor, then we may write

$$\mathrm{E}=\mathrm{H}(x-p)^m,$$

where H does not contain $x-p$.

Therefore

$$\frac{\delta\mathrm{E}}{\frac{d\mathrm{E}}{dx}}=\frac{(x-p)^m\delta\mathrm{H}-m\mathrm{H}(x-p)^{m-1}\delta p}{(x-p)^m\frac{d\mathrm{H}}{dx}+m\mathrm{H}(x-p)^{m-1}}=\frac{(x-p)\delta\mathrm{H}-m\mathrm{H}\delta p}{(x-p)\frac{d\mathrm{H}}{dx}+m\mathrm{H}} \quad \ldots\ldots \quad (13.)$$

The denominator of this expression does not contain $x-p$ as a factor. Hence there will be no factor of the above description in the denominator of the fraction within the brackets, and therefore no corresponding development in the performance of Θ. Thus the only factors which produce any effect are those found in the denominator of F. The part of the operation Θ expressed by $-C_{\frac{1}{x}}$ is of course unaffected by the nature of the function within the brackets.

On these accounts then the theorem (11.), seeing that its second member indicates the performance of the operation Θ on the function $F\frac{\delta E}{E}$, the interpretation of that operation being derived solely from the factor F, is reduced to the comparatively simple form

$$\Sigma F dx = \Theta[F]\frac{\delta E}{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

And in this form it constitutes the general theorem of transformation which it was required to demonstrate.

Application of the general Theorem of Transformation to the Comparison of Algebraical Transcendents.

16. In treating of the algebraical transcendents, I shall first exemplify the direct application of the general theorem of transformation to the solution of special problems, and for this application I shall select by preference examples known and familiar. I shall subsequently apply the theorem to the investigation of general formulæ from which the solution of all special problems may be derived.

There is no difficulty in the direct application of the theorem to special problems. The following directions will meet every case.

Let $\Sigma \int X dx$ be the expression whose finite value is to be found, the simultaneous values of x being determined by an equation

$$X = F(x, a_1, a_2 \ldots a_r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$a_1, a_2, \ldots a_r$ being the new variables in terms of which the value of the integral expression is to be obtained. The second member, which we shall represent by F, is supposed rational with respect to x. Let also the equation (1.), made rational and entire with respect to x by reduction to the form

$$p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} \ldots + p_n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

be represented by $E=0$.

Then observing that X does not become infinite when $E=0$, we have

$$\left.\begin{aligned} \Sigma \int X dx &= \Sigma \int F dx \\ &= \int \Theta[F]\frac{\delta E}{E} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

On performing the operation Θ in the second member, the function under the sign of integration will be an exact differential relatively to $a_1, a_2, \ldots a_r$, and, being integrated, will

give the value sought. If the number of integrals under the sign Σ is specified, suppose it n, the function F must be so chosen that the reduced equation $E=0$ may be of the nth degree.

The algebraic sign of each of the integrals in $\Sigma\int X dx$ will be the same as the sign of the corresponding function F, which, as being rational, is not ambiguous.

17. *Example* 1.—The following theorem is given by a writer in the Cambridge Mathematical Journal, vol. i. p. 268, as a generalization of a theorem of Mr. Fox Talbot's relating to the arcs of the equilateral hyperbola. The equation of any hyperbola referred to its asymptotes being $xy=\frac{a^2+b^2}{2}$, or for simplicity, $xy=m^2$, we have, supposing θ the angle between the asymptotes,

$$\text{arc} = \int \frac{\sqrt{x^4-2m^2x^2\cos\theta+m^4}}{x^2}\,dx.$$

Supposing then three values of x to be determined by the equation

$$\sqrt{x^4-2m^2x^2\cos\theta+m^4}=vx+m^2, \quad \ldots\ldots\ldots\ldots \quad (1.)$$

the sum of the corresponding arcs will be

$$\frac{3}{2}v+\frac{m^2\cos\theta}{v}+\text{const.} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.)$$

To demonstrate this theorem, it must be observed that the equation (1.), reduced to a rational and integral form, becomes

$$x^3-(2m^2\cos^2\theta+v^2)x-2m^2v=0, \quad \ldots\ldots\ldots\ldots \quad (3.)$$

which occupies the place of $E=0$, art. 16. By virtue of the same equation we have

$$\Sigma\int \frac{\sqrt{x^4-2m^2x^2\cos\theta+m^4}}{x^2}\,dx=\Sigma\int\frac{vx+m^2}{x^2}\,dx.$$

Thus we have, 1st, to transform the expression

$$\Sigma\frac{vx+m^2}{x^2}\,dx, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.)$$

the simultaneous values of x being determined by (3.); 2ndly, to integrate the result with respect to the new variable v.

Applying the general theorem of transformation, art. 12, we have

$$\Sigma\frac{vx+m^2}{x^2}\,dx=\Theta\left[\frac{vx+m^2}{x^2}\right]\frac{-(2xv+2m^2)\delta v}{x^3-(2m^2\cos\theta+v^2)x-2m^2v}. \quad \ldots\ldots\ldots \quad (5.)$$

Here we must develope the function

$$\frac{vx+m^2}{x^2}\times\frac{-(2xv+2m^2)\delta v}{x^3-(2m^2\cos\theta+v^2)x-2m^2v}. \quad \ldots\ldots\ldots\ldots \quad (6.)$$

in ascending powers of x, and take therein the coefficient of $\frac{1}{x}$. From this we must subtract the coefficient of $\frac{1}{x}$ in the development of the same function in descending powers of x.

Finally, we must integrate the result relatively to v.

Developing the two factors of (6.) by ordinary division in ascending powers of x, we may express it in the form

$$\left(\frac{m^2}{x^2}+\frac{v}{x}\right)\left(\frac{\delta v}{v}+\frac{v^2-2m^2\cos\theta}{2m^2v^2}x\delta v+\&c.\right).$$

Whence, on multiplication of the factors, we find for the coefficient of $\frac{1}{x}$,

$$\left(\frac{3}{2}-\frac{m^2\cos\theta}{v^2}\right)\delta v. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7.)$$

The coefficient of $\frac{1}{x}$ in the descending development will evidently be 0. Integrating, we have

$$\left(\frac{3}{2}v+\frac{m^2\cos\theta}{v}\right)+C \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8.)$$

for the value sought.

It must be observed that, in applying the theorem, the signs of the integrals under the symbol Σ, which would otherwise be ambiguous from the square root, are made determinate by the equivalent rational expression (4.). Each must be of the same sign as the corresponding value of the function $\frac{vx+m^2}{x^2}$.

18. In the example we have just considered, the equation of transformation connects x with but a single new variable v. In the following examples, which are intended to illustrate the doctrine of the comparison of the different orders of elliptic functions, two new variables are introduced.

Example 2.—Required the finite value of the expression

$$\Sigma\int\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.)$$

the simultaneous values of x being determined by the equation

$$\sqrt{(1-x^2)(1-c^2x^2)}=1+vx+wx^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.)$$

By virtue of this equation (1.) assumes the rational form

$$\Sigma\int\frac{dx}{1+vx+wx^2}.$$

Again, the equation (2.) becomes rational and integral when expressed in the form

$$(1+vx+wx^2)^2-(1-x^2)(1-c^2x^2)=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.)$$

and occupies the place of $E=0$ in the general theorem. Hence we have

$$\Sigma\int\frac{dx}{1+vx+wx^2}=\int\Theta\left[\frac{1}{1+vx+wx^2}\right]\frac{2(1+vx+wx^2)(xdv+x^2dw)}{(1+vx+wx^2)^2-(1-x^2)(1-c^2x^2)}.$$

Here the function

$$\frac{2(xdv+x^2dw)}{(1+vx+wx^2)^2-(1-x^2)(1-c^2x^2)}$$

is to be developed in ascending powers of the simple factors of $1+vx+wx^2$. Let $x-h$ be one of those factors. Then it is evident that there cannot be a term of the form $\frac{A}{x-h}$ in the development of the function in ascending powers of $x-h$, inasmuch as that

factor does not exist in the term $(1-x^2)(1-c^2x^2)$. Again, there cannot be a term of the form $\frac{A}{x}$ in the development of the function in descending powers of x. Hence the result of the operation Θ is 0, and we have, finally,

$$\Sigma\int\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}=\text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

The above constitutes in reality what is usually termed the fundamental theorem for the comparison of elliptic functions of the first order. The equation (3.), arranged according to the powers of x, gives

$$(w^2-c^2)x^2+2vwx^2+(v^2+2w+c^2+1)x+2v=0.$$

If we then represent the simultaneous values of x by x_1, x_2, x_3, we have

$$x_1+x_2+x_3=\frac{-2vw}{w^2-c^2}$$

$$x_1x_2+x_2x_3+x_3x_1=\frac{v^2+2w+c^2+1}{w^2-c^2}$$

$$x_1x_2x_3=\frac{2v}{w^2-c^2},$$

whence, eliminating v and w, we find

$$(1-x_1^2)(1-x_2^2)(1-x_3^2)=(2-x_1^2-x_2^2-x_3^2+c^2x_1^2x_2^2x_3^2)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

Now this is the form to which the known relation connecting the amplitudes φ, ψ, σ and the modulus c in the equation

$$\pm F_c(\varphi)\pm F_c(\psi)\pm F_c(\sigma)=0,$$

viz. the relation, $\cos\sigma=\cos\varphi\cos\psi-\sin\varphi\sin\psi\sqrt{1-c^2\sin^2\sigma}$, is reduced, when we therein make $\sin\varphi=x_1$, $\sin\psi=x_2$, $\sin\sigma=x_3$, and rationalize the resulting equation. The signs of the integrals will of course be determined by the signs of the function $1+vx+wx^2$.

To obtain a formula for the comparison of elliptic functions of the second order, we must deduce a finite expression for

$$\Sigma\int\sqrt{\frac{1-c^2x^2}{1-x^2}}\,dx,$$

subject to the relation (2.). We have then

$$\Sigma\int\sqrt{\frac{1-c^2x^2}{1-x^2}}\,dx=\Sigma\int\frac{1-c^2x^2}{1+vx+wx^2}\,dx$$

$$=\int\Theta\left[\frac{1-c^2x^2}{1+vx+wx^2}\right]\frac{2(1+vx+wx^2)(x\delta v+x^2\delta w)}{(1+vx+wx^2)-(1-x^2)(1-c^2x^2)}.$$

Here, as before, the effect of that part of the operation Θ which depends upon $1+vx+wx^2$ is 0, so that we have simply

$$\Sigma\int\sqrt{\frac{1-c^2x^2}{1-x^2}}\,dx=-\int C_{\frac{1}{x}}\frac{(1-c^2x^2)\times 2(x\delta v+x^2\delta w)}{(1+vx+wx^2)-(1-x^2)(1-c^2x^2)},$$

$C_{\frac{1}{x}}$ denoting the coefficient of $\frac{1}{x}$ in the development of the function in descending powers of x. Now developing the numerator and the denominator separately, we have

$$2(c^2x^3-1)(x^3\delta w+x\delta v)=2c^2x^4\delta w+2c^2x^3\delta v..$$

$$(1+vx+wx^3)^2-(1-x^3)(1-c^2x^2)=(w^2-c^2)x^4+2vwx^3..$$

Dividing the first of these by the second, we have a quotient

$$\frac{2c^2\delta w}{w^2-c^2}+\left(\frac{2c^2\delta v}{w^2-c^2}-\frac{4c^2wv\delta w}{(w^2-c^2)^2}\right)\frac{1}{x}+\&c.$$

Integrating the coefficient of $\frac{1}{x}$, which is a complete differential relatively to v and w, we have

$$\Sigma\int\sqrt{\frac{1-c^2x^2}{1-x^2}}\,dx=\frac{2c^2v}{w^2-c^2}+\text{constant}=c^2x_1x_2x_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

The signs of the three integrals under the sign Σ, are of course determined by the signs of the calculated values of $\frac{1-c^2x^2}{1+vx+wx^2}$.

We might in the same way deduce the known formulæ for the comparison of elliptic functions of the third order. Or we might at once investigate a formula for the comparison of elliptic functions of every order. For the latter purpose we should have to evaluate the expression

$$\Sigma\int\frac{(a+bx^2)dx}{(1+nx^2)\sqrt{(1-x^2)(1-c^2x^2)}},$$

under which the three canonical forms of elliptic functions are comprehended, in subjection to the condition (2.). By the general theorem of transformation this value is

$$\int\Theta\left[\frac{a+bx^2}{(1+nx^2)(1+vx+wx^2)}\right]\frac{2(1+vx+wx^2)(x\delta v+x^2\delta w)}{(1+vx+wx^2)^2-(1-x^2)(1-c^2x^2)},$$

which may be reduced at once to the form

$$\int\Theta\left[\frac{a+bx^2}{1+nx^2}\right]\frac{2x\delta v+2x^2\delta w}{(1+vx+wx^2)^2-(1-x^2)(1-c^2x^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

The rest of the solution involves no difficulty. We must develope the entire function following Θ in ascending powers of $x+\frac{1}{\sqrt{n}}\sqrt{-1}$ and $x-\frac{1}{\sqrt{n}}\sqrt{-1}$ successively, and take therein the coefficients of $\frac{1}{x+\frac{1}{\sqrt{n}}\sqrt{-1}}$ and $\frac{1}{x-\frac{1}{\sqrt{n}}\sqrt{-1}}$. From their sum we must subtract the coefficient of $\frac{1}{x}$ in the development of the same function in descending powers of x, and integrate the result as a complete differential with respect to v and w.

The above results are entirely founded upon the assumed theorem of transformation,

$$\sqrt{(1-x^2)(1-c^2x^2)}=1+vx+wx^2.$$

But any other transformation which would connect x with two new variables, through

the medium of an equation of the third degree with reference to x, would lead to results possessing the same degree of generality. Thus the equation

$$\sqrt{(1-x^3)(1-c^3x^3)}=v+wx+cx^3,$$

which connects x with v and w, and constitutes when freed from surds an equation of the third degree with reference to x, might have been employed. I am not aware that the above forms have been employed before. LEGENDRE, in deducing the properties of Elliptic Functions from ABEL'S theorem, sets out from a different assumption, and as I think a less simple one, leading however to equivalent results.

It is not my intention to enter here into the subject of the *connexion* of the different solutions which may thus be obtained. The theory of that connexion can however present no difficulties to those who are acquainted with the labours of JACOBI, RICHELOT and others, upon the differential equations on the integration of which the doctrine of the comparison of transcendents, as contemplated from another point of view, depends.

19. Before applying the general theorem of transformation to the investigation of general formulæ for the comparison of transcendents, I will say a few words upon ABEL'S theorem, as well as upon the class of theorems to which it belongs.

ABEL'S theorem is virtually an expression for

$$\Sigma\int\frac{f(x)dx}{(x-a)\sqrt{\varphi(x)}},$$

$f(x)$ and $\varphi(x)$ being polynomials, the simultaneous values of x in the several integrals being connected by the equation

$$\sqrt{\varphi(x)} = \chi(x),$$

or

$$\varphi(x) = \{\chi(x)\}^2,$$

where $\chi(x)$ is not restricted to being a polynomial, but is a rational function of x, in terms of the coefficients of which the value of the integral sum is to be determined. ABEL expresses $\varphi(x)$ as the product of two polynomials $\varphi_1(x)$, $\varphi_2(x)$, a form which is obviously given to it in order to meet the case in which a rational fraction occurs under the radical sign, since we have

$$\Sigma\int\frac{f(x)}{x-a}dx\sqrt{\frac{\varphi_1(x)}{\varphi_2(x)}} = \Sigma\int\frac{f(x)\varphi_1(x)dx}{(x-a)\sqrt{\varphi_1(x)\varphi_2(x)}}.$$

BROCH and others, including, I believe, ABEL himself, have considered very fully the more general case in which the polynomial $\varphi(x)$ is raised to any fractional power whatever, and to this case may be reduced the still more general one in which a rational fraction takes the place of the polynomial. The reduction is however obviously far more complex than in the simpler case in which the index is $\frac{1}{2}$. I intend here to discuss this problem in a form sufficiently general to render all such reductions unnecessary.

20. PROBLEM.—Required a finite expression for the integral sum

$$\Sigma\int\varphi\sqrt[m]{\psi}dx, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.)$$

φ and ψ being any rational functions of x, the simultaneous values of x in the different integrals being determined by the equation

$$\psi^{\frac{m}{n}}=\chi, \quad \ldots \quad (2.)$$

χ also denoting any rational function of x of the form

$$\frac{a_0+a_1x+a_2x^2..+a_mx^m}{b_0+b_1x+b_2x^2..+b_nx^n}.$$

Our object is to determine (1.) as a function of $a_0, a_1, ..a_m, b_0, b_1, ..b_n$, which are arbitrary in value, whereas the coefficients in φ and ψ are definite in value and are usually numerical.

Representing φ, ψ and χ in the forms

$$\varphi=\frac{p}{q}, \quad \psi=\frac{s}{t}, \quad \chi=\frac{u}{v}, \quad \ldots \quad (3.)$$

p, q, s, t, u, and v being polynomials in x, the transforming equation (2.) assumes the form

$$\left(\frac{s}{t}\right)^{\frac{m}{n}}=\frac{u}{v}.$$

Whence

$$s^mv^n-t^mu^n=0. \quad \ldots \quad (4.)$$

With this condition connecting the values of x in the several integrals with $a_0, a_1, ..b_0, b_1$, &c., we have to seek the value of the expression

$$\Sigma\int\frac{p}{q}\frac{u}{v}\,dx.$$

For to this form, rational with respect to x, the expression (1.) is reducible by virtue of the above relations.

Consider then $\Sigma\int\frac{pu}{qv}dx$ subject to (4.). To apply to this the general theorem, we must write therein

$$\mathrm{F}=\frac{pu}{qv}, \ \mathrm{E}=s^mv^n-t^mu^n.$$

We thus find

$$\Sigma\frac{pu}{qv}dx=\Theta\left[\frac{pu}{qv}\right]\delta\log(s^mv^n-t^mu^n)=n\Theta\left[\frac{pu}{qv}\right]\frac{s^mv^{n-1}\delta v-t^mu^{n-1}\delta u}{s^mv^n-t^mu^n} \quad \ldots \quad (5.)$$

We cannot in its present form integrate the second member, as the interpretation of Θ depends in part upon v, which contains some of the variables to which the integration has reference. As however the function which has to be developed in the performance of the operation Θ is a rational fraction relatively to u, viz.

$$\frac{pu}{qv}\frac{s^mv^{n-1}\delta v-t^mu^{n-1}\delta u}{s^mv^n-t^mu^n}, \quad \ldots \quad (6.)$$

we can resolve it into partial fractions, and this resolution will, in virtue of the properties of Θ, enable us to effect the integration required.

The partial fraction which has v for its demominator will be $\frac{p}{q}\frac{\partial u}{v}$. Separating this, the entire fraction (6.) will assume the form

$$\frac{p}{q}\frac{\partial u}{v}+\frac{p}{q}\frac{s^m v^{n-1}\partial u - s^m u v^{n-2}\partial v}{t^m u^n - s^m v^n}.$$

Thus (5.) becomes

$$\Sigma\frac{p}{q}\frac{u}{v}dx=n\Theta\frac{p}{q}\frac{\partial u}{v}+n\Theta\frac{p}{q}\frac{s^m v^{n-1}\partial u - s^m u v^{n-2}\partial v}{t^m u^n - s^m v^n}, \quad \ldots\ldots\ldots (7.)$$

Θ deriving its interpretation from q and v.

The first term in the second member, inasmuch as we may give to it the form

$$n\Theta\left[\frac{p\partial u}{qv}\right],$$

vanishes by virtue of (5.), art. 11. The second term may be reduced to the form

$$n\Theta\left[\frac{p}{q}\right]\frac{s^m v^{n-1}\partial u - s^m u v^{n-2}\partial v}{t^m u^n - s^m v^n}. \quad \ldots\ldots\ldots (8.)$$

In proof of this, I observe that the function to which Θ is applied cannot have any of the simple factors of v in its denominator. No such factor is involved in q. For by supposition all the coefficients in q are definite, while those in v are arbitrary. Neither, again, can any of those simple factors be involved in $t^m u^n - s^m v^n$, for if so it will be involved in $t^m u^n$, and therefore either in t or u. But it is not involved in t, as the constants in t are definite; and it is not involved in u, for if it were $\frac{u}{v}$ would not be in its lowest terms.

The expression (8.) may be written in the form

$$n\Theta\left[\frac{p}{q}\right]\frac{\left(\frac{s}{t}\right)^m\frac{v\partial u - u\partial v}{v^2}}{\left(\frac{u}{v}\right)^n-\left(\frac{s}{t}\right)^m}=n\Theta[\varphi]\frac{\psi^m\partial\chi}{\chi^n-\psi^m},$$

on replacing $\frac{p}{q}$, $\frac{s}{t}$ and $\frac{u}{v}$ by φ, ψ and χ.

Hence

$$\Sigma\varphi\psi^m dx=n\Theta[\varphi]\frac{\psi^m\partial\chi}{\chi^n-\psi^m}.$$

Therefore

$$\Sigma\int\varphi\psi^m dx=n\int\Theta[\varphi]\frac{\psi^m\partial\chi}{\chi^n-\psi^m}. \quad \ldots\ldots\ldots (9.)$$

The symbols Θ and $\int$ in the second member are now independent and may be transposed. We thus have

$$\Sigma\int\varphi\psi^m dx=n\Theta[\varphi]\psi^m\int\frac{\partial\chi}{\chi^n-\psi^m}, \quad \ldots\ldots\ldots (10.)$$

an expression of remarkable simplicity.

21. In applying this theorem we must effect the integration in the second member, regarding χ as the only variable, inasmuch as the variables a_0, b_0, &c. enter into the constitution of χ, but not into that of the other rational fractions φ and ψ. When the

integration is effected we must write for φ, ψ, and χ the several rational fractions for which they stand and then perform the operation Θ, as we are directed to do by the definition of that symbol.

On actual integration we have from (10.),

$$\Sigma\int\varphi\psi^{\frac{m}{n}}dx=\Theta[\varphi]\psi^{\frac{m}{n}}\Sigma\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\log\left\{\chi-\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\psi^{\frac{m}{n}}\right\}+n\Theta[\varphi]\psi^{\frac{m}{n}}C, \quad (11.)$$

the summation in the second member extending from $r=0$ to $r=n-1$. The last term in that member is equivalent to an arbitrary constant. For φ and $\psi^{\frac{m}{n}}$ do not contain the variables a_0, b_0, &c., which are only found in χ. Hence the coefficients of terms in the developments of the function $\varphi\psi^{\frac{m}{n}}$ will be determinate constants, and $n\Theta[\varphi]\psi^{\frac{m}{n}}C$, on account of the arbitrary factor C, will be itself an arbitrary constant.

22. We are thus led to the following theorem:—

THEOREM.—*The value of the expression* $\Sigma\int\varphi\psi^{\frac{m}{n}}dx$, *where* φ *and* ψ *are any rational functions of* x, *and the simultaneous values of* x *in the several integrals under the sign* Σ *are determined by the algebraic equation* $\psi^{\frac{m}{n}}=\chi$ *in which* χ *is a rational function of* x, will be expressed by the formula

$$\Sigma\int\varphi\psi^{\frac{m}{n}}dx=\Theta[\varphi]\psi^{\frac{m}{n}}\Sigma\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\log\left\{\chi-\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\psi^{\frac{m}{n}}\right\}+C,$$

the summation in the second member extending from $r=0$ to $r=n-1$.

23. In the particular case in which $m=1$, $n=2$, we have

$$\Sigma\int\varphi\sqrt{\psi}dx=\Theta[\varphi]\sqrt{\psi}\log\frac{\chi-\sqrt{\psi}}{\chi+\sqrt{\psi}}. \quad (12.)$$

Let us apply this theorem to the problem of Art. 17. We have

$$\varphi=\frac{1}{x^3}, \quad \psi=x^4-2m^2x^3\cos\theta+m^4, \quad \chi=vx+m^2;$$

$$\therefore \quad \Sigma\int\frac{\sqrt{x^4-2m^2x^3\cos\theta+m^4}}{x^3}dx$$

$$=\Theta\left[\frac{1}{x^3}\right]\sqrt{x^4-2m^2x^3\cos\theta+m^4}\log\frac{vx+m^2-\sqrt{x^4-2m^2x^3\cos\theta+m^4}}{vx+m^2+\sqrt{x^4-2m^2x^3\cos\theta+m^4}}.$$

First, then, we must develope the function in the second member in ascending powers of x, and seek the coefficient of $\frac{1}{x}$. Now

$$\sqrt{m^4-2m^2x^3\cos\theta+x^6}=m^2-x^3\cos\theta+\text{ &c.}$$

Substituting, the function becomes

$$\left(\frac{m^2}{x^3}-\cos\theta..\right)\log\frac{vx+x^2\cos\theta..}{2m^2+vx-x^2\cos\theta..}.$$

But
$$\log(vx+x^2\cos\theta\,..)=\log x+\log v+\frac{x\cos\theta}{v}+\&c.$$
$$\log(2m^2+vx-x^2\cos\theta\,..)=\log 2m^2+\frac{vx-x^2\cos\theta}{2m^2}+\&c.$$

Substituting, we have
$$\left(\frac{m^2}{x^2}-\cos\theta\,..\right)\left\{\log x+\log v-\log 2m^2+\left(\frac{\cos\theta}{v}-\frac{v}{2m^2}\right)x\,..\right\},$$
wherein the coefficient of $\frac{1}{x}$ in the product is
$$\frac{m^2\cos\theta}{v}-\frac{v}{2}. \qquad \ldots\ldots\ldots\ldots \quad (13.)$$

In the second place, developing the function in descending powers of x, we have
$$\sqrt{x^4-2m^2x^2\cos\theta+m^4}=x^2-m^2\cos\theta,\ \&c.,$$
which on substitution gives
$$\left(1-\frac{m^2\cos\theta}{x^2}\,..\right)\log\frac{-x^2+vx+m^2(1+\cos\theta)}{x^2+vx+m^2(1-\cos\theta)}$$
$$=\left(1-\frac{m^2\cos\theta}{x^2}\,..\right)\log\left(-1+\frac{2v}{x}\,..\right)$$
$$=\left(1-\frac{m^2\cos\theta}{x^2}\,..\right)\left(-\frac{2v}{x}\,..\right),$$
wherein the coefficient of $\frac{1}{x}$ is $-2v$. Hence, changing its sign and adding the result to (13.), we have
$$\Sigma\int\frac{\sqrt{x^4-2m^2x^2\cos\theta+m^4}}{x^2}\,dx=\frac{3}{2}v+\frac{m^2\cos\theta}{v}+C,$$
which agrees with (8.), art. 17.

It is, however, very much easier in the above problem, and perhaps in most others, to apply at once the fundamental theorem of transformation, as already exemplified in its solution.

24. ABEL's theorem is of course included in that of Art. 23. To deduce it, we must observe that its object is to determine the value of the expression
$$\Sigma\int\frac{f(x)dx}{(x-a)\sqrt{\varphi_1(x)\varphi_2(x)}}, \qquad \ldots\ldots\ldots\ldots \quad (1.)$$
the simultaneous values of x being connected by the following equation,—
$$\sqrt{\frac{\varphi_2(x)}{\varphi_1(x)}}=\frac{a_0+a_1x\,..+a_nx^n}{c_0+c_1x\,..+c_nx^n}.$$

To compare with the general theorem we must therefore write (1.) in the form
$$\Sigma\int\frac{f(x)}{(x-a)\varphi_2(x)}\sqrt{\frac{\varphi_2(x)}{\varphi_1(x)}}\,dx.$$
Hence we must make in (1.),
$$\varphi=\frac{f(x)}{(x-a)\varphi_2(x)},\quad \psi=\frac{\varphi_2(x)}{\varphi_1(x)},\quad \chi=\frac{a_0+a_1x\,..}{c_0+c_1x\,..}.$$

We thus find

$$\Sigma\int\frac{f(x)dx}{(x-a)\sqrt{\varphi_1(x)\varphi_2(x)}}=\Theta\left[\frac{f(x)}{(x-a)\varphi_2(x)}\right]\sqrt{\frac{\varphi_2(x)}{\varphi_1(x)}}\log\frac{\frac{a_0+a_1x..}{c_0+c_1x..}-\sqrt{\frac{\varphi_2(x)}{\varphi_1(x)}}}{\frac{a_0+a_1x..}{c_0+c_1x..}+\sqrt{\frac{\varphi_2(x)}{\varphi_1(x)}}}.$$

Here the function to be developed is

$$\frac{f(x)}{(x-a)\sqrt{\varphi_1(x)\varphi_2(x)}}\log\frac{(a_0+a_1x..)\sqrt{\varphi_1(x)}-(c_0+c_1x..)\sqrt{\varphi_2(x)}}{(a_0+a_1x..)\sqrt{\varphi_1(x)}+(c_0+c_1x..)\sqrt{\varphi_2(x)}},$$

the ascending developments having reference to $x-a$, and to the different simple factors, $x-h_1$, $x-h_2$.. of $\varphi_2(x)$. The coefficient of $\frac{1}{x-a}$ in the first development is evidently

$$\frac{f(a)}{\sqrt{\varphi_1(a)\varphi_2(a)}}\log\frac{(a_0+a_1a..)\sqrt{\varphi_1(a)}-(c_0+c_1a..)\sqrt{\varphi_2(a)}}{(a_0+a_1a..)\sqrt{\varphi_1a}\quad+(c_0+c_1a..)\sqrt{\varphi_2(a)}}.$$

The coefficients of $\frac{1}{x-h_1}$, $\frac{1}{x-h_2}$ in the latter developments are 0. Hence we have

$$\Sigma\int\frac{f(x)dx}{(x-a)\sqrt{\varphi_1(x)\varphi_2(x)}}=\frac{f(a)}{\sqrt{\varphi_1(a)\varphi_2(a)}}\log\frac{(a_0+a_1a..)\sqrt{\varphi_1(a)}-(c_0+c_1a..)\sqrt{\varphi_2(a)}}{(a_0+a_1a..)\sqrt{\varphi_1(a)}+(c_0+c_1a..)\sqrt{\varphi_2(a)}}$$
$$-C_{\frac{1}{x}}\frac{f(x)}{(x-a)\sqrt{\varphi_1(x)\varphi_2(x)}}\log\frac{(a_0+a_1x..)\sqrt{\varphi_1(x)}-(c_0+c_1x..)\sqrt{\varphi_2(x)}}{(a_0+a_1x..)\sqrt{\varphi_1(x)}+(c_0+c_1x..)\sqrt{\varphi_2(x)}},$$

which is ABEL'S theorem. There is, however, nothing gained by the peculiar form in which it supposes the integral to be expressed. The resolution of the polynomial under the radical into two factors $\varphi_1(x)$, $\varphi_2(x)$ is only a substitute, and an inconvenient one, for the more general hypothesis of the theorem of art. 22, which permits the function under the radical sign to be any rational fraction.

25. The theorem of art. 22 is, I believe, more general than any which have been investigated with relation to the same well-marked class of transcendents. BROCH, JURGENSEN, and MINDING have given formulæ directly applicable to the case in which $\sqrt{\ }$ is a polynomial*. Their results agree in substance with the above, under the particular restriction supposed, but they are far more complicated in expression. The introduction of the symbol Θ, definite in meaning and indicating the performance of operations which are always intelligible and always possible, greatly simplifies the expression of general theorems.

26. The most general form of the problem contemplated by ABEL in his theory of the comparison of transcendents may be thus expressed. Required a finite expression for $\Sigma\int f(x,y)dx$, $f(x,y)$ denoting a rational function of x and y, the latter of which quantities is itself an irrational function of x given by an algebraic equation of the form

$$p_0y^n+p_1y^{n-1}+p_2y^{n-2}..+p_n=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

wherein p_0, p_1,..p_n are rational and integral functions of x, the simultaneous values of x in the different integrals being determined by an equation of the form

$$y=r,$$

* CRELLE'S Journal, vol. xxiii.

wherein r is a rational function of x, and of any new variables $a_1, a_2, \ldots a_n$ in terms of which the value $\Sigma\int f(x,y)dx$ is to be found*. JURGENSEN has remarked that this problem may be reduced to that of the determination of the value of the expression

$$\Sigma\int f(x)\phi(x,y)dx,$$

$f(x)$ being a rational function of x and $\phi(x,y)$ a rational and integral function of x and y; and under this form, adding also the restriction that p_0 the coefficient of the highest power of y in (1.) shall be unity, he has solved the problem. MINDING has investigated the solution when the above restriction is not imposed, but his analysis is in reality founded upon a transformation in which $p_0 y$ takes the place of y†.

We can, both with increased generality and with that gain of simplicity which results from the employment of the symbol Θ, solve the same problem by the method of this section. But as the comparison of the algebraical transcendents is not the most important object of this paper, I do not propose to enter here upon the investigation in its most general form, but shall demonstrate a theorem which, while it is sufficiently general for all practical ends, will at the same time serve to throw light upon a peculiarity in the theorem of art. 22 already demonstrated.

27. PROBLEM.—*Required, in finite terms, the value of the integral expression*

$$\Sigma\int\phi(x)ydx, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.)$$

$\phi(\mathrm{x})$ *denoting a rational function of* x, *and* y *an irrational function of* x *determined by an equation of the* m*th degree,*

$$p_0 y^m + p_1 y^{m-1} + p_2 y^{m-2} \ldots + p_m = 0, \quad \ldots\ldots\ldots\ldots \quad (2.)$$

where $\mathrm{p}_1 \ldots \mathrm{p}_m$ *are rational and integral functions of* x. *We shall suppose the different simultaneous values of* x *under the sign* Σ *determined by an equation*

$$y = \chi, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.)$$

χ *being a rational function of* x *of the form* $\frac{\mathrm{u}}{\mathrm{v}}$, *in which* u *and* v *are polynomials.* The value of the integral expression (1.) is to be found in terms of the coefficients of those polynomials.

The equation (3.), cleared of the radicals contained in y, and arranged with respect to the powers of χ, will be

$$p_0 \chi^m + p_1 \chi^{m-1} \ldots + p_m = 0,$$

or writing for χ its value $\frac{u}{v}$, and clearing of fractions,

$$p_0 u^m + p_1 u^{m-1} v \ldots + p_m v^m = 0. \quad \ldots\ldots\ldots\ldots \quad (4.)$$

This equation, on substituting for u and v their values as polynomials, is rational and integral with respect to x, and occupies the place of the equation $\mathrm{E}=0$ in the general theorem of transformation. We shall suppose it of the nth degree. It may of course

* ABEL's Works, vol. ii. p. 66.

† CRELLE's Journal, vol. xxiii.

be exhibited in the form

$$p_0(u-vy_1)(u-vy_2)..(u-vy_m)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

$y_1, y_2..y_m$ being the different values of y, as determined by giving different signs to the radicals in its expression. Hence we have

$$\left.\begin{aligned}\Sigma\int\varphi(x)ydx=\Sigma\int\varphi(x)\tfrac{u}{v}\,dx&=\int\Theta\left[\varphi(x)\tfrac{u}{v}\right]\delta\log p_0(u-vy_1)(u-vy_2)..(u-vy_m)\\&=\int\Theta\left[\varphi(x)\tfrac{u}{v}\right]\Sigma\frac{\delta u-y_r\delta v}{u-y_rv}\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (6.)$$

the summation in the second member extending from $r=1$ to $r=m$.

Here, transposing the symbol Σ, the function upon which the operation Θ, *whose interpretation is derived from* $\varphi(x)\frac{u}{v}$, is to be performed, is

$$\varphi(x)\frac{u}{v}\frac{\delta u-y_r\delta v}{u-y_rv},$$

which may be resolved into

$$\varphi(x)y_r\frac{\delta u-y_r\delta v}{u-y_rv}+\frac{\varphi(x)}{v}(\delta u-y_r\delta v).$$

Hence

$$\Sigma\int\varphi(x)ydx=\Sigma\int\Theta\varphi(x)y_r\frac{\delta u-y_r\delta v}{u-y_rv}+\Sigma\int\Theta\frac{\varphi(x)}{v}(\delta u-y_r\delta v). \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

We are especially to remark, that while Σ in the first member has reference to the n different values of x furnished by the equation (3.) or (4.), Σ in the second member has reference to the different values of y furnished by the equation (2.), its numerical range being from $r=1$ to $r=m$.

Now considering the term $\quad \Sigma\int\Theta\varphi(x)y_r\frac{\delta u-y_r\delta v}{u-y_rv}$,

all that part of the operation Θ which depends upon v produces no effect. For none of the factors of v can enter in any way into y_r, since those factors contain the variables $a_0, a_1..a_r$, from which y_r is wholly free. Again, they cannot enter into the denominator $u-y_rv$, for then they would enter into u, and the fraction $\frac{u}{v}$ would not be in its lowest terms. Hence the first term in the second member of (7) becomes

$$\Sigma\int\Theta[\varphi(x)]y_r\frac{\delta u-y_r\delta v}{u-y_rv}.$$

We can now transpose the symbols $\int$ and Θ and integrate. The result is

$$\Theta[\varphi(x)]\Sigma y_r\log(u-y_rv). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

The last term of (7.) will, on account of the interpretation of Θ, be properly written in the form

$$\Sigma\Theta\left[\varphi(x)\frac{1}{v}\right](\delta u-y_r\delta v),$$

which may be resolved into

$$\Sigma\Theta\left[\varphi(x)\frac{1}{v}\right]\delta u-\Sigma\Theta\left[\varphi(x)\frac{1}{v}\right]y_r\delta v.$$

The first term vanishes by Art. 11. The second may be reduced as follows. We have

$$-\Sigma\Theta\left[\varphi(x)\frac{1}{v}\right]y_r\delta v=-\Theta\left[\varphi(x)\frac{1}{v}\right]\Sigma y_r\delta v=\Theta\left[\varphi(x)\frac{1}{v}\right]\frac{p_1}{p_0}\delta v.$$

Now
$$\Theta\left[\varphi(x)\frac{1}{v}\right]\frac{p_1}{p_0}\delta v+\Theta\left[\varphi(x)\frac{p_1}{p_0}\right]\frac{1}{v}\delta v=\Theta\left[\varphi(x)\frac{1}{v}\frac{p_1}{p_0}\right]\delta v+\Theta\left[\varphi(x)\right]\frac{1}{v}\frac{p_1}{p_0}\delta v. \quad . \quad . \quad . \quad (9.)$$

This is evident if we collect the different parts of the interpretation of Θ from the terms in each member, observing that in all cases it is upon the same function that Θ operates. Now the first term in the second member vanishes by Art. 11. Hence

$$\Theta\left[\varphi(x)\frac{1}{v}\right]\frac{p_1}{p_0}\delta v=\Theta\left[\varphi(x)\right]\frac{p_1}{p_0}\frac{\delta v}{v}-\Theta\left[\varphi(x)\frac{p_1}{p_0}\right]\frac{\delta v}{v}.$$

Attaching now the symbol of integration to the second member, and integrating, since $\int$ and Θ are now transposable, we have

$$\Theta[\varphi(x)]\frac{p_1}{p_0}\log v-\Theta\left[\varphi(x)\frac{p_1}{p_0}\right]\log v.$$

Writing in the first term of this expression $-\Sigma y_r$ for $\frac{p_1}{p_0}$ and adding the result to (8.), we have

$$\Theta[\varphi(x)]\{\Sigma y_r\log(u-y_rv)-\Sigma y_r\log v\}-\Theta\left[\varphi(x)\frac{p_1}{p_0}\right]\log v,$$

or

$$\Theta[\varphi(x)]\Sigma y_r\log\left(\frac{u}{v}-y_r\right)-\Theta\left[\varphi(x)\frac{p_1}{p_0}\right]\log v.$$

Hence replacing $\frac{u}{v}$ by χ and adding the constant of integration, we have

$$\Sigma\int\varphi(x)y\,dx=\Theta[\varphi(x)]\Sigma y_r\log(\chi-y_r)-\Theta\left[\varphi(x)\frac{p_1}{p_0}\right]\log v+C. \quad . \quad . \quad . \quad (10.)$$

the expression required. It will be observed that as $\varphi(x)$ and $\frac{p_1}{p_0}$ are always rational functions, the operation Θ may always be performed on each term of the second member.

As the summation $\Sigma y_r\log(\chi-y_r)$ can only be effected by connecting the several terms included under Σ by the sign of addition, it will be most convenient to express the solution in the following form,

$$\Sigma\int\varphi(x)y\,dx=\Sigma_{r=1}^{r=n}\Theta[\varphi(x)]y_r\log(\chi-y_r)-\Theta\left[\varphi(x)\frac{p_1}{p_0}\right]\log v+C. \quad . \quad . \quad . \quad (11.)$$

28. If $p_1=0$, we have

$$\Sigma\int\varphi(x)y\,dx=\Sigma_{r=1}^{r=n}\epsilon^{[\varphi(x)]y_r}\log(\chi-y_r)+C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

This includes as a particular case the theorem of Art. 22. For if $y=\sqrt[n]{\psi}$, we have

$$y^n-\psi^n=0,$$

of which any root y_r will be given by the formula

$$y_r=\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\sqrt[n]{\psi}.$$

Hence on substitution we have, since the number of values of y is n,

$$\Sigma\int\phi(x)\psi^{\frac{m}{n}}dx=\Sigma_{r=1}^{r=n}\Theta[\phi(x)]\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\psi^{\frac{m}{n}}\log\left\{\chi-\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\psi^{\frac{m}{n}}\right\},$$

which agrees with the result in question.

We now see why it is that, although in the investigation of the latter formula we had to take distinct account of the terms u and v in the fraction χ; in the final result they are recombined, and only present themselves implicitly as component parts of χ. It is due to the fact that the equation determining the irrational factor of the original integral wants a second term, *i. e.* that $p_1=0$.

If $p_0=1$, the term $-\Theta\left[\phi(x)_{p_0}^{p_1}\right]\log v$ in (11.) becomes $-\Theta[\phi(x)]p_1\log v$, or $-\Sigma\Theta[\phi(x)]y_r\log v$, whence the theorem assumes the following form,

$$\Sigma\int\phi(x)ydx=\Sigma_{r=1}^{r=n}\Theta[\phi(x)]y_r\{\log(\chi-y_r)+\log v\}=\Sigma_{r=1}^{r=n}\Theta[\phi(x)]y_r\log(u-y_rv).$$

We may remark in concluding this section, that all general theorems like the above for the comparison of algebraical transcendents are difficult of application, from the necessity which they impose of developing logarithmic and irrational functions. This difficulty we avoid by employing directly the theorem of transformation exemplified in the earlier problems of this section. The application of that theorem requires only the development of rational fractions, and this can always be effected by the operations of multiplication and division. When this form of procedure is adopted there remains, however, an integration to be performed. Circumstances must decide which of these methods is preferable, but generally I conceive it will be the latter one. I will only add, that the interest attaching to the entire subject of the comparison of algebraical transcendents appears to me to be chiefly of a speculative character. It is to be valued rather as affording evidence of the powers, and at the same time of the limitations of analysis, than as offering any prospect of increased command over the problems of physical science. Such at least seems to be the tenor of present indications.

Application of the General Theorem of Transformation to the reduction of Functional Transcendents.

29. Let us first apply the general theorem to the reduction of the expression $\Sigma\int\phi f(\psi)dx$, where ϕ and ψ are any rational functions of x, and f a general functional symbol, the simultaneous values of x in the several integrals being determined by an equation,

$$\psi=v, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

in which v is a new variable.

We have

$$\Sigma\int\phi f(\psi)dx=\Sigma\int f(v)\phi dx=\int\Sigma f(v)\phi dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

We must now seek the value of the expression $\Sigma f(v)\phi dx$, subject to the condition (1.), and then integrate with respect to v.

Now ψ as a rational function of x may be represented under the form $\frac{P}{Q}$, where P and Q are polynomials in x not involving v. The equation of transformation then becomes

$$\frac{P}{Q}=v,$$

and in its rational and integral form $Qv-P=0$.

Hence by the general theorem of transformation,

$$\Sigma f(v)\varphi dx=\Theta[f(v)\varphi]\delta\log(Qv-P)=\Theta[f(v)\varphi]\frac{Q\delta v}{Qv-P}.$$

Now the factor $f(v)$, inasmuch as it does not contain x, in nowise affects the interpretation of Θ. Hence it may be removed from within the brackets and the equation written in the form

$$\Sigma f(v)\varphi dx=f(v)\Theta[\varphi]\frac{Q\delta v}{Qv-P}=f(v)\Theta[\varphi]\frac{\delta v}{v-\psi},$$

on replacing $\frac{P}{Q}$ by ψ.

Hence, replacing the first member by the expression for which it is an equivalent, and attaching the sign of integration,

$$\Sigma\int\varphi f(\psi)dx=\int f(v)\Theta[\varphi]\frac{\delta v}{v-\psi}. \quad \ldots\ldots\ldots \quad (3.)$$

To this result we may also give the form

$$\Sigma\int\varphi f(\psi)dx=\Theta[\varphi]\int\frac{f(v)\delta v}{v-\psi}, \quad \ldots\ldots\ldots \quad (4.)$$

as the symbol Θ and the function φ are both independent of the variable v. And this would in fact be the best form if we could effect generally the integration in the second member. For the applications to which we shall proceed (3.), is, however, the form to be preferred.

We may express the results which have been arrived at in the following general theorem.

THEOREM.—*If φ and ψ are rational functions of* x, *and if the simultaneous values of* x *in the integrals included in the expression* $\Sigma\int\varphi f(\psi)$dx *are roots of an equation*

$$\psi=v,$$

v *being a variable quantity, then*

$$\Sigma\int\varphi f(\psi)dx=\int f(v)\Theta[\varphi]\frac{\delta v}{v-\psi}.$$

30. Apparently this is the most general theorem which exists with relation to that class of transcendents in which a perfectly arbitrary symbol of functionality occurs under the sign of integration. If we specify the form of f, so as to meet the case of the particular transcendents discussed in the previous sections of this paper, we shall obtain results accordant with, but less general than, those which have been there obtained. But the most important feature of the theorem is, that, without restricting the generality

of the functional symbol f, we may so determine the form of ψ as to cause the several integrals included under the sign Σ in the first member to *close up*, if the expression is permitted, into a single definite integral whose actual value will then be given by that of a far more simple integral in the second member.

Let the limits of v in the second member be p and q, and let the transforming equation

$$\psi = v$$

give $p_1, p_2 .. p_{n+1}$ for the values of x when $v=p$, and $q_1, q_2 .. q_{n+1}$ for its corresponding values when $v=q$. Then we have

$$\int_{p_1}^{q_1} \varphi f(\psi) dx + \int_{p_2}^{q_2} \varphi f(\psi) dx .. + \int_{p_{n+1}}^{q_{n+1}} \varphi f(\psi) dx = \int_p^q f(v) \Theta[\varphi]_{v=\psi} \frac{\delta v}{\delta \psi} \quad . \quad . \quad . \quad (5.)$$

Now let us give to ψ the form

$$x - \frac{a_1}{x-\lambda_1} - \frac{a_2}{x-\lambda_2} .. - \frac{a_n}{x-\lambda_n},$$

where $\lambda_1, \lambda_2, .. \lambda_n$ are real, and $a_1, a_2 .. a_n$ real and positive. The transforming equation is then

$$x - \frac{a_1}{x-\lambda_1} - \frac{a_2}{x-\lambda_2} .. - \frac{a}{x-\lambda_n} = v; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

whence representing still the first member by ψ,

$$\frac{d\psi}{dx} = 1 + \frac{a_1}{(x-\lambda_1)^2} + \frac{a_2}{(x-\lambda_2)^2} .. + \frac{a_n}{(x-\lambda_n)^2};$$

and as this expression is always positive, it follows,—1st, that ψ regarded as a function of x is never a maximum or minimum; 2ndly, that whenever ψ varies continuously while x increases, it varies by way of increase.

From these properties, and from the form of the equation (6.), it readily follows that if $\lambda_1, \lambda_2, .. \lambda_n$ are arranged in the order of increasing magnitude, then whatever real value v may have, the roots of (6.) will be real and will be disposed in the following manner, viz. one root less than λ_1, one root between λ_1 and λ_2.., one between λ_{n-1} and λ_n, and finally one between λ_n and ∞. To prove this in detail, let x vary from $-\infty$ to λ_1, then ψ, as is evident from its form, varies from $-\infty$ to ∞, and it varies continuously by way of increase so as never to resume a former value. Once therefore in its course it will be equal to v. Wherefore one root of (6.) lies between $-\infty$ and λ_1. Supposing x to continue to increase, the value of ψ suddenly changes when x passes over the value λ_1 from ∞ to $-\infty$, and as x varies from λ_1 to λ_2, ψ again varies continuously, and by way of increase from $-\infty$ to ∞, and therefore again becomes equal to v once in its course; wherefore a value of x lies between λ_1 and λ_2. In like manner there is a value of x between λ_2 and $\lambda_3 .. \lambda_{n-1}$ and λ_n. Finally, as x varies from λ_n to ∞, ψ once more passes over the value v. Whence the proposition is manifest.

The reality of all the roots of (6.) may also be readily shown in the following manner. Let $x = p + q\sqrt{-1}$. Then substituting in (6.), and reducing that equation to the form

$$A + B\sqrt{-1} = v,$$

which entails as a necessary consequence $A=v$, $B=0$, we find as the form of the equation $B=0$,

$$q\left\{1+\frac{a_1}{(p-\lambda_1)^2+q^2}+\frac{a_2}{(p-\lambda_2)^2+q^2}\cdots+\frac{a_n}{(p-\lambda_n)^2+q^2}\right\}=0.$$

Now as the function within the brackets is essentially positive, the above equation can only be satisfied by making $q=0$. But this indicates that all the roots are real.

Resuming (5.), it is evident from what precedes, that if the lower limits of integration $p_1, p_2 \cdots p_{n+1}$, corresponding to $v=p$, are arranged in the ascending order of magnitude, the upper limits $q_1, q_2 \cdots q_{n+1}$ will also be ranged in the same order. Moreover p_1 and q_1 will both be less than λ_1; p_2 and q_2 will lie between λ_1 and λ_2; finally, p_{n+1} and q_{n+1} will lie between λ_n and ∞. Hence, then, the elements in the different integrals in the first member of (5.) will be all different, the superior limit of each integral being less than the inferior limit of the integral which follows it.

31. Let us now examine the case in which the integration relative to v in the second member of (5.) is from $-\infty$ to ∞.

Let $p=-\infty$, and $q=\infty$, we then have

$$p_1=-\infty, \quad p_2=\lambda_1, \quad p_3=\lambda_2 \cdots p_{n+1}=\lambda_n,$$
$$q_1=\lambda_1, \quad q_2=\lambda_2, \quad q_3=\lambda_3 \cdots q_n=\lambda_n, \quad q_{n+1}=\infty.$$

Thus $q_1=p_2$, $q_2=p_3 \cdots q_n=p_{n+1}$, or the upper limit of each integral coincides with the lower limit of the integral which follows it.

It is more strict, however, to regard p and q as *tending* to the respective limits $-\infty$ and ∞. The first inferior limit, p_1, then tends to $-\infty$, and the last superior limit, q_{n+1}, to ∞, while the superior limit of each integral but the last tends upward to the same limiting value to which the inferior limit of the integral following tends downward. The different integrals close up into a single definite integral taken between the limits $-\infty$ and ∞. Thus we have

$$\int_{-\infty}^{\infty}\varphi f(\psi)dx=\int_{-\infty}^{\infty}f(v)\Theta[\varphi]_{v-\psi}^{\delta v} \quad \ldots\ldots\ldots \quad (7.)$$

The reasoning is evidently independent of the nature of the function symbolized by f. That function may either be continuous or discontinuous. We thus arrive at the following theorem, in the expression of which we shall restore to ψ its complete value, and shall replace the rational function φ by $\phi(x)$, and the symbol δ, which is no longer necessary for distinction, by d.

32. THEOREM.—*If* $\phi(x)$ *denote a rational function of* x, *and if* f *be a general functional symbol, then*

$$\int_{-\infty}^{\infty}\phi(x)f\left(x-\frac{a_1}{x-\lambda_1}-\frac{a_2}{x-\lambda_2}\cdots-\frac{a_n}{x-\lambda_n}\right)dx$$
$$=\int_{-\infty}^{\infty}dvf(v)\Theta[\phi(x)]\frac{1}{v-x+\frac{a_1}{x-\lambda_1}+\frac{a_2}{x-\lambda_2}\cdots+\frac{a_n}{x-\lambda_n}}, \quad \ldots\ldots \quad (1.)$$

provided that $a_1, a_2, \ldots a_n$ *are real and positive, and* $\lambda_1, \lambda_2 \ldots \lambda_n$ *real.*

This is the general theorem of definite integration to which reference has been made. The remainder of this paper will be devoted to its illustration.

The theorem is independent, as has been said, of the nature of the functional interpretation of f, and, even when the factor $f\left(x-\frac{a}{x-\lambda_1}\cdots-\frac{a}{x-\lambda_n}\right)$ becomes infinite for the limiting values $\lambda_1, \lambda_2 \ldots \lambda_n$, does not fail, but carries with it a correction for the discontinuity thence arising. We cannot otherwise attach a meaning to the expression $\int_a^b f(x)dx$, when for a value $x=\lambda$, included within the limits of integration, $f(x)$ becomes infinite, than by considering it as the limit to which the sum of the integrals

$$\int_a^{\lambda-\epsilon} f(x)dx+\int_{\lambda+\epsilon'}^{b} f(x)dx$$

tend as ϵ and ϵ' tend to 0. According to the nature of the function $f(x)$ and the modes in which ϵ and ϵ' tend to the limit 0, the integral may become, as CAUCHY has observed, finite or infinite, determinate or indeterminate. When $\epsilon'=\epsilon$, so that the approach on either side to the limiting value λ is made in the same manner, *i. e.* by equivalent infinitesimal variations of x, the value of the integral obtained will be that which CAUCHY terms its *principal* value. The equation (1.) will thus give the principal value of the integral in its first member, if we suppose v to approach by the same kind of variations to the limits $-\infty$ and ∞; in other words, if, representing the function under the sign of integration in the second member by $\mathrm{F}(v)$, we regard $\int_{-\infty}^{\infty}\mathrm{F}(v)dv$ as the limit of the value of $\int_{-a}^{a}\mathrm{F}(v)dv$, a becoming indefinitely great. For suppose x to be approaching the particular limit λ_1. The nearer its approach the more nearly (*vide* 6, art. 30) is the following equation realized, viz.—

$$\frac{-a_1}{x-\lambda_1}=v,$$

whence the more nearly have we

$$x=\lambda_1+\frac{a_1}{v};$$

and therefore, if v tend towards ∞ and $-\infty$ by equivalent variations, so also will x by equivalent variations approach from above and from below the limit λ_1.

Again, the larger x becomes the more nearly do x and v approach a ratio of equality, and therefore the mode of approach of x in the first member to the limits $-\infty$ and ∞ determines identically the mode of approach of v to $-\infty$ and ∞. Thus we may finally give to the theorem the following rigorous statement, viz.—

The two members of the equation

$$\int_{-a}^{a} dx\phi(x)f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)=\int_{-a}^{a} dvf(v)\Theta[\phi(x)]\frac{1}{v-x+\frac{a_1}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}}$$

approach a ratio of equality as a *approaches to infinity, provided that if*

$$f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)$$

become infinite for the critical values $\lambda_1, \lambda_2 \ldots \lambda_n$, *we suppose* x *to approach each critical value by equivalent infinitesimal variations.*

The most important, perhaps the only important, cases are those in which $f(v)$ vanishes when v is infinite.

33. I shall begin with noticing some particular deductions from the theorem, among which will be included certain known formulæ of analysis. I shall show that it enables us to deduce from any known definite integral the values of an infinite number of other definite integrals of progressively increasing complexity. I shall show that when the arbitrary function under the sign of integration is regarded as discontinuous, the first member of the equation becomes resolved into a number of definite integrals of continuous functions, and that we thus arrive at the same theorem for the comparison of functional transcendents, (5.), art. 30, from which the above theorem was itself derived. Finally, I shall apply the theorem to the extension of the theory of definite multiple integrals.

Special deductions may be obtained by limiting either,—1st, the form of the rational function $\varphi(x)$; or 2ndly, the interpretation of the functional sign f; or 3rdly, the number and value of the constants in the function under the sign f.

1st. Let $\varphi(x)=1$. Then in the second member of (1.), art. 32,

$$\Theta[\varphi(x)]\frac{dv}{v-x+\frac{a_1}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}}=-C_{\frac{1}{x}}\frac{dv}{v-x+\frac{a}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}}$$

$$=C_{\frac{1}{x}}\frac{dv}{x-v-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}}$$

$$=dv.$$

Hence

$$\int_{-\infty}^{\infty}f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)dx=\int_{-\infty}^{\infty}f(v)dv. \quad \ldots\ldots\ldots \quad (2.)$$

This was the theorem, or rather the most important case of the theorem referred to in Art. 1, as published in the 'Cambridge and Dublin Mathematical Journal,' No. XIX. The following are special applications of it, chiefly selected from the paper in question.

Since we have

$$\int_{-\infty}^{\infty}f\left(x-\frac{a}{x}\right)dx=\int_{-\infty}^{\infty}f(v)dv, \quad \ldots\ldots\ldots\ldots \quad (3.)$$

and

$$x^3+\frac{a^3}{x^3}=\left(x-\frac{a}{x}\right)^3+2a,$$

we shall have

$$\int_{-\infty}^{\infty}f\left(x^3+\frac{a^3}{x^3}\right)dx=\int_{-\infty}^{\infty}f(v^3+2a)dv. \quad \ldots\ldots\ldots \quad (4.)$$

Hence

$$\int_{-\infty}^{\infty}\varepsilon^{-\left(x^4+\frac{a^2}{x^2}\right)}dx=\int_{-\infty}^{\infty}\varepsilon^{-2a}\times\varepsilon^{-v^4}dv=\pi^{\frac{1}{2}}\varepsilon^{-2a}. \quad \ldots\ldots\ldots \quad (5.)$$

In like manner

$$\left.\begin{aligned}\int_{-\infty}^{\infty}\cos\left(x^2+\frac{a^2}{x^2}\right)dx&=\int_{-\infty}^{\infty}\cos(v^2+2a)dv\\&=\cos 2a\int_{-\infty}^{\infty}\cos(v^2)dv-\sin 2a\int_{-\infty}^{\infty}\sin(v^2)dv\\&=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\cos 2a-\sin 2a)\end{aligned}\right\}\quad\ldots\quad(6.)$$

and similarly,

$$\int_{-\infty}^{\infty}\sin\left(x^2+\frac{a^2}{x^2}\right)dx=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\cos 2a+\sin 2a),\quad\ldots\quad(7.)$$

which are known relations. We may by the same method deduce the relations

$$\int_{-\infty}^{\infty}dx\,e^{-\left(x^2+\frac{a^2}{x^2}\right)\cos\theta}\cos\left\{\left(x^2+\frac{a^2}{x^2}\right)\sin\theta\right\}=\pi^{\frac{1}{2}}e^{-2a\cos\theta}\cos\left(2a\sin\theta+\frac{\theta}{2}\right),\quad\ldots\quad(8.)$$

$$\int_{-\infty}^{\infty}dx\,e^{-\left(x^2+\frac{a^2}{x^2}\right)\cos\theta}\sin\left\{\left(x^2+\frac{a^2}{x^2}\right)\sin\theta\right\}=\pi^{\frac{1}{2}}e^{-2a\cos\theta}\sin\left(2a\sin\theta+\frac{\theta}{2}\right),\quad\ldots\quad(9.)$$

originally given by CAUCHY. All the above definite integrals are reduced by the theorem to immediate dependence on the fundamental theorem

$$\int_{-\infty}^{\infty}\epsilon^{-ax^2}dx=\frac{\pi^{\frac{1}{2}}}{\sqrt{a}}.$$

Again, let us consider the definite integral

$$u=\int_0^{\infty}\frac{dx.x^{n-\frac{1}{2}}}{(a+bx+cx^2)^n}.$$

Making $x^{\frac{1}{2}}=z$, we have

$$\begin{aligned}u&=2\int_0^{\infty}\frac{dz}{\left(b+cz^2+\frac{a}{z^2}\right)^n}\\&=\int_{-\infty}^{\infty}\frac{dz}{\left\{b+c\left(z^2+\frac{a}{cz^2}\right)\right\}^n}=\int_{-\infty}^{\infty}\frac{dv}{\left\{b+c\left(v^2+2\sqrt{\frac{a}{c}}\right)\right\}^n}\text{ by (4.)}\\&=\int_{-\infty}^{\infty}\frac{dv}{\left(b+2\sqrt{ac}+cv^2\right)^n}\\&=2\int_0^{\infty}\frac{dv}{\left(b+2\sqrt{ac}+cv^2\right)^n}.\end{aligned}$$

Now from the known theorem

$$\int_0^{\infty}\frac{dt.t^{a-1}}{(1+t)^r}=\frac{\Gamma(a)\Gamma(r-a)}{\Gamma(r)},$$

in which $r>a$, we readily deduce

$$\int_0^{\infty}\frac{dv.v^m}{(p+qv^2)^n}=\frac{1}{2p^{n-\frac{m+1}{2}}q^{\frac{m+1}{2}}}\cdot\frac{\Gamma\left(n-\frac{m+1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(n)},\quad\ldots\quad(10.)$$

whence, reducing the above expression for u, we find

$$\int_0^\infty \frac{dx.x^{n-\frac{1}{2}}}{(a+bx+cx^2)^n} = \frac{1}{c^{\frac{1}{2}}(b+2\sqrt{ac})^{n-\frac{1}{2}}} \cdot \frac{\Gamma(\frac{1}{2})\Gamma(n-\frac{1}{2})}{\Gamma(n)}; \quad \ldots\ldots (11.)$$

and hence on changing x into $\frac{1}{x}$, we find

$$\int_0^\infty \frac{dx.x^{n-\frac{1}{2}}}{(a+bx+cx^2)^n} = \frac{\Gamma(\frac{1}{2})\Gamma(n-\frac{1}{2})}{\Gamma(n)a^{\frac{1}{2}}(b+2\sqrt{ac})^{n-\frac{1}{2}}}. \quad \ldots\ldots (12.)$$

The two last theorems were discovered independently and about the same time by Mr. CAYLEY, Professor THOMSON* and SCHLÖMILCH†. It is to be noted that a, b, and c must be positive.

The above examples have been selected in the first instance because they relate to known results. But there is not one of the results arrived at which may not be generalized to an indefinite degree.

Thus, since we have

$$\int_{-\infty}^{\infty} dv e^{-v^n} = \frac{2}{n}\Gamma\left(\frac{1}{n}\right), \quad \ldots\ldots (13.)$$

we have

$$\int_{-\infty}^{\infty} dx e^{-\left(\frac{x-a}{s}\right)^n} = \frac{2}{n}\Gamma\left(\frac{1}{n}\right). \quad \ldots\ldots (14.)$$

If $n=2$, this gives

$$\int_{-\infty}^{\infty} dx e^{2ax-\left(x^2+\frac{a^2}{s^2}\right)} = \pi^{\frac{1}{2}},$$

whence

$$\int_{-\infty}^{\infty} e^{-\left(x^2+\frac{a^2}{s^2}\right)} dx = \pi^{\frac{1}{2}} e^{-2a}, \quad \ldots\ldots (15.)$$

agreeing with (5.). But let $n=4$, and we find

$$\int_{-\infty}^{\infty} e^{-\left(x^4+\frac{a^4}{s^4}\right)+4a\left(x^2+\frac{a^2}{s^2}\right)} dx = \tfrac{1}{2}\Gamma(\tfrac{1}{4}) e^{6a^2}, \quad \ldots\ldots (16.)$$

and so on indefinitely without even proceeding to employ the more general forms of (2.).

34. Let us examine the definite integral

$$u = \int_{-\infty}^{\infty} x^{2m} f\left(x - \frac{a}{x}\right) dx. \quad \ldots\ldots (17.)$$

By the general theorem (1.) we have

$$u = \int_{-\infty}^{\infty} dv f(v) \Theta[x^{2m}] \frac{1}{v - x + \frac{a}{x}} = \int_{-\infty}^{\infty} dv f v \mathrm{C}_{\frac{1}{x}} \frac{x^{2m+1}}{x^2 - vx - a} \quad \ldots\ldots (18.)$$

For, x^{2m} not being fractional, the interpretation of Θ is reduced simply to $-\mathrm{C}_{\frac{1}{x}}$. It is obviously desirable to express the term $\mathrm{C}_{\frac{1}{x}} \frac{x^{2m+1}}{x^2 - vx - a}$ in a series consisting of powers of v.

Now

$$\frac{x^{2m+1}}{x^2 - vx - a} = x^{2m+1}\left\{\frac{1}{x^2 - a} + \frac{vx}{(x^2-a)^2} + \frac{v^2x^2}{(x^2-a)^3} + \&c.\right\} = \frac{x^{2m+1}}{x^2-a} + v\frac{x^{2m+2}}{(x^2-a)^2} + v^2\frac{x^{2m+3}}{(x^2-a)^3} + \&c. \quad \ldots\ldots (19.)$$

* Cambridge and Dublin Mathematical Journal, vol. ii.

† CRELLE, vol. xxxii.

We are permitted to give this form to the expression before developing in descending powers of x, because, on thus developing the several terms in the second member, no higher power of x will present itself than would be obtained by developing the first member in descending powers of x.

The general term of (19.) is

$$\frac{v^i . x^{2n+(i+1}}{(x^2-a)^{i+1}}.$$

If i be odd, there will be no terms of the form $\frac{A}{x}$ in the development of this function in descending powers of x. Let i be even and be expressed by $2m$. Then we have to develope in descending powers of x a series of functions of the form

$$\frac{v^{2m}x^{2n+2m+1}}{(x-a)^{2m+1}}.$$

The coefficient of $\frac{1}{x}$ in the development of this expression is easily found to be

$$\frac{(2m+1)(2m+2)..(n+m)}{1.2..(n-m)}a^{n-m}v^{2m}.$$

Therefore

$$C_{\frac{1}{x}}\frac{x^{2n+1}}{x^2-vx-a}=\Sigma\frac{(2m+1)(2m+2)..(n+m)}{1.2..(n-m)}a^{n-m}v^{2m},$$

the summation extending from $m=0$ to $m=n$. Hence

$$\int_{-\infty}^{\infty}x^{2n}f\left(x-\frac{a}{x}\right)dx=\Sigma_{m=0}^{m=n}\frac{(2m+1)(2m+2)..(n+m)}{1.2..(n-m)}a^{n-m}\int_{-\infty}^{\infty}dvv^{2m}f(v). \quad . \quad (20.)$$

In the particular case in which the function denoted by f is even, we have, on replacing $f(x)$ by $\varphi(x^2)$,

$$\int_{-\infty}^{\infty}x^{2n}\varphi\left\{\left(x-\frac{a}{x}\right)^2\right\}dx=\Sigma_{m=0}^{m=n}\frac{(2m+1)(2m+2)..(n+m)}{1.2..(n-m)}a^{n-m}\int_{0}^{\infty}dvv^{2m}\varphi(v^2). \quad . \quad (21.)$$

This, when $a=1$, is CAUCHY'S theorem referred to in Art. 1. Some valuable illustrations of it will be found in the Corollaries to the memoir of which it forms the subject.

We may employ (20.) or (21.) to generalize the results given in (11.) and (12.). We may thus finitely determine the values of the integrals

$$\int_0^{\infty}\frac{dx.x^{n+i-\frac{1}{2}}}{(a+bx+cx^2)^n} \quad \int_0^{\infty}\frac{dx\,x^{n-i-\frac{1}{2}}}{(a+bx+cx^2)^n},$$

i being an integer. For the former integral we shall have the expression

$$\Sigma_{m=0}^{m=i}\frac{\overline{2m+1}.\overline{2m+2}..\overline{i+m}}{1.2..(i-m)}\left(\frac{a}{c}\right)^{\frac{i-m}{2}}\times\frac{1}{(b+2\sqrt{ac})^{n-\frac{2i+1}{2}}c^{\frac{2i+1}{2}}}\times\frac{\Gamma\left(n-\frac{2i+1}{2}\right)\Gamma\left(\frac{2i+1}{2}\right)}{\Gamma(n)} \quad . \quad (22.)$$

For the latter integral we shall only have to change in the above, a into c and c into a.

The results in (11.) and (12.), and the more general conclusions just obtained, are of importance in some of the more difficult problems connected with the mathematical theory of electricity. It is probable that a result equivalent to (22.) may be obtained

by some formulæ of Mr. CAYLEY's connected with the reduction of the integrals which occur in certain problems of this class. I have not, however, attempted a verification.

34. Although the list which I have given of results obtainable by other methods might be increased, it is still only in comparatively rare instances that the means of independent verification present themselves. We might by transformations such as CAUCHY has employed, verify the theorem

$$\int_{-\infty}^{\infty} \frac{dx \cos m\left(x-\frac{a}{x}\right)}{1+\left(x-\frac{a}{x}\right)^2} = \pi\epsilon^{-m};$$

but it would not be easy by any such process to verify the theorem

$$\int_{-\infty}^{\infty} \frac{dx \cos\left\{m\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)\right\}}{1+\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)^2} = \pi\epsilon^{-m},$$

a_1 and a_2, &c. being any positive quantities, and λ_1, λ_2, &c. any real quantities whatever. I shall not, however, dwell any longer upon special results, but shall briefly state some of the general consequences which flow from the application of the primary theorem (1.).

1st. *The evaluation of any definite integral*

$$\int_{-\infty}^{\infty} \varphi(x) f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right) dx,$$

in which $\varphi(x)$ *is a rational and integral function of* x, *is reducible to that of the definite integral*

$$\int_{-\infty}^{\infty} \psi(v) f(v) dv, \qquad \ldots\ldots\ldots\ldots \quad (1.)$$

in which $\psi(v)$ *is a rational and integral function of an order not higher than the order of* $\varphi(x)$.

For, by the general theorem,

$$\int_{-\infty}^{\infty} dx \varphi(x) f\left(x-\frac{a_1}{x-\lambda_1}-\frac{a_2}{x-\lambda_2}\cdots-\frac{a_n}{x-\lambda_n}\right) = \int_{-\infty}^{\infty} dv f(v) \Theta[\varphi(x)] \frac{1}{v-x+\frac{a_1}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}}$$

$$= \int_{-\infty}^{\infty} dv f(v) C_{\frac{1}{x}} \frac{\varphi(x)}{x-v-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}},$$

since $\varphi(x)$ is integral. If we develope the fractions $\frac{a_1}{x-\lambda_1}$, $\frac{a_2}{x-\lambda_2}$, &c. in descending powers of x, we shall have

$$\frac{\varphi(x)}{x-v-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}} = \frac{\varphi(x)}{x-v-\frac{\Sigma a_r}{x}-\frac{\Sigma a_r\lambda_r}{x^2}\cdots}.$$

Suppose m to be the highest index of x in $\varphi(x)$, then the development of the right-hand member, in descending powers of x by division, will assume the following form,—

$$B_0x^{m-1}+B_1x^{m-2}+B_2x^{m-3}..+B_mx^{-1}+B_{m+1}x^{-2}..+\&c.,$$

B_0 not containing v, B_1 containing the first power of v, B_2 the first and second power, &c. Hence B_m, the coefficient of x^{-1}, will involve powers of v up to the mth. Let this function be represented by $\psi(v)$, and we have

$$\int_{-\infty}^{\infty} dx\varphi(x)f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty}\psi(v)f(v)dv, \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

$\psi(v)$being of the same order with respect to v as $\varphi(x)$ with respect to x.

The following are particular examples:—

$$\int_{-\infty}^{\infty} dx.xf\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty} vf(v)dv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

$$\int_{-\infty}^{\infty} dx.x^2f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty}(v^2+a_1+a_2..+a_n)f(v)dv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

$$\int_{-\infty}^{\infty} dx.x^2f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty}\left\{v^2+2(a_1..+a_n)v+a_1\lambda_1..+a_n\lambda_n\right\}f(v)dv. \quad . \quad . \quad (5.)$$

2ndly. *The evaluation of the definite integral*

$$\int_{-\infty}^{\infty} dx\varphi(x)f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right),$$

where $\varphi(x)$ *is a rational fraction, is reducible to that of the definite integral*

$$\int_{-\infty}^{\infty} dv\psi(v)f(v), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

where $\psi(v)$*is a rational fraction of the same order as* $\varphi(x)$.

By a rational fraction of the same order, I mean one whose numerator is of the same degree, and whose denominator involves the same number of simple factors elevated to the same powers, the only difference arising from the constant coefficients.

By the general theorem we have

$$\int_{-\infty}^{\infty} dx\varphi(x)f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty} dvf(v)\Theta[\varphi(x)]\frac{1}{v-x+\frac{a_1}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}}, \quad . \quad . \quad . \quad (7.)$$

in which, on account of the distributive character of the symbol Θ, we may resolve $\varphi(x)$ into its component terms, and give to Θ in succession the respective interpretations which they afford.

The component terms of $\varphi(x)$ will be of the forms ax^i and $\frac{a}{(x-p)^i}$, i being an integer. We have just considered the effect of the first class of terms, and it only remains to consider that of the second class.

Now
$$\Theta\left[\frac{a}{(x-h)^i}\right]\frac{1}{v-x+\frac{a_1}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}}$$
$$=\frac{a}{1.2..(i-1)}\left(\frac{d}{dx}\right)^{i-1}\frac{1}{v-h+\frac{a_1}{h-\lambda_1}\cdots+\frac{a_n}{h-\lambda_n}}$$
$$+C_i\frac{a}{(x-h)^i\left(x-v-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)},$$

the former of the two terms in the second member being the coefficient of $\frac{1}{x-h}$ in the development of the function in ascending powers of $x-h$.

It is evident that the latter of the terms in the second member vanishes; for the first term of the development in descending powers of x being $\frac{a}{x^{i+1}}$, there will be no term of the form $\frac{A}{x}$. Hence we have merely to consider the term

$$\frac{a}{1.2..(i-1)}\cdot\left(\frac{d}{dh}\right)^{i-1}\frac{1}{v-H}, \quad \ldots\ldots\ldots\ldots \quad (8.)$$

H standing for

$$h-\frac{a_1}{h-\lambda_1}\cdots-\frac{a_n}{h-\lambda_n}.$$

If $i=1$, (8.) becomes

$$\frac{a}{v-H}.$$

Let $i=2$, and (8.) becomes

$$a\frac{dH}{dh}$$
$$(v-H)^2.$$

If $i=3$, (8.) becomes

$$\frac{a}{1.2}\left\{\frac{\frac{d^2H}{dh^2}}{(v-H)^2}+\frac{2\left(\frac{dH}{dh}\right)^2}{(v-H)^3}\right\}.$$

Hence, generally,

$$\frac{a}{1.2..(i-1)}\left(\frac{d}{dh}\right)^{i-1}\frac{1}{v-H}=\frac{H_1}{v-H}+\frac{H_2}{(v-H)^2}\cdots+\frac{H_i}{(v-H)^i},$$

$H_1, H_2, \ldots H_i$ being independent of v. The second member of the above equation, on addition of its several terms, becomes a rational fraction whose denominator is $(v-H)^i$, and whose numerator is a rational and integral function of v of an order not higher than $i-1$. Hence the theorem is demonstrated.

35. The conclusion to which these investigations lead is a remarkable one, and may be thus expressed. *The evaluation of the definite integral*

$$\int_{-\infty}^{\infty}\frac{p_0+p_1x+p_2x^2\ldots+p_ix^i}{q_0+q_1x+q_2x^2\ldots+q_jx^j}f\left(x-\frac{a_1}{x-\lambda_1}-\frac{a_2}{x-\lambda_2}\cdots-\frac{a_n}{x-\lambda_n}\right)dx$$

is reducible to that of a definite integral of the form

$$\int_{-\infty}^{\infty}\frac{P_0+P_1v+P_2v^2\ldots+P_iv^i}{Q_0+Q_1v+Q_2v^2\ldots+Q_jv^j}f(v)dv,$$

$P_0, P_1, \ldots Q_0, Q_1, \ldots$ *being constants whose values in terms of* $p_1, p_2, \ldots q_1, q_2, \ldots$ *can always be finitely determined.*

As particular examples of the above, we should have

$$\int_{-\infty}^{\infty} \frac{dx}{x-h} f\left(x - \frac{a_1}{x-\lambda_1} \cdots - \frac{a_n}{x-\lambda_n}\right) = \int_{-\infty}^{\infty} \frac{f(v)dv}{v-h+\frac{a_1}{h-\lambda_1} \cdots + \frac{a_n}{h-\lambda_n}}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x-h)^2} f\left(x - \frac{a_1}{x-\lambda_1} \cdots - \frac{a_n}{x-\lambda^n}\right) = \left\{1 + \frac{a_1}{(h-\lambda_1)^2} \cdots + \frac{a_n}{(h-\lambda_n)^2}\right\} \int_{-\infty}^{\infty} \frac{f(v)dv}{\left(v-h+\frac{a_1}{h-\lambda_1} \cdots + \frac{a_n}{h-\lambda_n}\right)^2}.$$

In the last theorem the particular case in which $h=\lambda_1$, the function $f(v)$ being at the same time supposed small for very large values of v, is interesting. We see that as h approaches λ_1 the terms $\frac{a_1}{(h-\lambda_1)^2}$ and $\frac{a_1}{h-\lambda_1}$ become large in comparison with those with which they are connected by addition. Thus the second member approaches the value

$$\frac{a_1}{(h-\lambda_1)^2} \int_{-\infty}^{\infty} \frac{f(v)dv}{\left(\frac{a_1}{h-\lambda_1}\right)^2} = \frac{1}{a_1} \int_{-\infty}^{\infty} f(v)dv,$$

so that in the limit

$$\int_{-\infty}^{\infty} \frac{dx}{(x-\lambda_1)^2} f\left(x - \frac{a_1}{(x-\lambda_1)} \cdots - \frac{a_n}{x-\lambda_n}\right) = \frac{1}{a_1} \int_{-\infty}^{\infty} f(v)dv.$$

It may be worth while to verify this theorem in a particular instance. We have from it

$$\int_{-\infty}^{\infty} \frac{dx}{(x-\lambda)^2} f\left(x - \frac{a}{x-\lambda}\right) = \frac{1}{a} \int_{-\infty}^{\infty} f(v)dv.$$

Now assume in the first member

$$\frac{-a}{x-\lambda} = y-\lambda.$$

The transformed integral is easily found to be

$$\frac{1}{a} \int dy f\left(y - \frac{a}{y-\lambda}\right).$$

As to the limits, when x varies from $-\infty$ to λ, y varies from λ to ∞; and when x varies from λ to ∞, y varies from $-\infty$ to λ. Thus, by mere transposition of the two portions of the integral, the limits of y become the same as those of x, and we have

$$\int_{-\infty}^{\infty} \frac{dx}{(x-\lambda)^2} f\left(x - \frac{a}{x-\lambda}\right) = \frac{1}{a} \int_{-\infty}^{\infty} dy f\left(y - \frac{a}{y-\lambda}\right) = \frac{1}{a} \int_{-\infty}^{\infty} f(v)dv \text{ by (2.), art. 32.}$$

As a particular deduction from the above we shall have.

$$\int_{-\infty}^{\infty} \frac{dx}{x^{\frac{1}{2}}} e^{-\left(x^2 + \frac{a^2}{x^2}\right)} = \frac{\pi^{\frac{1}{2}} e^{-2a}}{a},$$

which may also be verified by differentiating (5.), art. 32, with respect to a. We are permitted thus to differentiate with respect to a, because the function under the sign of integration does not become infinite within the limits. This condition must be strictly attended to in all similar attempts at verification.

3rdly. *From the value of any known definite integral, we can, by the general theorem, deduce either the values of other definite integrals taken between the limits* $-\infty$ *and* ∞, *or relations among the values of those integrals taken between other limits.*

To accomplish the first object, we have only to transform the given integral into one whose limits are $-\infty$ and ∞, and then apply directly the general theorem. The method requires no illustration.

To accomplish the second object, we must express the function under the sign of integration, not as a continuous function taken between the given limits, but as a discontinuous function taken between the limits $-\infty$ and ∞, the character of its discontinuity being such, that for all values within the *given* limits of integration it shall assume the form specified, and for all values without the given limits shall vanish.

Thus let $\int_p^q f(v)dv$ be the definite integral whose value is given. We may extend the integration from $-\infty$ to ∞, provided that we regard $f(v)$ as vanishing when v falls without the limits p and q. We shall thus have

$$\int_{-\infty}^{\infty} f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)dx=\int_{-\infty}^{\infty} f(v)dv=\int_p^q f(v)dv,$$

provided that $f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)$ vanishes whenever $x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}$ falls without the limits p and q. Let the roots of the equation

$$x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}=p,$$

taken in ascending order of magnitude, be $p_1, p_2, \ldots p_{n+1}$, and the roots of

$$x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda^n}=q,$$

taken in the same order, be $q_1, q_2 \ldots q_{n+1}$. We suppose $\lambda_1, \lambda_2 \ldots \lambda_n$ also to be in ascending order of magnitude. Thus (art. 30) p_1 and q_1 lie between $-\infty$ and λ_1, p_2 and q_2 between λ_1 and λ_2, and so on. Also q_1 is greater than p_1, q_2 than p_2, &c. Hence we see that $x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}$ will only fall within the limits p and q when x falls either between p_1 and q_1, or between p_2 and q_2, &c. Thus we have in fact

$$\Sigma\int_{p_i}^{q_i} dx f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)=\int_p^q f(v)dv,$$

and still more generally, $\varphi(x)$ being a rational function of x,

$$\Sigma\int_{p_i}^{q_i}\varphi(x)f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)dx=\int_p^q dv f(v)\Theta[\varphi(x)]\frac{1}{v-x+\frac{a_1}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}},$$

which is a reproduction of (5.), art. 30. I deem it, however, an important fact, that in the comparison of functional transcendents, formulæ involving the sign of summation

may be dispensed with; a more general conception of the nature of a function supplying their place.

36. One remarkable theorem must still be noticed. Since

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots,$$

we have, on taking the logarithms of both sides and differentiating,

$$\cot x = \frac{1}{x}+\frac{1}{x+\pi}+\frac{1}{x-\pi}+\frac{1}{x+2\pi}+\frac{1}{x-2\pi}+\text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Hence

$$x-\cot x = x-\frac{1}{x}-\frac{1}{x+\pi}-\frac{1}{x-\pi}-\frac{1}{x+2\pi}-\text{ \&c.}$$

Whence, by (2.), art. 32,

$$\int_{-\infty}^{\infty} dx f(x-\cot x) = \int_{-\infty}^{\infty} dv f(v). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

The result may however be generalized. For from (1.),

$$a_1 \cot(x-\lambda_1) = \frac{a_1}{x-\lambda_1}+\frac{a_1}{x-\lambda_1+\pi}+\frac{a_1}{x-\lambda_1-\pi}+\text{ \&c.}$$

Taking the sum of any series of such terms, we shall evidently have

$$x-a_1\cot(x-\lambda_1)-a_2\cot(x-\lambda_2)\cdots-a_n\cot(x-\lambda_n)$$
$$= x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}-\frac{a_1}{x+\pi-\lambda_1}\cdots-\frac{a_n}{x+\pi-\lambda_n}\cdots,$$

which agrees in form with the function under the sign f in (2.), art. 32. Hence

$$\int_{-\infty}^{\infty} dx f\{x-a_1\cot(x-\lambda_1)\cdots-a_n\cot(x-\lambda_n)\} = \int_{-\infty}^{\infty} dv f(v). \quad . \quad . \quad . \quad . \quad (3.)$$

If we treat in the same way the theorem

$$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\left(1-\frac{4x^2}{5^2\pi^2}\right)\cdots,$$

we shall arrive at the theorem

$$\int_{-\infty}^{\infty} dx f\{x+a_1\tan(x-\lambda_1)\cdots+a_n\tan(x-\lambda_n)\} = \int_{-\infty}^{\infty} dv f(v). \quad . \quad . \quad . \quad (4.)$$

Essentially, however, this result is involved in (3.), the analogy of which with (2.), Art. 32, will be most apparent if we place it in the form

$$\int_{-\infty}^{\infty} dx f\left\{x-\frac{a_1}{\tan(x-\lambda_1)}\cdots-\frac{a_n}{\tan(x-\lambda_n)}\right\} = \int_{-\infty}^{\infty} dv f(v).$$

As before, the quantities $a_1, a_2 \ldots a_n$ must be positive.

The verification of these theorems by some independent method seems desirable.

Application of the General Theorem to the evaluation of multiple definite integrals.

37. The form in which multiple definite integrals present themselves in the application of mathematics to natural philosophy, is usually the following. The value of a triple integral,

$$\iiint \phi(x, y, z)dx\,dy\,dz, \quad \ldots\ldots\ldots\ldots \quad (1.)$$

is required to be found, the integration extending, in some instances, to all positive values, but more generally to all values whatever of the variables x, y, z which satisfy a condition

$$\psi(x, y, z) \leqq 1. \quad \ldots\ldots\ldots\ldots \quad (2.)$$

The most general method of treating this class of problems is due to M. LEJEUNE DIRICHLET. It consists in converting $\phi(x, y, z)$ into a discontinuous function which vanishes whenever the variables x, y, z transcend the limits assigned in (2.), and which is equal to $\phi(x, y, z)$ whenever those variables satisfy the above condition. This transformation being effected, we are permitted to regard the integrations relatively to x, y, z as independent, and as individually taken between the limits $-\infty$ and ∞.

From this circumstance the progress of our knowledge of multiple definite integrals must be in some degree coordinate with the extension of our command over single definite integrals taken between the limits $-\infty$ and ∞. I propose in this section both to illustrate, by one or two examples, the theory of multiple integrals as above stated, and to show how it gains extension from the theorem of definite integration demonstrated in the preceding pages.

There are several different forms in which the application of the principle of discontinuity to the evaluation of multiple integrals may be presented. The form which I shall adopt in this paper is similar to one originally given by me in the Transactions of the Royal Irish Academy (vol. xxi. pt. 1), but is more convenient in application. It depends essentially upon the employment of FOURIER'S theorem, viz.—

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} da\,.\,dv\,.\cos(av - xv)f(a).$$

If we write the cosine in its exponential form, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} dadv\left\{\varepsilon^{(av-xv)\sqrt{-1}} + \varepsilon^{-(av-xv)\sqrt{-1}}\right\}f(a). \quad \ldots\ldots\ldots \quad (3.)$$

Now by known theorems

$$\frac{1}{t^i} = \frac{\varepsilon^{\frac{i\pi}{2}\sqrt{-1}}}{\Gamma(i)}\int_{0}^{\infty} dw\,.\,w^{i-1}\varepsilon^{-tw\sqrt{-1}} = \frac{\varepsilon^{-\frac{i\pi}{2}\sqrt{-1}}}{\Gamma(i)}\int_{0}^{\infty} dw\,.\,w^{i-1}\varepsilon^{tw\sqrt{-1}}. \quad \ldots\ldots\ldots \quad (4.)$$

Multiply the terms of (3.) by those of (4.) taken in the same order, and, converting the exponentials into sines and cosines, we have

$$\frac{f(x)}{t^i} = \frac{1}{\pi\Gamma(i)}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} dadvdw\cos\left(av - xv - tw + \frac{i\pi}{2}\right)w^{i-1}f(a), \quad \ldots\ldots \quad (5.)$$

the theorem employed in the Irish Transactions.

Now v remaining unchanged, let $w=vs$. Then transforming in the usual way, we have

$$dvdw=vdvds,$$

whence, on substitution,

$$\frac{f(x)}{t^i}=\frac{1}{\pi\Gamma(i)}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}dadvds\cos\left\{(a-x-ts)v+\frac{i\pi}{2}\right\}vs^{i-1}f(a). \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

In the application of this theorem to the reduction of multiple integrals, x and t will be replaced by functions of the variables involved in those integrals. Its advantages are the following. Like FOURIER'S theorem, from which it is derived, it enables us to express any species of discontinuity in the function $f(x)$. Thus if $f(x)$ is to vanish for all values of x which lie without the limits p and q, we have only to substitute p and q for $-\infty$ and ∞ in the integration relative to a. At the same time the theorem presents x and t in a functional connexion, which in all the most important cases renders possible the subsequent integrations without any new transformation.

One subsidiary theorem remains to be noticed. Since we have

$$\int_{-\infty}^{\infty}dy\cos(a\pm cy^2)=\frac{\pi^{\frac{1}{2}}}{c^{\frac{1}{2}}}\cos\left(a\pm\frac{\pi}{4}\right),$$

we have, by successive applications of this theorem,

$$\int_{-\infty}^{\infty}dy_1dy_2..dy_n\cos(a\pm\Sigma_{r=1}^{n}c_ry_r^2)=\frac{\pi^{\frac{n}{2}}}{c_1^{\frac{1}{2}}c_2^{\frac{1}{2}}..c_n^{\frac{1}{2}}}\cos\left(a\pm\frac{n\pi}{4}\right). \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

38. Example 1st. Let it be required to evaluate the multiple definite integral

$$V=\int..dx_1dx_2..dx_n\frac{f(l_1x_1+l_2x_2..+l_nx_n)}{\{h^2+(a_1-x_1)^2+(a_2-x_2)^2..+(a_n-x_n)^2\}^i}, \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

the integration extending to all values of the variables which satisfy the condition

$$l_1x_1+l_2x_2..+l_nx_n\begin{matrix}<q\\>p\end{matrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

If for convenience we represent $l_1x_1+l_2x_2..+l_nx_n$ by Σl_rx_r and $(a_1-x_1)^2+(a_2-x_2)^2..+(a_n-x_n)^2$ by $\Sigma(a_r-x_r)^2$, we have by the theorem (6.),

$$\frac{l_1x_1+l_2x_2..+l_nx_n}{\{h^2+(a_1-x_1)^2+(a_2-x_2)^2..+(a_n-x_n)^2\}^i}$$

$$=\frac{1}{\pi\Gamma i}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}dadvds\cos\left\{\left(a-\Sigma l_rx_r-(h^2+\Sigma\overline{a_r-x_r}^2.s)\right)v+\frac{i\pi}{2}\right\}v^is^{i-1}f(a).$$

The conditions relative to the limits will be fulfilled by introducing p and q for $-\infty$ and ∞ in the above. Effecting this change and extending, as we then may do, the integrations relative to $x_1, x_2..x_n$ from $-\infty$ to ∞, we have

$$V=\frac{1}{\pi\Gamma(i)}\int_{p}^{q}\int_{0}^{\infty}\int_{0}^{\infty}dadvdsv^is^{i-1}f(a)T,$$

where $$T=\int_{-\infty}^{\infty}..dx_1dx_2..dx_n\cos\left\{\left(a-h^2s-\Sigma(l_rx_r+s.\overline{a_r-x_r}^2)\right)v+i\frac{\pi}{2}\right\}.$$

Now $l_rx_r+s(a_r-x_r)^2=s\left(x_r-a_r+\frac{l_r}{2s}\right)^2+l_ra_r-\frac{l_r^2}{4s}=sy_r^2+l_ra_r-\frac{l_r^2}{4s}$ if $y_r=x_r-a_r+\frac{l_r}{2s}$. Sub-

stituting and observing that the limits of y_r are $-\infty$ and ∞, we have

$$\mathrm{T}=\int_{-\infty}^{\infty}..dy_1dy_2..dy_n\cos\left\{\left(a-h^2s-\Sigma\left(sy_r^2+l_ra_r-\frac{l_r^2}{4s}\right)\right)v+i\frac{\pi}{2}\right\}$$
$$=\int_{-\infty}^{\infty}..dy_1dy_2..dy_n\cos\left\{\left(a-h^2s-\Sigma l_ra_r+\frac{\Sigma l_r^2}{4s}\right)v+\Sigma vsy_r^2+\frac{i\pi}{2}\right\}.$$

Let $$\sigma=h^2s+\Sigma l_ra_r-\frac{\Sigma l_r^2}{4s}. \qquad (3.)$$

Then $$\mathrm{T}=\int_{-\infty}^{\infty}..dy_1dy_2..dy_n\cos\left\{(a-\sigma)v+\Sigma vsy_r^2+i\frac{\pi}{2}\right\}=\frac{\pi^{\frac{n}{2}}}{(vs)^{\frac{n}{2}}}\cos\left\{(a-\sigma)v+i\frac{\pi}{2}-n\frac{\pi}{4}\right\},$$

whence

$$\mathrm{V}=\frac{\pi^{\frac{n}{2}-1}}{\Gamma(i)}\int_p^q\int_0^{\infty}\int_0^{\infty}dadvds\,v^{i-\frac{n}{2}}s^{i-\frac{n}{2}-1}\cos\left\{(a-\sigma)v+\left(i-\frac{n}{2}\right)\frac{\pi}{2}\right\}f(a)=\frac{\pi^{\frac{n}{2}}}{\Gamma(i)}\int_0^{\infty}ds\,s^{i-\frac{n}{2}-1}\mathrm{Q},$$

where

$$\mathrm{Q}=\frac{1}{\pi}\int_p^q\int_0^{\infty}dadv\,v^{i-\frac{n}{2}}\cos\left\{(a-\sigma)v+\left(i-\frac{n}{2}\right)\frac{\pi}{2}\right\}f(a)$$
$$=\left(-\frac{d}{d\sigma}\right)^{i-\frac{n}{2}}\frac{1}{\pi}\int_p^q\int_0^{\infty}dadv\cos\{(a-\sigma)v\}f(a)$$
$$=\left(-\frac{d}{d\sigma}\right)^{i-\frac{n}{2}}f(\sigma)$$

by FOURIER'S theorem, $f(\sigma)$ vanishing when σ does not fall within the limits p and q. Thus, finally,

$$\mathrm{V}=\frac{\pi^{\frac{n}{2}}}{\Gamma(i)}\int_0^{\infty}ds\,s^{i-\frac{n}{2}-1}\left(-\frac{d}{d\sigma}\right)^{i-\frac{n}{2}}f(\sigma), \qquad (4.)$$

the fully expressed value of σ being

$$\sigma=h^2s+l_1a_1+l_2a_2..+l_na_n-\frac{l_1^2+l_2^2..+l_n^2}{4s}. \qquad (5.)$$

We may remark, that as

$$\frac{d\sigma}{ds}=h^2+\frac{l_1^2+l_2^2..+l_n^2}{4s^2},$$

σ increases with s. As s varies from 0 to ∞, σ varies from $-\infty$ to ∞. Thus, whatever may be the values of p and q within which the variation of σ is confined, there will exist corresponding positive values within which the variation of s will be confined, and those limits must take the place of 0 and ∞ in the expression of (4.).

39. In strictness there is no need of referring to the limit in the statement of general theorems like the above involving an arbitrary symbol of functionality. The consistent interpretation of that symbol will suffice. Thus the results at which we have arrived are virtually included in the theorem

$$\int_{-\infty}^{\infty}..dx_1dx_2..dx_n\frac{f(l_1x_1+l_2x_2..+l_nx_n)}{\{h^2+(a_1-x_1)^2+(a_2-x_2)^2..+(a_n-x_n)^2\}^i}=\frac{\pi^{\frac{n}{2}}}{\Gamma(i)}\int_0^{\infty}ds.s^{i-\frac{n}{2}-1}\left(-\frac{d}{d\sigma}\right)^{i-\frac{n}{2}}f(\sigma), \qquad (6.)$$

σ having the value given above. For if $l_1x_1+l_2x_2..+l_nx_n$ is confined within given limits, $f(l_1x_1+l_2x_2..+l_nx_n)$ may be regarded as vanishing whenever $l_1x_1+l_2x_2..+l_nx_n$ transcends

those limits, and therefore, by consistency of interpretation, $f(\sigma)$ as vanishing whenever σ transcends the same limits.

40. I shall not enter into any discussion of the above solution, but shall briefly point out in what way it may be generalized by the theorem of definite integration of Art. 32. It is evident that if we change in (6.)

$$x_1 \text{ into } x_1 - \frac{b_1}{x_1-\lambda_1} - \frac{b_2}{x_1-\lambda_2} \cdots - \frac{b_m}{x_1-\lambda_m},$$

$$x_2 \text{ into } x_2 - \frac{c_1}{x_2-\mu_1} - \frac{c_2}{x_2-\mu_2} \cdots - \frac{c_m}{x_2-\mu_m},$$

&c. &c. &c.

or into any of the remarkable forms thence derived, Art. 36, leaving $dx_1 dx_2 \ldots dx_n$ unchanged, the actual value of the multiple integral will be unaltered. Thus, as a particular illustration, if we suppose

$$V = \iiint dx dy dz \frac{f\left\{l\left(x-\frac{1}{x}\right)+m\left(y-\frac{1}{y}\right)+n\left(z-\frac{1}{z}\right)\right\}}{\left\{h^2+\left(a-x+\frac{1}{x}\right)^2+\left(b-y+\frac{1}{y}\right)^2+\left(c-z+\frac{1}{z}\right)^2\right\}^i},$$

the integrations being limited only by the condition

$$l\left(x-\frac{1}{x}\right)+m\left(y-\frac{1}{y}\right)+n\left(z-\frac{1}{z}\right) \lessgtr 0,$$

we should find

$$V = \frac{\pi^{\frac{3}{2}}}{\Gamma(i)} \int_0^\infty ds . s^{i-\frac{3}{2}-1} \left(\frac{d}{d\sigma}\right)^{i-\frac{3}{2}} f(\sigma),$$

where

$$\sigma = la+mb+nc-\frac{l^2+m^2+n^2}{4s},$$

provided that $f(\sigma)=0$ when σ does not fall within the limits 0 and 1.

41. Example 2nd. Let

$$V = \int \ldots dx_1 dx_2 \ldots dx_n \frac{f\left\{l_1^2\left(x_1^2+\frac{a_1^2}{x_1^2}\right)\cdots+l_n^2\left(x_n^2+\frac{a_n^2}{x_n^2}\right)\right\}}{\left\{h^2+m_1^2\left(x_1^2+\frac{b_1^2}{x_1^2}\right)\cdots+m_n^2\left(x_n^2+\frac{b_n^2}{x_n^2}\right)\right\}^i},$$

the integration extending to all values of the variables which satisfy the condition

$$l_1^2\left(x_1^2+\frac{a_1^2}{x_1^2}\right)\cdots+l_n^2\left(x_n^2+\frac{a_n^2}{x_n^2}\right) \lessgtr 1.$$

Here, after reductions similar to those which have been exemplified in the preceding problem, but more complicated, we find

$$V = \frac{\pi^{\frac{n}{2}}}{\Gamma(i)} \int_0^\infty \frac{ds . s^{i-1}\left(-\frac{d}{d\sigma}\right)^{i-\frac{n}{2}} f(\sigma)}{(l_1^2+m_1^2 s)^{\frac{1}{2}} \cdots (l_n^2+m_n^2 s)^{\frac{1}{2}}}, \quad \ldots\ldots\ldots (1.)$$

wherein $\sigma = h^2 s + 2(l_1 + m_1^2 s)^{\frac{1}{2}}(l_1 a_1^2 + m_1^2 b_1^2 s_1)^{\frac{1}{2}} + \cdots + 2(l_n + m_n^2 s)^{\frac{1}{2}}(l_n a_n^2 + m_n^2 b_n^2 s)^{\frac{1}{2}}$,

$f(\sigma)$ being supposed to vanish when σ transcends the limits 0 and 1. This theorem admits of the same kind of generalization as the preceding one. It was communicated by me, some years ago, to Mr. CAYLEY, and published by him, at my desire, in LIOUVILLE'S 'Journal,' vol. xiii.

If in the above theorem we make $a_1 = 0, \ldots a_n = 0$, we have an expression for the value of the multiple integral

$$\int \ldots dx_1 dx_2 \ldots dx_n \frac{f(l_1^2 x_1^2 + l_2^2 x_2^2 \cdots + l_n^2 x_n^2)}{\left\{h^2 + m_1^2\left(x_1^2 + \frac{b_1^2}{x_1^2}\right) \cdots + m_n^2\left(x_n^2 + \frac{b_n^2}{x_n^2}\right)\right\}^i},$$

the integrations being extended through the mass of the ellipsoid whose equation is

$$l_1^2 x_1^2 + \cdots + l_n^2 x_n^2 = 1.$$

The following example, originally published by me, but without any intimation of its possible extension by means of the general theorem of definite integration, in the memoir already referred to in the 'Transactions of the Royal Irish Academy,' is of great practical importance.

42. Example 3rd. Required the value of the multiple integral

$$V = \int \ldots dx_1 dx_2 \ldots dx_n \frac{f\left(\frac{x_1^2}{h_1^2} + \frac{x_2^2}{h_2^2} \cdots + \frac{x_n^2}{h_n^2}\right)}{\{h^2 + (a_1 - x_1)^2 + (a_2 - x_2)^2 \cdots + (a_n - x_n)^2\}^i} \quad \ldots \quad (1.)$$

the integrations extending to all values of the variables which satisfy the condition

$$\frac{x_1^2}{h_1^2} + \frac{x_2^2}{h_2^2} \cdots + \frac{x_n^2}{h_n^2} \leqq 1. \quad \ldots \quad (2.)$$

Here V will be found by integrating with respect to $x_1, x_2, \ldots x_n$ between the limits $-\infty$ and ∞ the expression

$$\frac{1}{\pi\Gamma(i)} \int_0^1 \int_0^\infty \int_0^\infty da\,ds\,dv\, v^i s^{i-1} \cos\left\{\left(a - \frac{\Sigma x_r^2}{h_r^2} - s\left(h^2 + \Sigma(a_r - x_r)^2\right)\right)v + \frac{i\pi}{2}\right\} f(a).$$

Hence changing, as before, the order of integration,

$$V = \frac{1}{\pi\Gamma(i)} \int_0^1 \int_0^\infty \int_0^\infty da\,ds\,dv\, v^i s^{i-1} f(a) T,$$

where

$$T = \int_{-\infty}^{\infty} \ldots dx_1 dx_2 \ldots dx_n \cos\left\{\left(a - h^2 s - \Sigma\left(\frac{x_r^2}{h_r^2} + s(a_r - x_r)^2\right)\right)v + \frac{i\pi}{2}\right\}.$$

Now

$$\frac{x_r^2}{h_r^2} + s(a_r - x_r)^2 = \frac{1 + h_r^2 s}{h_r^2} y_r^2 + \frac{s a_r^2}{1 + h_r^2 s}$$

if we make

$$y_r = x_r - \frac{h_r^2 s a_r}{1 + h_r^2 s}.$$

Substituting and integrating with respect to $y, y_1 .. y_n$ between the limits $-\infty$ and ∞, we have

$$T=\frac{h_1 h_2 .. h_n \pi^{\frac{n}{2}}}{(1+h_1^2 s)^{\frac{1}{2}} .. (1+h_n^2 s)^{\frac{1}{2}} v^{\frac{n}{2}}} \cos\left\{\left(a-h^2 s-\Sigma\frac{a_r^2 s}{1+h_r^2 s}\right)v+\left(i-\frac{n}{2}\right)\frac{\pi}{2}\right\}; \quad . \quad . \quad . \quad (3.)$$

whence, if we make

$$\sigma=h^2 s+\frac{a_1^2 s}{1+h_1^2 s} \cdots +\frac{a_n^2 s}{1+h_n^2 s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

we have

$$\left.\begin{aligned} V&=\frac{\pi^{\frac{n}{2}-1} h_1 h_2 .. h_n}{\Gamma(i)} \int_0^1 \int_0^\infty \int_0^\infty \frac{dadvds\, s^{i-1} v^{i-\frac{n}{2}} \cos\left\{(a-\sigma)+\left(i-\frac{n}{2}\right)\frac{\pi}{2}\right\} f(a)}{(1+h_1^2 s)^{\frac{1}{2}}(1+h_2^2 s)^{\frac{1}{2}} .. (1+h_n^2 s)^{\frac{1}{2}}} \\ &=\frac{\pi^{\frac{n}{2}} h_1 h_2 .. h_n}{\Gamma(i)} \int_0^\infty \frac{ds.s^{-1}}{(1+h_1^2 s)^{\frac{1}{2}} .. (1+h_n^2 s)^{\frac{1}{2}}} \frac{1}{\pi} \int_0^1 \int_0^\infty dadv\, v^{i-\frac{n}{2}} \cos\left\{(a-\sigma)v+\left(i-\frac{n}{2}\right)\frac{\pi}{2}\right\} f(a) \quad . \quad (5.) \\ &=\frac{\pi^{\frac{n}{2}} h_1 h_2 .. h_n}{\Gamma(i)} \int_0^\infty \frac{ds.s^{i-1}\left(-\frac{d}{d\sigma}\right)^{i-\frac{n}{2}} f(\sigma)}{(1+h_1^2 s)^{\frac{1}{2}}(1+h_2^2 s)^{\frac{1}{2}} .. (1+h_n^2 s)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\}$$

Here σ increases continuously with s. As s varies from 0 to ∞, σ also varies from 0 to ∞. To any positive limits of σ will correspond positive limits of s, and these, as will hereafter appear, will in certain cases replace the limits 0 and ∞ in the expression for V.

It is also deserving of note that σ may be placed in the form

$$\sigma=\frac{a_1^2}{h_1^2} \cdots +\frac{a_n^2}{h_n^2}+h^2 s-\frac{\frac{a_1^2}{h_1^4}}{s+\frac{1}{h_1^2}} \cdots -\frac{\frac{a_n^2}{h_n^4}}{s+\frac{1}{h_n^2}},$$

which only differs by a constant term from the form of the function ψ in the general theorem of definite integration. It follows at once from this that all the values of s corresponding to a given value of σ will be real and that only one of them will be positive, the others lying between limits expressed by $-\infty$, $-\frac{1}{h_1^2}$, $-\frac{1}{h_2^2}, \cdots -\frac{1}{h_n^2}$, supposing these ranged in order of ascending magnitude.

The above example admits of the same generalization as the first which we considered. The value of V remains unchanged if, both in the original integral (1.) and in the equation of the limits (2.), we substitute $x_1-\frac{a_1}{x_1-\lambda_1}-\frac{a_2}{x_1-\lambda_2} \cdots -\frac{a_m}{x_1-\lambda_m}$ for x_1, with similar but not necessarily the same transformations for $x_2, x_3 .. x_n$, leaving $dx_1, dx_2 .. dx_n$ unaltered; or if we employ the derived transformations of Art. 36.

I do not conceive that these extensions possess any kind of prospective value or importance, beyond what must attach to all real additions to our knowledge of the Integral Calculus. Upon such questions it is, however, almost always unsafe to specu-

late. We can do little more than address ourselves patiently to follow the tracks which open before us, without attempting to prescribe their direction or conjecture their end.

The interpretation of the formulæ which have been arrived at in this section is so far distinct from the general course and design of the paper, that I have thrown into a note such observations as I have to offer upon the subject. And I have the rather taken this course, because those observations have been to some extent anticipated in a former publication. See Note B.

I have not attempted the further extension of the theory of multiple integrals, which would seem to be involved in the general theorem of Art. 32, when other values than unity are assigned to the function $\phi(x)$. Neither have I attempted to extend that theorem to the case in which $\phi(x)$ is rational—a case evidently of some importance from its formal connexion with the last member of (5.).

Note A.

On the Connexion between the Symbol Θ *and* Cauchy's *Symbol* $\mathcal{E}$, *employed in the Calculus of Residues.*

It has been explained that the operation denoted by the symbol $\mathcal{E}$ is equivalent to that portion of the operation represented by Θ which depends upon the ascending developments of the subject function. There is, however, a difference in the mode of statement.

Thus $\mathcal{E}\frac{x^2}{[(x-a)^2(x-b)]}$ would denote, according to Cauchy's definition, the sum of the coefficients of $\frac{1}{x}$ in the developments of the respective functions $\frac{(x+a)^2}{x^2(x+a-b)}$ and $\frac{(x+b)^2}{(x+b-a)^2x}$, in ascending powers of x. This would be the same as the sum of the coefficients of $\frac{1}{x-a}$ and $\frac{1}{x-b}$ in the respective ascending developments of the primitive function $\frac{x^2}{(x-a)^2(x-b)}$, in ascending powers of $x-a$ and $x-b$ respectively. The operation Θ would add to the above the coefficient with changed sign of $\frac{1}{x}$ in the development of the same function in descending powers of x.

We shall perhaps best exemplify the connexion thus established between the two symbols, by applying the new symbol Θ to the solution of some of the problems which Cauchy has treated by the Calculus of Residues. I select for the purpose,—1st, the problem of the integration of linear differential equations with constant coefficients; 2ndly, the problem of the integration of rational fractions.

Both these applications depend on a transformation of the rational function $f(v)$.

1st. It follows from the subsidiary theorem (6.), Art. 13, that if we have

$$v-x=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

as an equation connecting x and v, then

$$\Sigma f(x)=\Theta[f(x)]\frac{1}{v-x}.$$

But since from (1.) x has only one value, viz. v, it is evident that $\Sigma f(x)=f(v)$. Hence

$$f(v)=\Theta[f(x)]\frac{1}{v-x}. \quad \ldots\ldots\ldots \quad (2.)$$

Now let $f\left(\frac{d}{d\theta}\right)u=\mathrm{U}$ represent any differential equation with constant coefficients, θ being the independent, u the dependent variable, and U a given function of θ. To secure the utmost generality, I will suppose $f\left(\frac{d}{d\theta}\right)$ a rational fraction. Thus if the equation were $\frac{d^2u}{d\theta^2}+u=\frac{d\mathrm{U}}{d\theta}$, we should have

$$f\left(\frac{d}{d\theta}\right)=\frac{\frac{d^2}{d\theta^2}+1}{\frac{d}{d\theta}}.$$

Now

$$u=\left\{f\left(\frac{d}{d\theta}\right)\right\}^{-1}\mathrm{U}=\Theta[f(x)^{-1}]\left(\frac{d}{d\theta}-x\right)^{-1}\mathrm{U}\ \text{by (2.)}=\Theta[f(x)^{-1}]\varepsilon^{x\theta}\left\{\int\varepsilon^{-x\theta}\mathrm{U}d\theta+\psi(x)\right\}, \quad (3.)$$

$\psi(x)$ being an arbitrary function of x. This is the complete solution of the differential equation proposed. The reader will have no difficulty in applying it to particular cases. The arbitrary constants have their origin in the complementary function $\psi(x)$.

In the Calculus of Residues the theorem corresponding to (2.) is the following:—

$$f(v)=\mathrm{E}^{\frac{[f(x)]}{v-x}}+\mathcal{E}\frac{f\left(\frac{1}{x}\right)}{[x](1-vx)}. \quad \ldots\ldots\ldots \quad (4.)$$

The first term in this expression for $f(v)$ is equivalent to the result of the first part of the operation Θ in the second member of (2.), viz. that which depends upon the developments which are effected in ascending powers of the simple factors of the denominator of $f(x)$. The second term is a transformation of the result of the second part of the operation Θ. It would be more conveniently expressed in the form

$$-\mathrm{C}_{\frac{1}{v}}\frac{f(x)}{v-x}.$$

The solution of the equation $f\frac{d}{d\theta}u=\mathrm{U}$, furnished by (4.), will evidently be

$$u=\mathrm{E}^{[f(x)^{-1}]}\varepsilon^{x\theta}\left\{\int\varepsilon^{-x\theta}\mathrm{U}d\theta+\psi(x)\right\}-\mathcal{E}\frac{\frac{1}{x}f\left(\frac{1}{x}\right)}{[x]}\varepsilon^{\frac{\theta}{x}}\left\{\int\varepsilon^{-\frac{\theta}{x}}\mathrm{U}d\theta+\psi(x)\right\}, \quad \ldots \quad (5.)$$

$\psi(x)$ being an arbitrary function of x.

When, as indeed is usually the case, $f\left(\frac{d}{d\theta}\right)$ is a rational and integral function of the symbol $\frac{d}{d\theta}$, the second member of (5.) reduces to its first term. The symbols Θ and $\mathcal{E}$ then become identical.

2ndly. From the equation $v-x=0$, we have, by the general theorem of transformation,

$$\Sigma f(x)dx=\Theta[f(x)]\frac{dv}{v-x}.$$

Whence, x having only one value in terms of v,

$$f(v)dv=\Theta[f(x)]\frac{dv}{v-x},$$

and integrating,

$$\int f(v)dv=\Theta[f(x)]\{\log(v-x)+\psi(x)\},$$

$\psi(x)$ being an arbitrary function of x. It is easily seen that $\Theta[f(x)]\psi(x)$ may be represented by C, whence

$$\int f(v)dv=\Theta[f(x)]\log(v-x)+C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

CAUCHY has very extensively employed the Calculus of Residues in the evaluation of definite integrals taken between the limits 0 and ∞. These applications are among the most valuable portions of his writings. They have no connexion, however, with the researches of this paper, and I have not even examined whether they would be in any degree generalized by the adoption of the symbol Θ.

NOTE B.

On the Interpretation of the Formulæ for the Evaluation of Multiple Integrals.

The three principal formulæ, (4.) art. 38, (1.) art. 41, and (5.) art. 42, evidently possess a common type. In each of them we recognize under the sign $\int$, a function $f(\sigma)$ which may be discontinuous within the limits of integration, and which is at the same time subject to an operation of general differentiation. This is a combination which is at least unusual in analysis. I purpose to consider here some of the questions of interpretation which it suggests. To some extent, indeed, these questions have been considered in my previous memoir, already referred to; but one of the most important of them, the effect of discontinuity in the function $f(\sigma)$ upon the integral in which it is involved, admits of being presented in a more satisfactory light. I do not propose to enter upon a complete investigation of the latter question, but only to examine one or two special and well-marked cases, in the hope of directing the attention of others to the subject.

When there is but one variable, and the index of differentiation is 0, the formulæ reduce in effect to ordinary integral transformations. And it is quite worthy of observation, that in this way the formula (4.), Art. 38, leads to known *modular* transformations of the elliptic functions. Thus if we make $n=1$, $h=1$, $i=\frac{1}{2}$, $a_1=0$ and drop the suffix from x_1 and l_1, we have

$$\int\frac{dxf(lx)}{(1+x^2)^{\frac{1}{2}}}=\int\frac{ds}{s}f(\sigma), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

where $\sigma=s-\frac{l^2}{4s}$ and $f(\sigma)$ vanishes when σ transcends the limits of lx. But this amounts to saying that

$$\int\frac{dxf(lx)}{(1+x^2)^{\frac{1}{2}}}=\int\frac{dsf\left(s-\frac{l^2}{4s}\right)}{s},$$

provided that

$$lx=s-\frac{l^2}{4s},$$

a result easily verified. Now let $f(lx)=\frac{1}{\sqrt{1+l^2x^2}}$, we have

$$\int\frac{dx}{\sqrt{(1+x^2)(1+l^2x^2)}}=\int_s\frac{ds}{\sqrt{1+\left(s-\frac{l^2}{4s}\right)^2}}=\int\frac{ds}{\sqrt{\frac{l^2}{16}+\left(1-\frac{l^2}{2}\right)s^2+s^4}}.$$

The second member is a function of the same kind as the first, differing only in the constants. If we assume $s=mt$, we can so determine m as to reduce the equation to the form

$$\int\frac{dx}{\sqrt{(1+x^2)(1+l^2x^2)}}=\mathrm{L}\int\frac{dt}{\sqrt{(1+t^2)(1+l'^2t^2)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

We shall find

$$\mathrm{L}=\frac{2}{1+\sqrt{1-l^2}},\quad l'=\frac{1-\sqrt{1-l^2}}{1+\sqrt{1+l^2}};$$

the relation between x and t being

$$t=l^2(x\sqrt{x^2+1}).$$

If in the above we make $x=\tan\phi$, $t=\tan\theta$, $1-l^2=k^2$, $1-l'^2=k'^2$, we find

$$\int\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}=\frac{2}{1+k}\int\frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

provided that

$$k=\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} \text{ and } \tan\theta=\sqrt{\frac{1+k}{1-k}}\,(\tan\phi+\sec\phi).$$

These are of course known relations.

In the following example, which is valuable only for the sake of the principle involved, the differential coefficient of a discontinuous function occurs under the sign of integration.

In the same equation (4.), Art. 38, let $h=0$, $i=\frac{3}{2}$, $n=1$, $l_1=1$; then dropping the suffix from the single variable retained, we have

$$\int\frac{dxf(x)}{(a-x)^{\frac{3}{2}}}=2\int_0^{\infty}ds\left(-\frac{d}{d\sigma}\right)f(\sigma), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

wherein $\sigma=a-\frac{1}{4s}$ and $f(\sigma)$ vanishes whenever σ transcends the limits of x. Suppose those limits 0 and 1, and let a be greater than 1. Now σ and s increase together, and $s=\frac{1}{4(a-\sigma)}$. Hence $\sigma=0$ gives $s=\frac{1}{4a}$, and $\sigma=1$ gives $s=\frac{1}{4(a-1)}$. Therefore

$$\int_0^1\frac{dxf(x)}{(a-x)^{\frac{3}{2}}}=-2\int_0^{\infty}ds\,\frac{d}{d\sigma}f(\sigma), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

provided that $f(\sigma)$ be regarded as a discontinuous function defined in the following manner, viz.—

$$\text{From } s=0 \text{ to } s=\frac{1}{4a} \qquad f(\sigma)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

$$\text{From } s=\frac{1}{4a} \text{ to } s=\frac{1}{4a-1} \quad f(\sigma)=f\left(a-\frac{1}{4s}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

$$\text{From } s=\frac{1}{4a-1} \text{ to } s=\infty \quad f(\sigma)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

Let us now examine the corresponding values of the element $ds\frac{d}{d\sigma}f(\sigma)$ under the sign of integration in (4.).

1st. From $s=0$ to $s=\frac{1}{4a}$ that element is 0 by (6.).

2ndly. At the point $s=\frac{1}{4a}$ we have

$$ds\frac{d}{d\sigma}f(\sigma)=\frac{ds}{d\sigma}\frac{d}{ds}f(\sigma)ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

But $\frac{d}{ds}f(\sigma)\,ds$ is the increment of $f(\sigma)$ corresponding to an infinitesimal increment ds in the value of s. At the break, where $s=\frac{1}{4a}$, $f(\sigma)$ changes in value from 0 to $f(0)$, the initial value of $f\left(a-\frac{1}{4s}\right)$, and the increment of $f(\sigma)$ is $f(0)$, *whatever the value of* ds *may be*, provided only that it is infinitesimal. Hence at this point we have

$$ds\frac{d}{d\sigma}f(\sigma)=\frac{ds}{d\sigma}f(0)=\frac{1}{4(a-\sigma)^2}f(0)=\frac{f(0)}{4a^2}.$$

3rdly. From $s=\frac{1}{4a}$ to $s=\frac{1}{4(a-1)}$, $ds\frac{d}{d\sigma}f(\sigma)=dsf'\left(a-\frac{1}{4s}\right)$.

4thly. At the second break, where $s=\frac{1}{4(a-1)}$ and $\sigma=1$, we find $ds\frac{d}{d\sigma}f(\sigma)=\frac{-f(1)}{4(a-1)^2}$.

5thly. From $s=\frac{1}{4(a-1)}$ to $s=\infty$ we have $ds\frac{d}{d\sigma}f(\sigma)=0$. Thus on recapitulation

$$\int_0^\infty ds\frac{d}{d\sigma}f(\sigma)$$

comprises two finite elements whose sum is

$$\frac{f(0)}{4a^2}-\frac{f(1)}{4(a-1)^2},$$

and a series of infinitesimal elements which give by integration

$$\int_{\frac{1}{4a}}^{\frac{1}{4(a-1)}} dsf'\left(a-\frac{1}{4s}\right),$$

whence (5.) becomes

$$\int_0^a \frac{f(x)dx}{(a-x)^2}=2\left\{\frac{f(1)}{4(a-1)^2}-\frac{f(0)}{4a^2}-\int_{\frac{1}{4a}}^{\frac{1}{4(a-1)}} dsf'\left(a-\frac{1}{4s}\right)\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

Now this result may be verified by integrating the first member of the equation by parts, and transforming the integral which remains by assuming $x=a-\frac{1}{4s}$.

I think that this example very clearly shows, that, regarding the integral

$$\int_0^\infty ds\left(\frac{d}{d\sigma}\right)f(\sigma)$$

as made up of elements some of which are finite and perfectly determinate in value, the others infinitesimal and subject to the conditions of ordinary integration, we can interpret the general theorem (4.), Art. 38, in accordance with known truth. In determining, as we have done, the value of the element $ds\left(\frac{d}{d\sigma}\right)f(\sigma)$ at each break in the discontinuous function, we attach no new signification to a differential coefficient. We regard that element as the limiting value to which $\Delta s\frac{\Delta}{\Delta\sigma}f(\sigma)$ approaches as Δs approaches to 0. In the present instance this value is finite. Usually it is 0.

The following examples, which are adopted with some improvements from my previous memoir, will illustrate the more important applications of (5.), Art. 42.

Let $n=3$, $i=\frac{1}{3}$, $h=0$, and let us substitute x, y, z for x_1, x_2, x_3, and a, b, c for a_1, a_2, a_3; then

$$\mathrm{V}=\iiint\frac{dxdydzf\left(\frac{x^2}{h_1^2}+\frac{y^2}{h_2^2}+\frac{z^2}{h_3^2}\right)}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{1}{2}}}, \qquad (11.)$$

the limits being given by the conditions

$$\frac{x^2}{h_1^2}+\frac{y^2}{h_2^2}+\frac{z^2}{h_3^2}\leqq 1.$$

The value of V becomes

$$\mathrm{V}=-h_1h_2h_3\pi\int_0^\infty\frac{ds.s^{-\frac{1}{2}}\left(\frac{d}{d\sigma}\right)^{-1}f(\sigma)}{(1+h_1^2s)^{\frac{1}{2}}(1+h_2^2s)^{\frac{1}{2}}(1+h_3^2s)^{\frac{1}{2}}}, \qquad (12.)$$

wherein

$$\sigma=\frac{a^2s}{1+h_1^2s}+\frac{b^2s}{1+h_2^2s}+\frac{c^2s}{1+h_3^2s}. \qquad (13.)$$

Now the attraction, according to the law of nature, of the ellipsoid whose equation is

$$\frac{x^2}{h_1^2}+\frac{y^2}{h_2^2}+\frac{z^2}{h_3^2}=1,$$

and whose internal density is expressed by the function

$$\varrho f\left(\frac{x^2}{h_1^2}+\frac{y^2}{h_2^2}+\frac{z^2}{h_3^2}\right)$$

upon the point (a, b, c), will be $-\varrho\frac{d\mathrm{V}}{da}$. Observing that, in the value of V given in (12.), a only appears as involved in σ, we have

$$\frac{d}{da}=\frac{d\sigma}{da}\frac{d}{d\sigma} \text{ and } \frac{d}{da}\left(\frac{d}{d\sigma}\right)^{-1}f(\sigma)=\frac{d\sigma}{da}f(\sigma)=\frac{2as}{1+h_1^2s}f(\sigma),$$

whence

$$-\varrho\frac{d\mathrm{V}}{da}=h_1h_2h_3\pi\varrho\int_0^\infty\frac{2as^{\frac{1}{2}}f(\sigma)ds}{(1+h_1^2s)(1+h_1^2s)^{\frac{1}{2}}(1+h_2^2s)^{\frac{1}{2}}(1+h_3^2s)^{\frac{1}{2}}}$$

$$=h_1h_2h_3\pi\varrho\int_0^\infty\frac{2as}{1+h_1^2s}.\frac{s^{-\frac{1}{2}}f(\sigma)ds}{(1+h_1^2s)^{\frac{1}{2}}(1+h_2^2s)^{\frac{1}{2}}(1+h_3^2s)^{\frac{1}{2}}}$$

Now $f(\sigma)$ is to vanish when σ falls without the limits 0 and 1, and s as falling between the limits 0 and ∞ is to be positive. But from the expression for σ, it appears that when $s=0$ $\sigma=0$, and when $s=\infty$ $\sigma=\frac{a^2}{h_1^2}+\frac{b^2}{h_2^2}+\frac{c^2}{h_3^2}$; also σ increases with s (Art. 42.), and therefore passes over the value 1 when $\frac{a^2}{h_1^2}+\frac{b^2}{h_2^2}+\frac{c^2}{h_3^2}>1$, but reaches not that value otherwise.

The former condition is realized when the attracted point is external. Representing by η the value of s for which $\sigma=1$, we have

$$-\varrho\frac{d\mathrm{V}}{da}=2h_1h_2h_3\pi\varrho a\int_0^{\eta}\frac{ds.s^{\frac{1}{2}}f\left(\frac{a^2s}{1+h_1^2s}+\frac{b^2s}{1+h_2^2s}+\frac{c^2s}{1+h_3^2s}\right)}{(1+h_1^2s)^{\frac{1}{2}}(1+h_2^2s)^{\frac{1}{2}}(1+h_3^2s)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

η being the positive root of the equation

$$\frac{a^2s}{1+h_1^2s}+\frac{b^2s}{1+h_2^2s}+\frac{c^2s}{1+h_3^2s}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.)$$

When the attracted point is internal we have only to substitute ∞ for η in the upper limit of integration; for all positive values of σ then satisfy the condition $\sigma<1$, and positive values only are admissible.

Both these cases may also be derived from the more general theorem deducible from (5.), Art. 42,

$$-\varrho\frac{d}{da}\iiint\frac{dxdydz}{\{h^2+(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{1}{2}}}=2h_1h_2h_3\pi\varrho a\int_0^{\eta}\frac{ds.s^{-\frac{1}{2}}f\left(h^2s+\frac{a^2s}{1+h_1^2s}+\frac{b^2s}{1+h_2^2s}+\frac{c^2s}{1+h_3^2s}\right)}{(1+h_1^2s)^{\frac{1}{2}}(1+h_2^2s)^{\frac{1}{2}}(1+h_3^2s)^{\frac{1}{2}}}, \quad (16.)$$

where η is the positive root of the equation

$$h^2s+\frac{a^2s}{1+h_1^2s}+\frac{b^2s}{1+h_2^2s}+\frac{c^2s}{1+h_3^2s}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

When h approaches to 0 this root approaches to the positive root of the equation (15.) if $\frac{a^2}{h_1^2}+\frac{b^2}{h_2^2}+\frac{c^2}{h_3^2}$ is greater than 1, but tends to ∞ if $\frac{a^2}{h_1^2}+\frac{b^2}{h_2^2}+\frac{c^2}{h_3^2}$ is equal to or less than 1.

If $f(\sigma)=1$, the expressions are easily reducible to elliptic functions and agree with known results.

Lastly, let the law of force be that of the inverse fourth power of the distance and equal to ϱ. The expression for the attraction on an external point (a, b, c) is

$$-\frac{\varrho}{3}\frac{d\mathrm{V}}{da},$$

where

$$\mathrm{V}=\iiint\frac{dxdydz}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{1}{2}}}=2\pi h_1h_2h_3\int_0^{\infty}\frac{ds.s^{\frac{1}{2}}f(\sigma)}{(1+h_1^2s)^{\frac{1}{2}}(1+h_2^2s)^{\frac{1}{2}}(1+h_3^2s)^{\frac{1}{2}}},$$

therefore

$$-\frac{\varrho}{3}\frac{df(\sigma)}{da}=-\frac{\varrho}{3}\frac{d(\sigma)}{da}\frac{df(\sigma)}{d(\sigma)}=-\frac{2\varrho as}{3(1+h_1^2s)}.\frac{df(\sigma)}{d(\sigma)}=-\frac{2\varrho as}{3(1+h_1^2s)}.\frac{ds}{d(\sigma)}.\frac{df(\sigma)}{ds}.$$

Therefore

$$-\frac{s}{3}\frac{dV}{da}=-\frac{4}{3}\pi\rho a h_1 h_2 h_3\int_0^\infty \frac{s^{\frac{1}{2}}\frac{da}{d\sigma}\cdot\frac{df(\sigma)}{ds}ds}{(1+h_1^2 s)^{\frac{1}{2}}(1+h_2^2 s)^{\frac{1}{2}}(1+h_3^2 s)^{\frac{1}{2}}}. \quad \ldots \quad (18.)$$

Let η be the positive root of the equation

$$\frac{a^2 s}{1+h_1^2 s}+\frac{b^2 s}{1+h_2^2 s}+\frac{c^2 s}{1+h_3^2 s}=1,$$

and let the density be uniform; then $f(\sigma)=1$ or 0, according as s is less or greater than η. Before and after the break, therefore, $\frac{df(\sigma)}{ds}=0$. At the break we have, by the reasoning of a previous section, $\frac{df(\sigma)}{ds}ds=-1$. We must therefore substitute this value in (18.), and in the rest of the expression under the integral sign change s into η. Observing that this substitution converts $\frac{d\sigma}{ds}$ into $\frac{a^2}{(1+h_1^2 s)^2}+\frac{b^2}{(1+h_2^2 s)^2}+\frac{c^2}{(1+h_3^2 s)^2}$, and that the integral being reduced to a single finite element we may reject the integral sign, we have

$$-\frac{\rho}{3}\cdot\frac{dV}{da}=\frac{\frac{4}{3}\pi\rho a h_1 h_2 h_3\eta^{\frac{3}{2}}}{\left\{\frac{a^2}{(1+h_1^2\eta)^2}+\frac{b^2}{(1+h_2^2\eta)^2}+\frac{c^2}{(1+h_3^2\eta)^2}\right\}(1+h_1^2\eta)^{\frac{1}{2}}(1+h_2^2\eta)^{\frac{1}{2}}(1+h_3^2\eta)^{\frac{1}{2}}}.$$

This result is due, I believe, to Mr. CAYLEY, but was originally obtained by an entirely different analysis.

It only remains to add, that when the index of differentiation is fractional we must revert to the first expression in (5.), Art. 42, and effect the integrations separately. The integration with respect to v may always be performed. The possibility of the two others will depend upon the nature of the problem under consideration. Thus writing the expression in the form

$$V=\frac{\pi^{\frac{n}{2}-1}h_1 h_2 \ldots h_n}{\Gamma(i)}\int_0^1\int_0^\infty \frac{da.ds.s^{i-1}f(a)}{(1+h_1^2 s)^{\frac{1}{2}}\ldots(1+h_n^2 s)^{\frac{1}{2}}}\int_0^\infty dv.v^{i-\frac{n}{2}}\cos\left\{(a-\sigma)v+\left(i-\frac{n}{2}\right)\frac{\pi}{2}\right\},$$

it is easily shown that, according as a is greater or less than σ,

$$\int_0^\infty dv.v^{i-\frac{n}{2}}\cos\left\{(a-\sigma)v+\left(i-\frac{n}{2}\right)\frac{\pi}{2}\right\}=\frac{\Gamma\left(i-\frac{n}{2}+1\right)\sin\left(i-\frac{n}{2}+1\right)\pi}{(a-\sigma)^{i-\frac{n}{2}+1}} \text{ or } 0,$$

$$=\frac{\pi}{\Gamma\left(\frac{n}{2}-1\right)(a-\sigma)^{i-\frac{n}{2}+1}} \text{ or } 0,$$

by a known property of the function Γ. The latter form gives

$$V=\frac{\pi^{\frac{n}{2}}h_1 h_2 \ldots h_n}{\Gamma(i)\Gamma\left(\frac{n}{2}-i\right)}\int_0^1\int_0^\infty \frac{da.ds.s^{i-1}f(a)}{(1+h_1^2 s)^{\frac{1}{2}}(1+h_2^2 s)^{\frac{1}{2}}\ldots(1+h_n^2 s)^{\frac{1}{2}}(a-\sigma)^{i-\frac{n}{2}+1}}.$$

This transformation, or one in effect equivalent to it, is due to Mr. CAYLEY*, who has applied it to obtain the value of a remarkable definite integral which occurs in the mathematical theory of electricity.

* Cambridge and Dublin Mathematical Journal, vol. ii. p. 219.

XXXVII. *Experimental Researches on the Conductive Powers of various Substances, with the application of the Results to the Problem of Terrestrial Temperature.*

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In presenting this memoir to the Society, I feel myself called upon to acknowledge the liberal aid which I have received from the Committee for the disposal of the Annual Government Grant for the Advancement of Science, in the prosecution of the experimental researches in which I have been engaged for a considerable time. The present paper contains an account of a part only of my experiments, with the application of the results of them to the problem of terrestrial temperature. An account of the remaining experiments will be reserved for a future paper. I am likewise bound to express in the strongest terms my obligations to my friends Mr. Fairbairn and Mr. Joule. Without the aid of the former of these gentlemen I should have been unable even to commence the series of experiments which I have now nearly concluded; and among the many ways in which this assistance has been so promptly rendered, I may mention his having constantly placed at my disposal the invaluable services of one of his principal workmen, William Ward, without whose untiring activity and mechanical resources, I should have utterly despaired of bringing my experiments to any successful issue. The value of Mr. Joule's assistance, especially in the commencement of these experiments, will be understood by those who are acquainted with his sound philosophical knowledge and experimental skill. More specific acknowledgement of his services will be called for in my next communication. I now proceed to the subject of this paper.

§ I. *General Experimental Results respecting the Conductive Power of various Substances, and the circumstances by which it is affected.*

1. That I may be able to state the more clearly the nature of the quantity which measures the conductivity of any proposed substance with reference to heat, and the experimental method I have adopted for the determination of its value, it will be necessary to recall the solution of the problem, under one of its simplest aspects, the object of which is to ascertain the manner in which heat is transmitted by conduction through a body under certain conditions. Conceive the body to be bounded by two parallel plane surfaces of indefinite extent, the distance between them being h. Suppose one of these bounding surfaces (which, for convenience, may be termed the *lower* one) to be kept at a uniform and constant temperature t_1, while the temperature of the free space into which the heat radiates from the *upper* surface, is constant and $=\tau$. Also let ζ

denote the temperature at the distance z from the lower surface, when the temperature has become *steady*. The differential equation for the determination of ζ will then be

$$\frac{d}{dz}\left(k\frac{d\zeta}{dz}\right)=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$\therefore \quad k\frac{d\zeta}{dz}=\mathrm{C}, \text{ a constant.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

The expression $-k\frac{d\zeta}{dz}$ measures the quantity of heat which passes through a unit of area parallel to either bounding surface, in a unit of time. We may conceive k to be a function of ζ or z, but it is here considered constant. It is the quantity which is always taken to measure the *conductivity*, or *conductive power*, of the mass through which the heat is transmitted, and which can only be determined for different substances by experiment.

Again, let t_2 be the temperature of the upper bounding surface; then will the quantity of heat which radiates from a unit of area of that surface in a unit of time be

$$p(t_2-\tau),$$

p being constant, and independent of the temperatures t_2 and τ, at least for considerable ranges of those temperatures. It measures the *radiating power* of the upper surface of the mass*. Now since the same quantity of heat must pass through a unit of the upper surface as through any unit of area parallel to that surface in the interior of the mass, equation (2.) will become

$$-k\frac{d\zeta}{dz}=p(t_2-\tau). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Integrating again

$$\mathrm{C}-k\zeta=p(t_2-\tau)z;$$

and since $\zeta=t_1$ when $z=0$,

$$k(t_1-\zeta)=p(t_2-\tau)z; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

and since $\zeta=t_2$ when $z=h$,

$$\frac{k}{p}=\frac{t_2-\tau}{t_1-t_2}h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

If h be known and the temperatures t_1, t_2 and τ be observed, this equation will determine $\frac{k}{p}$, the ratio of the conductivity of the substance to the radiating power of its surface, which is very different for different substances. If, however, the upper surface be covered by a thin stratum of any other matter which will assume the temperature of the upper surface of the transmitting mass, p will then be the radiating power of this super-

* I have adopted the approximate law of radiation, as much more simple and convenient than the more exact law of Dulong and Petit, and sufficiently accurate for my purpose. All the experiments described in this paper aim only at *comparative* results, and most of them have been made under nearly the same thermal conditions. Those requiring any considerable accuracy have been made in the form of *differential* experiments. Hence the use of the approximate law of radiation can lead to no error of any importance in the experimental results.

imposed matter. I made use of mercury for this purpose in my own experiments; and thus, if c denote the radiating power of mercury, we have

$$\frac{k}{c}=\frac{t_2-\tau}{t_1-t_2}h; \quad \ldots \ldots \ldots \ldots \quad (6.)$$

and c being the same, whatever may be the substance experimented on, this formula enables us to compare the conductive powers of different substances, or to determine the absolute numerical values of those powers, when c the radiating power of mercury is known. The determination of these comparative values of k, and not those of $\frac{k}{p}$ (which have more usually been determined), has formed the object of these experimental investigations.

My experiments have been made on a great variety of mineral substances, and on some others also: I shall reserve certain details respecting them for a subsequent part of this paper, giving here the general results at which I have arrived. It should be observed, that k being a linear quantity, like h $\left(\text{as appears from the expression for } \frac{k}{c}\right)$, its numerical value will depend on the unit of length. This unit has been assumed to be *one foot**.

2. Experiments were made for the purpose of ascertaining the conductive powers of calcareous, argillaceous, and siliceous masses in a state of dry powder. The first was obtained from a piece of pure chalk rock, the second from a piece of clay which appeared to have very little admixture of other elements, and the third was obtained from a piece of New Red Sandstone. All were thoroughly dried. The results were as follows:—

	Values of $\frac{k}{c}$
Calcareous powder	·056
Argillaceous powder	·07
Siliceous powder	·15
A mixture of the two last in equal quantities	·11

3. The following results were obtained for different kinds of rocks:—

Calcareous rocks.

Chalk, as it exists in the general mass of chalk, but well dried	·17
Clunch, from the lower portion of the chalk formation (very moist)	·30
Oolites from three different beds in the Ancaster quarries	·38 ·37 ·37
Statuary marble	·53
Very hard blue mountain limestone from Derbyshire	·55

* The temperatures are expressed in degrees of FAHRENHEIT's thermometer.

	Values of $\frac{k}{c}$.
Argillaceous substances.	
Dry clay	·27
Very dry	·23
Moist	·37
Siliceous rocks.	
New Red Sandstone (dry)	·25
New Red Sandstone (saturated with water)	·60
Sandstone for building (freestone)	·33
Sandstone for building	·43
Millstone-grit (partially decomposed)	·376
Millstone-grit	·58
Millstone-grit from the coal-shaft at Duckenfield, at depth of 120 feet	·51
Millstone-grit from deeper beds	·65
Millstone-grit from the depth of 1300 feet, and very hard	·726
Millstone-grit from Chapel-le-Frith, used for paving-stone at Manchester, very hard	·75
Millstone-grit from Chapel-le-Frith, used for paving-stone at Manchester, very hard	·76
Old Sedimentary rocks.	
Blue, hard, close-grained rock from Penmaenmaur, used for paving-stone	·5
A similar specimen	·6
Blue, hard, compact slate from Charnwood Forest	·61
Igneous rocks.	
Granite	·53
Scotch granite, used for paving at Manchester	·55
A hard compact rock from North Wales (called *Welsh granite*)	·60
Scotch granite, rather large-grained	·75
Basalt from near Loch Katrine	·53
Syenite from Charnwood Forest	·85
A very hard, close-grained rock from Charnwood Forest	·99
Igneous rock from Loch Katrine	1·0
Basalt from the same locality	·59
Mountsorrel granite	·8

A great number of experiments were also made to determine the influence of pressure, discontinuity, temperature and moisture on the conductive powers of various substances. I proceed to state the general results of them.

4. *Influence of Pressure.*

(1) A block of spermaceti, solidified under a pressure of 850 lbs. per square inch, gave

$$\frac{k}{c} = \cdot 086.$$

Another block, solidified under a pressure of 7500 lbs. per square inch, gave so nearly the same value as only to admit the conclusion that pressure had no sensible influence on the conductive power of this substance.

Instead of taking blocks solidified under the above pressures, I took another compressed, *after solidification*, with a weight of 7500 lbs. per square inch. The result was identical with that obtained from the block solidified under the same pressure.

(2) Wax.—This substance, when uncompressed, gave

$$\frac{k}{c} = \cdot 072,$$

and when compressed by a pressure of 7500 lbs. per square inch,

$$\frac{k}{c} = \cdot 079.$$

The increase is too small to exceed the probable limits of error.

(3) Clay.—The results in this case were as follows:—

Uncompressed clay	$\frac{k}{c} = \cdot 26$
Compressed with a weight of 4300 lbs. per square inch . . .	$\frac{k}{c} = \cdot 30$
Compressed with a weight of 7500 lbs. per square inch . . .	$\frac{k}{c} = \cdot 33$

Here we have a considerable increase due to pressure.

(4) Chalk.—No appreciable increase of conductivity was observed in this case with pressures equal to those above cited.

(5) Mixture of sand and clay:—

Compressed with a weight of 4300 lbs. per square inch . . .	$\frac{k}{c} = \cdot 36$
Compressed with a weight of 7500 lbs. per square inch . . .	$\frac{k}{c} = \cdot 378$

5. *Influence of Temperature.*

The experiments for determining the influence of temperature were confined to two substances which melt at low temperatures, bees'-wax and spermaceti. Denoting the temperatures of the lower and upper surfaces of the blocks respectively by t_1 and t_2, I obtained the following results:—

$$\left.\begin{aligned} t_1 &= 105^\circ \cdot 3 \\ t_2 &= 85^\circ \cdot 6 \end{aligned}\right\}, \quad \frac{k}{c} = \cdot 088$$

$$\left.\begin{matrix} t_1 = 115°{\cdot}5 \\ t_2 = 91°{\cdot}5 \end{matrix}\right\}, \frac{k}{c} = {\cdot}09$$

$$\left.\begin{matrix} t_1 = 126°{\cdot}5 \\ t_2 = 95°{\cdot}0 \end{matrix}\right\}, \frac{k}{c} = {\cdot}077$$

$$\left.\begin{matrix} t_1 = 139° \\ t_2 = 100° \end{matrix}\right\}, \frac{k}{c} = {\cdot}074.$$

The temperature of complete melting of the wax was about 140°, so that in the last experiments the lower part of the block was very nearly in a melting state, and the molecular condition of a considerable portion of it was doubtless affected. This change is indicated by a decrease of conductivity. Exactly similar results were obtained for spermaceti. So long as the temperature was insufficient to produce any apparent effect on the constitution of these substances, my experiments did not detect any sensible effect on their conductivity. Biot remarked the diminution of conductivity of a bar of fusible alloy when one extremity was maintained very nearly at its fusing temperature.

With respect to mineral substances of very high temperatures of fusion, there can be little doubt, I conceive, that their conductivity remains unaffected by any temperature, for instance, not exceeding that of boiling mercury. Some experiments were made, partly with the view of corroborating this conclusion, but more especially for testing, in the case of mercury, the approximate truth of the law which asserts that the intensity of radiation from a given surface is proportional to the excess of the temperature of that surface above that of the surrounding space. The following results were obtained by experimenting at different temperatures with the same block of sandstone; t_1 and t_2 denote the same temperatures as above, and τ the temperature of the surrounding atmosphere:—

(1) $\left.\begin{matrix} t_1 = 144°{\cdot}5 \\ t_2 = 123°{\cdot}2 \end{matrix}\right\}, \tau = 66°, \quad \frac{k}{c} = {\cdot}45.$

(2) $\left.\begin{matrix} t_1 = 241°{\cdot}5 \\ t_2 = 193°{\cdot}0 \end{matrix}\right\}, \tau = 69°, \quad \frac{k}{c} = {\cdot}435.$

(3) $\left.\begin{matrix} t_1 = 334°{\cdot}0 \\ t_2 = 260°{\cdot}5 \end{matrix}\right\}, \tau = 78°, \quad \frac{k}{c} = {\cdot}425.$

(4) $\left.\begin{matrix} t_1 = 447°{\cdot}8 \\ t_2 = 326°{\cdot}0 \end{matrix}\right\}, \tau = 81°, \quad \frac{k}{c} = {\cdot}34.$

(5) $\left.\begin{matrix} t_1 = 522°{\cdot}0 \\ t_2 = 371°{\cdot}0 \end{matrix}\right\}, \tau = 85°{\cdot}5, \frac{k}{c} = {\cdot}33.$

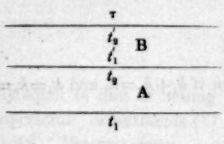
Hence if we assume k to be constant for these different temperatures, c the radiating power of mercury must increase with the increase of the difference $t_2 - \tau$. The increase however is not considerable while $t_2 - \tau$ varies from 50° to 120° or 130°, as in the two first of the above experiments. Within those limits the law of radiation, as usually assumed, will be approximately true. It was desirable to corroborate this conclusion

with reference to mercury, on account of its being the substance from which the radiation took place in all my experiments. The increase of radiation indicated by the above experiments is in general accordance with the law of DULONG and PETIT, but it is smaller in amount.

6. *Effect of Discontinuity.*

In all the preceding cases the substances experimented on were in unbroken continuous small masses. It is important to inquire how far their conductivity is affected by a breach of this perfect continuity. Several experiments were made for this purpose; but before I proceed to state these experimental results, it may be better (as in art. 1) to deduce the theoretical results with which they must be compared for the purpose of determining the additional coefficient, or constant, which must be introduced on the hypothesis that the discontinuity in the conducting substance produces a discontinuity in the law according to which the temperature decreases as the heat passes from one bounding surface to the other. These surfaces being supposed parallel and of indefinite extent, as before, for a mass of one substance (A), conceive another mass (B) of different conductivity, and bounded also by parallel plane surfaces, to be placed upon the former. Let t_1 denote the constant temperature at which the lower surface of A (fig. 1) is maintained; t_2 that of its upper surface; t'_1 the temperature of the lower surface of B in contact with the upper surface of A; t'_2 that of the upper surface of B; and τ that of surrounding space. I shall assume that the quantity of heat which flows through a unit of area of the surfaces of junction, in a unit of time,

Fig. 1.



$$=q(t_2-t'_1),$$

where q is independent of t_2 and t'_1. Then, since the same quantity of heat when the temperature is steady, must pass through each parallel surface, we must have

$$q(t_2-t'_1)=p(t'_2-\tau), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

p being the radiating power of the upper surface of B. We have also (art. 1.)

$$-k_1\frac{d\zeta}{dz}=p(t'_2-\tau),$$

and therefore

$$C-k_1\zeta=p(t'_2-\tau)z,$$

$$C-k_1t_1=0,$$

$$\therefore \quad k_1(t_1-\zeta)=p(t'_2-\tau)z,$$

and

$$k_1(t_1-t_2)=p(t'_2-\tau)h_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

where h_1=thickness of A.

In like manner we obtain

$$k_2(t'_1-t'_2)=p(t'_2-\tau)h_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

where h_2=thickness of B.

Equations (1.), (2.) and (3.) will determine t'_2, t_1 and t'_3, in terms of t_1 and τ, and the other quantities involved. They may be written

$$\left.\begin{aligned} t_1-t_2&=\frac{p}{k_1}(t'_2-\tau)h_1,\\ t_2-t'_1&=\frac{p}{q}(t'_2-\tau),\\ t'_1-t'_3&=\frac{p}{k_2}(t'_2-\tau)h_2. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (\alpha.)$$

Adding them together, we have

$$t_1-t'_3=\left(\frac{ph_1}{k_1}+\frac{ph_2}{k_2}+\frac{p}{q}\right)(t'_2-\tau),$$

or writing $t_1-\tau-(t'_2-\tau)$ for $t_1-t'_2$,

$$t_1-\tau=\left(\frac{ph_1}{k_1}+\frac{ph_2}{k_2}+\frac{p}{q}+1\right)(t'_2-\tau);$$

$$\therefore \quad \frac{ph_1}{k_1}+\frac{ph_2}{k_2}+\frac{p}{q}=\frac{t_1-\tau}{t'_2-\tau}-1$$

$$=\frac{t_1-t'_2}{t'_2-\tau},$$

$$\therefore \quad \frac{p}{q}=\frac{t_1-t'_2}{t'_2-\tau}-p\left(\frac{h_1}{k_1}+\frac{h_2}{k_2}\right),$$

or if $h_1+h_2=h$, and $k_1=k_2=k$,

$$\frac{p}{q}=\frac{t_1-t'_2}{t'_2-\tau}-\frac{ph}{k}. \quad \ldots\ldots\ldots\ldots \quad (4.)$$

If (t_1), (t'_2) and (τ) denote the values which t'_1, t'_2 and τ would have in an experiment with an ***undivided*** block of equal length and of the same substance, we shall have

$$\frac{ph}{k}=\frac{(t_1)-(t_2)}{(t_2)-\tau} \quad \text{(art. 1, equation (5.))},$$

and therefore

$$\frac{p}{q}=\frac{t_1-t'_2}{t'_2-\tau}-\frac{(t_1)-(t_2)}{(t_2)-(\tau)}\ldots\ldots \quad \ldots\ldots\ldots\ldots \quad (5.)$$

7. To apply this formula to the experimental determination of $\frac{p}{q}$, I took two equal blocks of the same piece of sandstone. One of these was split transversely, and the experiments conducted as hereafter described (see page 845). The undivided block gave

$$\frac{(t_1)-(t_2)}{(t_2)-(\tau)}=\cdot 315.$$

(1) In the first experiment with the divided block, one portion of it was placed in close contact with the other, without cement of any kind between them. This gave

$$\frac{t_1-t'_2}{t'_2-\tau}=\cdot 44,$$

and therefore $$\frac{p}{q}=\cdot125.$$

(2) In the next experiment, the two portions of the divided block were cemented together with plaster of Paris, applied moist and left to dry and harden, and forming a layer of about $\frac{1}{10}$th of an inch in thickness. I thus obtained

$$\frac{t_1-t'_2}{t'_2-\tau}=\cdot365,$$

and

$$\frac{p}{q}=\cdot05.$$

(3) Fine dry powdered clay was then put between the two portions of the divided block. The result was

$$\frac{t_1-t'_2}{t'_2-\tau}=\cdot415,$$

and

$$\frac{p}{q}=\cdot1.$$

(4) Moist clay was then used to cement the two portions of the block, which were pressed together for twelve hours. In this case we had

$$\frac{t_1-t'_2}{t'_2-\tau}=\cdot384,$$

and

$$\frac{p}{q}=\cdot069.$$

These experiments were made with a temperature t_1 nearly equal to that of boiling water. To ascertain whether q was independent of the temperature, experiments (3) and (4) were repeated at lower temperatures, t_1 being about 150°. The first gave

$$\frac{p}{q}=\cdot085,$$

and the second,

$$\frac{p}{q}=\cdot058.$$

These values are respectively about one-sixth less than those obtained in experiments (3) and (4). They may be considered as corroborating the approximate truth of the assumed law that the coefficient q is independent of the temperature, at least when the difference of temperature is not too great.

Hence, then, the values of $\frac{q}{p}$, in the four cases above stated, are

(1) $\frac{q_1}{p}=8,$

(2) $\frac{q_2}{p}=20,$

(3) $\frac{q_3}{p}=10,$

(4) $\frac{q_4}{p}=14.$

The cases (2) and (4) are those in which the junction was doubtless rendered most complete, by means of plaster of Paris in the first case (applied in a moist state and allowed to dry), and in the latter case, by means of moistened clay, applied under considerable pressure. These cases, it will be observed, correspond to the largest values of q, and, therefore, to the most rapid transmission of heat through the surface of junction for any given difference of temperature on each side of it.

8. The influence of this kind of discontinuity in the conducting mass is manifestly of great importance in the conduction of terrestrial heat from the interior of the earth to its surface through a series of different strata. If we have several superimposed strata, each having the same conductivity, but separated by planes of discontinuity, as in the preceding experiments, it may be convenient to know what must be the conductive power of a *continuous* mass of equal thickness, in order that the temperatures of the upper surfaces of the continuous and discontinuous masses may be the same, the temperatures of their lower surfaces being equal, as well as the radiating powers of their upper surfaces. The quantities of heat transmitted in the two cases will then be the same.

Let us suppose the conducting mass to have the same conductivity throughout, but to be divided into $\overline{n+1}$ portions by n planes of discontinuity, h_1, h_2, ... h_n being their respective thicknesses. Let us also suppose the coefficient q to be the same for each discontinuity. Instead of the equations (α) (art. 6), we shall have the following system:—

$$\left.\begin{aligned}
t_1 - t_2 &= \tfrac{p}{k}(t_2^{(n)} - \tau)h, \\
t_2 - t_1' &= \tfrac{p}{q}(t_2^{(n)} - \tau), \\
t_1' - t_2' &= \tfrac{p}{k}(t_2^{(n)} - \tau)h_1, \\
t_2' - t_1'' &= \tfrac{p}{q}(t_2^{(n)} - \tau), \\
\&c. &= \quad \&c. \\
t_1^{(n-1)} - t_2^{(n-1)} &= \tfrac{p}{k}(t_2^{(n)} - \tau)h_{n-1}, \\
t_2^{(n-1)} - t_1^{(n)} &= \tfrac{p}{q}(t_2^{(n)} - \tau), \\
t_2^{(n)} - t_2^{(n)} &= \tfrac{p}{k}(t_2^{(n)} - \tau)h_n;
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (\beta)$$

and adding,

$$t_1 - t_2^{(n)} = \tfrac{p}{k}(t_2^{(n)} - \tau)(h + h_1 + h_2 + \ldots + h_n) + n\tfrac{p}{q}(t_2^{(n)} - \tau),$$

$$t_1 - t_2^{(n)} = \tfrac{p}{k}\left(h + h_1 + h_2 + \ldots + n\tfrac{p}{q}\right)(t_2^{(n)} - \tau),$$

$$= \left(\tfrac{p}{k}\mathrm{H} + n\tfrac{p}{q}\right)(t_2^{(n)} - \tau),$$

when $H=h+h_1+h_2+..+h_n$; and writing $t_1-\tau-(t_2^{(n)}-\tau)$ for $t_1-t_2^{(n)}$, we have

$$\frac{p}{k}H+n\frac{p}{q}=\frac{t_1-\tau}{t_2^{(n)}-\tau}-1$$

$$=\frac{t_1-t_2^{(n)}}{t_2^{(n)}-\tau}.$$

Let k' be the required conductivity of a mass whose thickness is H, and of which the extreme temperatures are t_1 and $t_2^{(n)}$ as in the divided mass. Then, since the radiating power (p) is also the same,

$$\frac{t_1-t_2^{(n)}}{t_2^{(n)}-\tau}=\frac{p}{k'}\cdot H, \text{ (art. 1),}$$

and therefore

$$\frac{p}{k}H+n\frac{p}{q}=\frac{p}{k'}\cdot H,$$

$$\therefore \quad p\left(\frac{1}{k'}-\frac{1}{k}\right)=n\frac{p}{q}\cdot\frac{1}{H},$$

$$\frac{k-k'}{k'}=n\frac{p}{q}\cdot\frac{k}{p}\cdot\frac{1}{H},$$

and therefore,

$$\frac{k'}{k}=\frac{1}{1+n\frac{p}{q}\cdot\frac{k}{pH}} \qquad \text{(6.)}$$

As an example of this formula, let us take $\frac{p}{q}=\frac{1}{10}$ as determined in case (3.) of art. 7; and $\frac{k}{p}=\cdot 52$ (its value for the sandstone used in the experiments), the unit of length being one foot. Then

$$\frac{k'}{k}=\frac{1}{1+\frac{n}{20H}}.$$

Suppose H=100 feet, and that there are 100 discontinuities,

$$\frac{k'}{k}=\frac{1}{1+\frac{1}{20}}=1-\frac{1}{20} \text{ nearly;}$$

so that with a discontinuity on the average for every foot, the effect would only be equivalent to a diminution of $\frac{1}{20}$th of the conductive power; and a discontinuity every 6 inches would be equivalent to a diminution of about $\frac{1}{10}$th of that power, in the particular case now selected for illustration.

9. In the practical application of these researches to the case of the earth's crust, (equation 6.) (art. 8) is so important that it may be worth while to verify it by a somewhat different process. It appears by equation (4.) that the temperature t_2 is entirely independent of h_1 or h_n, the distances of the plane of discontinuity from the terminal planes, since the equation only involves the sum of h_1 and h_n, or h. We may therefore conceive the plane of discontinuity indefinitely near to the terminal plane of which the temperature is t_1. In like manner we may suppose all the planes of discontinuity, in the

general case, placed indefinitely near to the same terminal plane. But we shall then have, in the limit,

$$\begin{aligned} t_2 &= t_1 \\ t'_2 &= t'_1 \\ &\&c. \\ t_2^{(n)} &= t_1^{(n)}; \end{aligned}$$

and consequently the alternate equations of the system of equations (β.) (art. 8) will become

$$\begin{aligned} t_1 - t'_1 &= \frac{p}{q}(t_2^{(n)} - \tau), \\ t'_1 - t''_1 &= \frac{p}{q}(t_2^{(n)} - \tau), \\ \&c. \qquad &= \&c. \\ t_1^{(n-1)} - t_1^{(n)} &= \frac{p}{q}(t_2^{(n)} - \tau). \end{aligned}$$

Adding, we have $$t_1 - t_1^{(n)} = n\frac{p}{q}(t_2^{(n)} - \tau),$$

or $$t_1 - t_2^{(n)} - (t_1^{(n)} - t_2^{(n)}) = n\frac{p}{q}(t_2^{(n)} - \tau);$$

$$\therefore \quad \frac{t_1 - t_2^{(n)}}{t_2^{(n)} - \tau} = \frac{t_1^{(n)} - t_2^{(n)}}{t_2^{(n)} - \tau} = n\frac{p}{q}.$$

The conductivity of the actual mass being k, and the thickness H, we have

$$\frac{p\mathrm{H}}{k} = \frac{t_1^{(n)} - t_2^{(n)}}{t_2^{(n)} - \tau} \text{ (art. 1)};$$

and k' being the conductivity of a mass of the same thickness, and of which the terminal temperatures are, by hypothesis, t_1 and $t_2^{(n)}$, we have

$$\frac{p\mathrm{H}}{k'} = \frac{t_1 - t_2^{(n)}}{t_2^{(n)} - \tau}.$$

Hence $$p\left(\frac{1}{k'} - \frac{1}{k}\right) = n\frac{p}{q}\cdot\frac{1}{\mathrm{H}},$$

and $$\frac{k'}{k} = \frac{1}{1 + n\frac{p}{q}\cdot\frac{k}{p\mathrm{H}}},$$

as before.

10. *Effect of Moisture.*

The following experiments were made to ascertain the influence of *moisture* on the conductive powers of rocks.

(1) *Calcareous Rocks.*—I took a block of chalk from the lower part of the Chalk formation near Cambridge. It is provincially called *clunch*, and is frequently used as building-stone, where it can be well defended against the disintegrating influences of the

atmosphere. It is more compact than ordinary chalk. The specimen used in the experiment was very moist. It gave

$$\frac{k}{c}=\cdot 3;$$

and when thoroughly saturated with moisture, it gave again

$$\frac{k}{c}=\cdot 3.$$

The same block, when thoroughly dried, gave

$$\frac{k}{c}=\cdot 19.$$

In the first case, the block weighed

6808·5 grains

immediately after the experiment. In the second case, when perfectly saturated, the weight was 7602 grains before the experiment, and 7503 grains after it, while, when thoroughly dried, it was

6187·5 grains.

Hence in the first case the block may be considered to have contained 621 grs. of water, nearly $\frac{1}{10}$th of the weight of the dry chalk; and in the second, 1315·5 grs., while the conductive power in the two cases was the same. It should be observed, however, that the temperature of the lower surface of the block in the first experiment was 211°·5, and that of the upper surface about 163°; while in the latter the two temperatures were only 123°·5 and 101°·4 respectively. It is probable that the higher temperature might increase the conductivity by the more rapid conversion of the water into vapour.

These results were verified by almost identical results obtained from another similar block of chalk.

The specimen of *clunch* on which I experimented did not, as I conceived, afford any certain test of the quantity of moisture ordinarily contained in the Chalk as it exists in the general mass of the chalk formation. To obtain a better test, I afterwards procured a specimen from the same place near Cambridge as that from which the above-mentioned specimens were obtained. It was taken from a spot which had not been immediately exposed to the atmosphere, but it contained a quantity of moisture sufficient to render its colour much darker than the white of dry chalk. Soon after it was taken from the mass it weighed

4966·4 grains,

and after being thoroughly dried by a fire for several days, it weighed

4154·1 grains.

It therefore lost 812·3 grs., or nearly one-fifth of its weight when dry. This proportion is about the same as in the case above mentioned, in which the chalk was *saturated* with water. The preceding value (·3) of $\frac{k}{c}$ may therefore be considered as applicable to the general mass of the Lower Chalk near Cambridge. This, I have no doubt, is near the

maximum value for chalk, the general structure of this *clunch* of the Lower Chalk being much more compact, and approaching nearer to the character of *rock* than the mass of the Upper Chalk. I should estimate the value of $\frac{k}{c}$ for chalk in the general mass at about ·25.

Similar experiments were made on an Oolitic block from the quarries at Ancaster. It gave, when saturated with moisture,

$$\frac{k}{c}=\cdot 34,$$

and when dry,

$$\frac{k}{c}=\cdot 30.$$

In the first case the weight was

8633 grains

immediately after the experiment, and

8041·5 grains

when dry. The quantity of water contained in it, when saturated, may therefore be estimated at

591·5 grains,

so that the effect on the conductivity in this instance was much less in proportion to the quantity of moisture, than in chalk. The result was verified by similar experiments on another block. Both blocks had given $\frac{k}{c}=\cdot 37$ when first tried. They had probably imbibed considerable moisture, the season being at the time very wet. These experiments were made at about the lower temperatures above mentioned.

(2) *Arenaceous Rocks.*—A block of New Red Sandstone was very much affected in its conductivity by being saturated with moisture. When saturated it gave

$$\frac{k}{c}=\cdot 60,$$

and when dry,

$$\frac{k}{c}=\cdot 25.$$

In the first case the weight, after the experiment, was

7442 grains,

and when dry, its weight was

6883 grains;

and therefore, the weight of the contained water was

559 grains.

The effect on the conductivity was much greater than in any other case. The quantity of water imbibed in the saturated state was about 350 grains greater than the last-stated amount. These 350 grains were lost during the experiment, though made at temperatures little exceeding 100°.

Another block of harder sandstone than the preceding gave, when dry,

$$\frac{k}{c}=\cdot 49,$$

when its weight was

9221·5 grains.

When moist, it gave

$$\frac{k}{c}=\cdot 65,$$

its weight being then

9434 grains

after the experiment. And again, when saturated with moisture, it gave

$$\frac{k}{c}=\cdot 62,$$

its weight being

9547 grains

after the experiment. The temperature of the lower surface of the block in each of these experiments was 211°·0.

We have the same conclusion here as with the chalk—that an increase of conductivity is produced by an increase of moisture, but that the conductivity ceases to increase considerably before we arrive at the state of saturation, when it would appear to be somewhat less than its maximum value. We have also the same conclusion from a similar experiment on a block of sandstone from the Millstone Grit, and made at like temperatures with the preceding one. It gave, when dry,

$$\frac{k}{c}=\cdot 55,$$

and then weighed

8999·5 grains.

When moist,

$$\frac{k}{c}=\cdot 65,$$

and it weighed

9210 grains.

When saturated,

$$\frac{k}{c}=\cdot 61,$$

its weight being

9291 grains.

The quantity of water absorbed in both these last cases was much less than in the Chalk and New Red Sandstone, as well as the effect on the conductivity.

(3) *Argillaceous Rocks.*—A block of very dry clay gave

$$\frac{k}{c}=\cdot 23,$$

and a moist block gave

$$\frac{k}{c}=\cdot 37.$$

The weight of the moist block *before* the experiment was

6088·5 grains.

Its weight immediately after the experiment was accidentally not observed, but twelve hours afterwards it was

5687·5 grains.

It probably lost about 150 grains during the experiment, which would make the weight taken as in the preceding cases equal to about

5840 grains.

Its weight after being thoroughly dried was

5251 grains.

Consequently, the absorption of a quantity of water = 589 grains, which was nearly one-tenth of the weight of the block, increased the conductive power from ·23 to ·37.

(4) *Very hard Rocks.*—A block of Millstone Grit gave the following results:—when dry,

$$\frac{k}{c} = \cdot 71,$$

weight = 9925 grains.

After being immersed in water about two days,

$$\frac{k}{c} = \cdot 69,$$

weight = 9956 grains.

From a block of close Palæozoic rock I obtained, when dry,

$$\frac{k}{c} = \cdot 52,$$

weight = 11085 grains;

and after immersion in water the same time as the last block,

$$\frac{k}{c} = \cdot 53,$$

weight = 11098 grains.

In these cases the absorbing power was very small, and the difference of the conductive powers indicated by the experiments scarcely exceeds the limits of error.

General Summary of the preceding Results.

11. The conductive powers of calcareous, argillaceous, and siliceous substances—those which compose the great mass of the earth's crust—are in the order in which I have now named these substances, when masses of them are formed by the simple aggregations of each substance previously reduced to dry powder, the first being the worst conductor. The conductive power of Chalk, as it exists in mass, may be estimated

as varying from 2·5 to 3·0; and very hard and compact Mountain Limestone may be estimated at 5·5. These, according to my results, may be taken as the approximate limits for any rocks containing a sufficient quantity of lime to be designated as calcareous rocks.

Arenaceous and siliceous rocks have a somewhat wider range of conductivity. That of friable New Red Sandstone was about ·25, while that of very hard Millstone Grit rises to as much as ·75. With respect to argillaceous rocks, my experiments have been less complete. They show, however, that when not indurated, and in that state of dryness or moisture in which they may be supposed generally to exist in mass, their conductive power may be taken at about ·3. I have not tried any hard clay rock, but, judging from other cases, I should expect it to rise to about ·6.

The igneous rocks have uniformly high conductive powers. In ten cases they varied from ·53 to 1, the mean of them being ·72. In the old sedimentary rocks (Cambrian or Silurian), not calcareous, they appear to vary from about ·5 to ·6, or probably still higher.

It is manifest that whatever may be the mineral constituents of a rock, its conductive power depends very much on its degree of hardness and compactness, that kind of induration which results from some chemical action; while that compactness and comparative hardness which arise merely from mechanical pressure, have less influence than might have been anticipated. In such substances as wax and spermaceti, it is not appreciable in my experiments; nor does it appear to be so in pure calcareous masses like chalk. In clay, however, the effect is very sensible. Dry sand is not sufficiently adhesive to be formed into blocks convenient for the purpose of experiment, even under very heavy pressure. Pressure would probably have little effect on the conductivity.

Temperature has a considerable effect on the conductivity, when it approaches the temperature of fusion. Biot showed this to be the case with certain alloys which are easily fusible; and I have also found the same true for wax and spermaceti. There seems no reason, however, to suppose that temperatures differing sufficiently from those of fusion, produce much effect on the conductivity.

The conduction of heat is very sensibly affected by *discontinuity* in the conducting mass. I have shown, however, that in the case of a sandstone of about a mean conducting power, a division of the mass into beds of one foot in average thickness would not diminish the conductive power by more than about one-twentieth part.

In rocks which are great absorbents of water, the conductivity is much affected by the degree of moisture. The effect increases with the quantity of moisture up to a certain point, which, however, appears to be considerably short of *saturation*. It would appear slightly to decrease again. In chalk, friable sandstone, and clay, the conductive power is increased in a considerable ratio. Hard sandstone, and any highly indurated rocks, are comparatively bad absorbents, and their conductivity appears to be little affected by any moisture they are capable of imbibing.

§ II. *Comparison of Theoretical Deductions from the preceding Experimental Results, with Observations on Terrestrial Temperature at various depths.*

12. A considerable number of observations have been made (as is well known) to ascertain the temperature of the earth at considerable depths beneath its surface, and the law according to which that temperature increases in descending. The existence of increased heat at considerable depths has long been established beyond all doubt, and the law of increase in a considerable number of localities may be considered as approximately and somewhat roughly determined to be—that the increase of temperature above that of the mean temperature at the surface in any proposed locality, is proportional to the depth beneath the surface.

Now I shall not attempt to discuss the numerous observations by which this approximate law has been established, but shall merely cite a few instances in which the observations may be fully relied on, and in which also the great depth to which they were extended leaves less liability to serious error than in many other cases.

The Puits de Grenelle at Paris is an Artesian well, which extends to the depth of 546 metres, through strata which must all have low conductive powers. It first penetrates through upwards of 41·5 metres of the older tertiaries; afterwards through the chalk to the depth of nearly 500 metres, and finally through argillaceous beds for the remaining depth. The whole process and its results are very fully described in the sixth volume of Arago's Works (p. 399). Commencing at the depth of the caves of the Observatory at Paris, where the temperature is constant, and equal to 11°·7 C., the increase of temperature for the whole depth was at the rate of 1° Fahr. for 60 feet. The following Table will also explain the extent of the deviation from uniformity in the progressive increase of temperature in descending (p. 388). It gives for different depths the increase of depth corresponding to an increase of 1° C.

Depths. metres. metres.	Increase of Depth for 1° C. metres.
From 28 to 66	31·1
From 66 to 173	30·6
From 173 to 248	20·8
From 248 to 298	22·8
From 298 to 400	62·5
From 400 to 505	38·9
From 505 to 548	33·0

This Table exhibits a remarkable anomaly in the rate of increase in the central beds of the chalk, for which there appears to be no adequate reason in the nature of the beds themselves as described by Arago (p. 426).

Arago also gives an account, sent to him by Humboldt, of an Artesian well at Neu-Saltzwerk in Westphalia. It had reached the depth of 644·5 metres, nearly 100 metres

deeper than the Puits de Grenelle. The observed temperatures give an increase of about 1° Fahr. for 54 feet of depth. Unfortunately, no geological details are given respecting the nature of the strata through which this well has been sunk. It is merely stated, incidentally, that it penetrated the lower beds of the lias. Another bore is also referred to by Humboldt in his description of the above, in which also the temperature appears to have been very carefully observed by MM. De la Rive and Marcet, near Geneva. The depth was 225 metres, and the observed temperature gave an increase of 1° Fahr. for about 55 feet. Here again no mention is made of the beds which were penetrated.

A very deep Artesian well also exists at Mondorff, in the Grand Duchy of Luxembourg. The account of it is given by Arago (p. 397). We have in this case the geological formations through which the well has penetrated.

	metres.
Lias	54·11
Keuper	206·02
Muschelkalk	142·17
Grès bigarré and Grès Vosgien	311·46
Old Schistose Rocks	16·24
Whole depth	730·00

The increase of temperature was at the rate of 1° Fahr. for about 57 feet.

In our own country there are two deep shafts of coal-mines which have afforded good opportunities of making observations of this kind. One is at Monkwearmouth near Sunderland. Professor Phillips found an increase there of 1° Fahr. for about 60 feet in depth. The mouth of this shaft must, I presume, be in the Magnesian limestone, which, with the subjacent formations of the depth of about 1700 feet, must have been penetrated by the shaft.

There is also a coal shaft at Duckenfield, near Manchester, which is to be carried to the depth of upwards of 2000 feet. It has already attained the depth of more than 1400 feet, and from the depth of 700 feet downwards, the proprietor, Mr. Astley, has obligingly had a series of careful observations made at my request. The rate of increase of temperature between the depths of 700 feet and 1330, was 1° Fahr. for about 65 feet. The shaft penetrates through a number of beds of sandstone and shale, of which some are extremely hard and compact. They belong to the Millstone Grit and Coal formations.

I have selected the above instances as those in which observations fully entitled to our confidence have been made at great depths, and therefore likely to be, as far as possible, free from the influences of local causes. But I may also add, that a considerable number of observations made in the coal-mines of this country give the rate of increase as equal to about 1° Fahr. for 60 feet of depth; and the same result has been

arrived at by M. WALFERDIN respecting the rate of increase in the Chalk throughout a large portion of Northern France. Each of these results must be considered as the mean rate of increase in the particular well or shaft in which it has been obtained. In most cases the exact temperatures at different depths have not been observed at the only time when they can be obtained with accuracy, *i. e.* during the sinking of the shaft or well. It will be remarked, however, that there are great irregularities in the rate of increase in the 'Puits de Grenelle,' and observation appears to indicate similar variations in other localities; but it is easily conceivable that such variations may be due to local and superficial causes, and that they may disappear at sufficient depths beneath the surface. Assuming this to be the case, and neglecting the effects of local causes, the preceding results of observation would lead to the conclusion that the rate of increase of temperature in descending beneath the earth's surface is nearly uniform in each locality, and nearly the same in different localities. At the same time, many observations indicate material deviations from this equality of the rate of increase in different places, and to these I shall again refer; but the general conclusion which has usually been deduced from observations on terrestrial temperature is that above enunciated.

Now if an enormous sphere, like the earth, were originally heated to any degree, and were then left to cool by radiation into surrounding space, for a sufficient length of time; and supposing, moreover, the conductive power of the mass, or at least of its more superficial portion, to be uniform—then the law of temperature above enunciated as deduced from observation, would be the actual law in the case now supposed, at points not remote from the surface of the sphere. This presumed coincidence between the results of theory and observation has naturally led to a very general adoption of the theory which assigns the existing terrestrial temperature entirely to a primitive heat, of which the remains, though producing a comparatively feeble effect within the range of those depths to which we can penetrate, may yet produce an enormous temperature in the more central portions of the globe. The investigations on this subject, however, have hitherto been very imperfect with respect to the determination of the law of increase of temperature in that stratified envelope of the earth which consists of so many layers of substances possessing, as I have now shown, such different conductive powers. I shall endeavour to supply this deficiency; and for this purpose we must first solve the following problem:—

13. If any number of strata, bounded by parallel surfaces of indefinite extent, be superimposed on each other, the conductive power (k) for each stratum, and the transmitting power (q) for each discontinuity, being different, to find the law of temperature in the mass, the lowest surface being maintained at the constant temperature t_1, and the temperature of the atmosphere being τ.

Recurring to the equations (β.) (art. 8), we shall have, giving to k and q their different values, k, k_1, k_2 ..... k_n and q_1, q_2 ..... q_n respectively,

$$\left.\begin{aligned}
t_1-t_2&=\frac{p}{k}(t_2^{(n)}-\tau)h,\\
t_2-t_1'&=\frac{p}{q_1}(t_2^{(n)}-\tau),\\
t_1'-t_2'&=\frac{p}{k_1}(t_2^{(n)}-\tau)h_1,\\
t_2'-t_1''&=\frac{p}{q_2}(t_2^{(n)}-\tau),\\
&\text{\&c.}\quad=\quad\text{\&c.}\\
t_1^{(n-1)}-t_2^{(n-1)}&=\frac{p}{k_{n-1}}(t_2^{(n)}-\tau)h_{n-1},\\
t_2^{(n-1)}-t_1^{(n)}&=\frac{p}{q_n}(t_2^{(n)}-\tau),\\
t_1^{(n)}-t_2^{(n)}&=\frac{p}{k_n}(t_2^{(n)}-\tau)h_n;
\end{aligned}\right\}\quad\ldots\ldots\quad(\gamma.)$$

and adding, we have

$$t_1-t_2^{(n)}=p(t_2^{(n)}-\tau)\left\{\frac{h}{k}+\frac{h_1}{k_1}+\ldots+\frac{h_n}{k_n}+\frac{1}{q_1}+\frac{1}{q_2}+\ldots+\frac{1}{q_n}\right\};$$

when, writing $t_1-\tau-(t_2^{(n)}-\tau)$ for $t_1-t_2^{(n)}$),

$$t_1-\tau=(t_2^{(n)}-\tau)\left\{1+\frac{ph}{k}+\frac{ph_1}{k_1}\ldots+\frac{ph_n}{k_n}+\frac{p}{q_1}+\frac{p}{q_2}\ldots\ldots+\frac{p}{q_n}\right\}$$

$$=(t_2^{(n)}-\tau)\left\{1+\Sigma\left(\frac{ph}{k}\right)+\Sigma\left(\frac{p}{q}\right)\right\},$$

$$\therefore\quad t_2^{(n)}-\tau=\frac{t_1+\tau}{1+\Sigma\left(\frac{ph}{k}\right)+\Sigma\frac{p}{q}}.$$

The value of $t_2^{(n)}-\tau$ being thus determined, the successive equations of (γ.) will determine t_2, t_1', t_2' ..... $t_1^{(n)}$.

For the $(r+1)$th stratum we have

$$-\frac{d\zeta}{dz}=\frac{p}{k_r}(t_2^{(n)}-\tau)$$

$$=\frac{p}{k_r}\cdot\frac{t_1-\tau}{1+\Sigma\left(\frac{ph}{k}\right)+\Sigma\left(\frac{p}{q}\right)},\quad\ldots\ldots\quad(1.)$$

which gives the rate of decrease of temperature in the $(r+1)$th stratum in ascending.

Integrating this equation, we have

$$t_1'-\zeta_0=\frac{p}{k_r}\cdot\frac{t_1-\tau}{1+\Sigma\left(\frac{ph}{k}\right)+\Sigma\left(\frac{p}{q}\right)}\cdot z,$$

whence, substituting for $t_1^{(r)}$, ζ is determined for any point in the $(r+1)$th stratum.

14. Let us now suppose that we have, in another locality, another system of superimposed strata similar to the above, in which the quantities p k h &c. are designated by p' k' h' &c., and for which t_1 and τ are the same as in the former system. Then shall

we have

$$-\frac{d\zeta'}{dz'}=\frac{p'}{k'_r}\cdot\frac{t_1-\tau}{1+\Sigma\left(\frac{p'h'}{k'}\right)+\Sigma\left(\frac{p'}{q'}\right)}; \quad \ldots\ldots\ldots\ldots \quad (2.)$$

which gives the rate at which the temperature in the $(r'+1)$th stratum decreases in ascending. Comparing this with the rate in the former system, we have

$$\frac{-\frac{d\zeta}{dz}}{-\frac{d\zeta'}{dz'}}=\frac{p}{p'}\cdot\frac{k'_r}{k_r}\cdot\frac{1+\Sigma\frac{p'h'}{k'}+\Sigma\frac{p'}{q'}}{1+\Sigma\frac{ph}{k}+\Sigma\frac{p}{q}}. \quad \ldots\ldots\ldots\ldots \quad (3.)$$

To form a general estimate of the numerical values of $\Sigma\frac{ph}{k}$ and $\Sigma\frac{p}{q}$, let us suppose the value of $\frac{k}{p}$ for each of the $n+1$ strata to equal ·5, which is nearly a mean value of $\frac{k}{c}$, as above determined, one foot being the unit of length. Then will $\frac{p}{k}=2$, and

$$\Sigma\frac{ph}{k}=\frac{p}{k}\cdot\Sigma h,$$

$$=2\mathrm{H},$$

when H is the aggregate thickness of the system of strata. Again the number of discontinuities in the $n+1$ strata will $=n$; so that if we take a mean value of $\frac{p}{q}$, we shall have

$$\Sigma\frac{p}{q}=n\frac{p}{q};$$

and if we take $\frac{p}{q}=\frac{1}{10}$, which is greater than the mean of the values given above (art. 7), we have

$$\Sigma\frac{p}{q}=\frac{n}{10}.$$

Hence if twice the aggregate thickness of the strata, expressed in feet, be much greater than one-tenth of the number of strata, and also much greater than unity, the denominator of the expression in equation (3.) will reduce itself nearly to $\Sigma\frac{ph}{k}$; and in like manner, under similar conditions, the numerator will be reduced nearly to $\Sigma\frac{p'h'}{k'}$. Consequently, we shall thus have

$$\frac{-\frac{d\zeta}{dz}}{-\frac{d\zeta'}{dz'}}=\frac{p}{p'}\cdot\frac{k'_r}{k_r}\cdot\frac{\Sigma\frac{p'h'}{k'}}{\Sigma\frac{ph}{k}} \text{ nearly.} \quad \ldots\ldots\ldots\ldots \quad (4.)$$

15. There are two particular cases which it is important to notice: first, that in which the strata particularized by the conductive powers k_r and k'_r, are each of *large thickness* compared with the aggregate thickness of the remaining strata; and, secondly, that in which these two strata are of very *small thickness* compared with the aggregate thickness of the whole mass. Writing $\Sigma\frac{ph}{k}$ and $\Sigma\frac{p'h'}{k'}$ at full length, we obtain

$$\frac{-k_r\frac{d\zeta}{dx}}{-k'_r\frac{d\zeta'}{dx'}}=\frac{\frac{H'}{P}+\frac{h'_1}{k'_1}+\ldots+\frac{h'_r}{k'_r}+\ldots.}{\frac{h}{k}+\frac{h_1}{k_1}+\ldots+\frac{h_r}{k_r}+\ldots.}; \quad \ldots\ldots\ldots\ldots \quad (5.)$$

and if h_r and h'_r be each large, according to our first hypothesis, we shall have

$$\frac{-k_r\frac{d\zeta}{dx}}{-k'_r\frac{d\zeta'}{dx'}}=\frac{\frac{k'_r}{h'_r}}{\frac{k_r}{h_r}}=\frac{k_r}{k'_r}\cdot\frac{k'_r}{h_r}; \quad \ldots\ldots\ldots\ldots \quad (6.)$$

and

$$\frac{\frac{d\zeta}{dx}}{\frac{d\zeta'}{dx'}}=\frac{k'_r}{h_r}. \quad \ldots\ldots\ldots\ldots \quad (7.)$$

The first of these equations shows that, in the case before us, the quantity of heat transmitted will be in a ratio compounded of the ratios of the conductive powers, and the inverse ratio of the thicknesses; and the second equation shows that the increase of temperature in descending, or its decrease in ascending, will be inversely as the thickness of the strata, and independent of the conductive powers.

In the second case, the numerator and denominator of the fraction on the right-hand side of equation (5.) may be considered independent of k_r and k'_r, on account of the comparative smallness of h_r and h'_r. Hence, writing for the numerator $\frac{H'}{K}$ and for the denominator $\frac{H}{K}$,

$$\frac{-k_r\frac{d\zeta}{dx}}{-k'_r\frac{d\zeta'}{dx'}}=\frac{K}{K'}\cdot\frac{H'}{H}, \quad \ldots\ldots\ldots\ldots \quad (8.)$$

and

$$\frac{\frac{d\zeta}{dx}}{\frac{d\zeta'}{dx'}}=\frac{K}{K'}\cdot\frac{H'}{H}\cdot\frac{k'_r}{k_r}. \quad \ldots\ldots\ldots\ldots \quad (9.)$$

Hence, in this case, the quantities of heat transmitted will be independent of the conductive powers; while the rate of increase of temperature descending through any two strata situated respectively in the two groups of strata, will be in the inverse ratio of the conductive powers.

16. Let us now examine how far this investigation may apply to the case of the earth, assuming the actual terrestrial temperature to be due to the remains of a primitive heat. If the earth, considered spherical, were formed of matter of uniform conductive power, and the mean temperature at every point of its surface were the same, the surfaces of equal temperature would, after a sufficient lapse of time, be spherical surfaces concentric with the external surface; and in the actual case, making due allowance for the differ-

ences of mean superficial temperatures in different latitudes, and for possible variations in the conductive powers at different points, it would still seem extremely improbable that the above law respecting the surfaces of equal temperature should not be approximately true at depths beneath the surface at which such superficial causes as are merely local cannot be supposed to exercise a very sensible influence. Let us suppose then that there is a surface of equal temperature concentric with the earth's surface at the depth, for example, of fifty miles; and let us also take a portion of the surface of 100 miles in diameter, with one set of sedimentary strata, the thickness of which is small compared with the above depth; and another smaller area with another similar set of strata. If these portions of the earth's crust be contiguous, the flow of heat near their common boundary will not be entirely *vertical* (as supposed in our investigations), but in some degree *lateral* also; because the conductive powers being by hypothesis different, the temperatures at the same depth in the two sets of strata respectively will not be exactly the same, except at the lowest surface of each set, where they are here assumed to be the same. This difference of temperatures, however, at equal depths will be too small to produce any sensible lateral flow of heat except at points near the common junction of the two sets of strata, or therefore to affect sensibly the vertical flow of heat in all the more central parts of these two contiguous portions of the earth's superficial crust. Also, it is manifest that the surfaces of the strata having no more horizontal extent than here supposed, may be considered as plane instead of spherical. Hence, the hypothesis of the flow of heat being vertical will be very approximately true in the example now proposed.

Again, on account of the great pressure to which the portions of the earth's mass at considerable depths must be subjected, and the absence of stratification in the portion beneath the sedimentary deposits, it would seem impossible that the influence of discontinuity in the lower parts of the mass should not be considerably smaller than in the superincumbent stratified mass; and since it is small, as I have shown, in the latter, it must *à fortiori* be so in the former. It also follows from the great thickness of the whole mass here assumed, and the general values of $\frac{k}{p}$ determined in our own experiments $\left(\text{supposing them much the same as those of } \frac{k}{c}\right)$, that the value of $\Sigma\left(\frac{ph}{k}\right)$ must be very large. Hence $1+\Sigma\left(\frac{p}{q}\right)$ may be neglected in comparison of $\Sigma\left(\frac{ph}{k}\right)$; and for the same reason $1+\Sigma\left(\frac{p'}{q'}\right)$ may be neglected in comparison with $\Sigma\left(\frac{p'h'}{k'}\right)$. Also, if we suppose the two strata denoted by their conductive powers k_r and $k'_{r'}$ to be any two sedimentary strata, their thicknesses h_r and $h'_{r'}$ must be very small compared with the depth, which, for greater distinctness of conception, I have supposed to be fifty miles. Hence the equations (8, 9.) will be applicable to our present case. Moreover it seems extremely improbable that portions of the earth's mass beneath the sedimentary beds, and for considerable depths, should vary much in their conductive powers. All the

igneous rocks I have tried are of high, but not very variable conductivity; and from all we can know by observation, we have no reason to suppose that the unstratified portion of the earth's crust presents any variation of general structure which would be likely to influence very materially its conductivity in different localities. For these reasons we may, I conceive, confidently assume that for any such depth as fifty miles, we shall have

$$\Sigma\left(\frac{H'}{P}\right)=\Sigma\left(\frac{h}{k}\right),$$

or

$$\frac{H'}{K'}=\frac{H}{K},$$

very approximately. In this case equation (9.) becomes

$$\frac{\frac{d\zeta}{dz}}{\frac{d\zeta'}{dz'}}=\frac{k'}{k};$$

i. e. the rates of change of temperature in ascending or descending will be very nearly in the inverse ratio of the conductive powers. This interpretation applies immediately to two strata in the two separate groups of strata; but it manifestly applies also to any two strata in the same group; since for each group the flow of heat is constant, and therefore for every stratum

$$k\frac{d\zeta}{dz}=\text{const.}$$

We also observe, that when

$$\frac{H'}{K'}=\frac{H}{K},$$

as above supposed, equation (8.) shows that the flow of heat is the same in both groups of strata.

17. The result at which we have arrived respecting the rate of change of temperature in ascending or descending through different strata, is of so much importance in testing the theory of terrestrial temperature which attributes the actual temperature of the earth's crust entirely to central heat, as we are now supposing, that it may be worth while to elucidate the above general conclusions by a particular and simple case.

Let us suppose, then, a single stratum AB (fig. 2) of sedimentary matter of about 100

Fig. 2.

miles in radius as above supposed, superimposed on the unstratified mass (C) of the earth's crust, which is supposed to remain uncovered with sedimentary beds in all the neighbouring region. Let us also suppose the thickness of our sedimentary bed to be about 1000 feet, and its conductive power to be one-third of that of the unstratified mass. Now let us first conceive the entire absence of the stratified mass. The temperature at the depth of 1000 feet, according to the actual observed rate of increase of temperature as we descend (about 1° FAHR. for every 60 feet), would be nearly

$$=T+16°;$$

and the temperature at the depth of fifty miles would be nearly

$$=T+4400^{\circ}.$$

Let us now conceive the 1000 feet of sedimentary mass of lower conductive power to be deposited upon the unstratified mass. The escape of the heat would at first be impeded till the temperature of the bottom of the sedimentary mass should exceed the former temperature ($T+16^{\circ}$) at the depth of 1000 feet, by a quantity which would compensate for the lower conductive power of the sedimentary portion. This would require the rate of increase of temperature in descending through this mass to be three times as great as in the unstratified mass, since the conductive power of the former is here assumed to be one-third of that of the latter. Thus, when the increase in descending 1000 feet should become 48° instead of 16°, the quantity of heat conducted through the upper stratum would be equal to that conducted to it through the lower mass, and the temperatures would again be stationary; but the rate of increase of temperature in the sedimentary mass would be three times as great as in the unstratified mass either beneath the sedimentary portion, or in the surrounding region beyond its boundaries. This is the result which our general formula applied to this case immediately affords.

In this explanation it has been assumed that the quantity of heat conducted through the sedimentary mass to the surface would be as great as that conducted to the surface immediately by the unstratified and more highly conductive mass. That this would be very approximately true is easily shown. The quantity of heat conducted through the lower and unstratified mass, of which we have assumed the thickness to be fifty miles, will depend on the difference of temperatures at its lower and upper surfaces. Now if the sedimentary mass did not exist, the temperature of the upper surface of the unstratified mass would $=T^{\circ}$; and when it did exist, the temperature of the upper surface of the unstratified mass immediately below the sedimentary mass would $=T+48^{\circ}$ when the temperature of the whole should have become *steady*, as above explained. Hence the quantity of heat conducted through the lower mass without the sedimentary bed, will be, to that conducted through it with the sedimentary bed, in the ratio of $4400^{\circ}-T^{\circ}$ to $4400^{\circ}-(T^{\circ}+48^{\circ})$, which is very nearly a ratio of equality.

18. This numerical example may also enable us to explain very simply the amount of lateral conduction. The temperature at the depth of 1000 feet at C would be nearly, as we have seen, $T^{\circ}+48^{\circ}$; while at the same depth beyond A and B, it would be $T^{\circ}+16^{\circ}$, the rate of increase there in descending being, by hypothesis, only one-third of that rate at C. Consequently the decrease of temperature in the horizontal distance CB, which we have supposed to be some 100 miles, would be about 32°; while the decrease for half the same distance (50 miles) along the vertical line through C, would be about 4300°. These two numbers enable us to judge how very small must be the lateral conduction of heat, in a case like that here considered, in comparison with the vertical conduction, and not only at the centre C of the area occupied by the stratum of smaller conductivity, but also throughout all the central portion of that area.

19. We are now prepared to compare our theoretical results with those obtained by observation respecting the terrestrial temperature at various depths beneath the earth's surface, as already described. Those observations, as before stated, extend to the greatest depths at which satisfactory observations on terrestrial temperature (so far as I am acquainted with them) have been made. We see that, according to our theoretical results, the rate of increase of temperature in different strata, either in the same or in different localities, ought to vary inversely as the conductive powers of the strata; whereas the observed rates of increase, in the instances above cited, exhibit a striking uniformity, although the observations are made in masses in which the conductive powers are unquestionably very different. I would direct attention especially to two of these cases, the Puits de Grenelle at Paris, and the coal-shaft at Duckenfield. In the former a great depth of chalk was penetrated, and in the latter a like depth of strata, which for the most part are hard arenaceous rocks. The chalk of the northern region of France and that of this country are extremely similar in composition and structure; we cannot therefore be much in error, I conceive, if we estimate its conductive power at from ·25 to ·27 (art. 10). Nor should I estimate the conductive powers of the superincumbent beds (of comparatively small thickness) at much higher values. At Duckenfield the mass which has been penetrated is more distinctly stratified, and consists of beds of a much greater variety than is found in a mass of chalk. They have, however, for the most part, the common character of being very siliceous; and many of the thickest beds, the aggregate of which composes a large portion of the mass through which the shaft passes, are almost entirely siliceous, and extremely hard and compact. I obtained the conductive powers of a considerable number of these rocks. They varied between ·51 and ·726. I estimate their mean value at about ·62, and the conductive power of the aggregate mass of all the strata at not less than ·5. The number of discontinuities between successive strata is not sufficient to diminish materially the conductivity of the whole; but whatever effect may be produced by them, it is probably more than counterbalanced by the effect of moisture in those strata which are less compact and of smaller conductive power *.

According to these estimates, then, the conductive power of the mass through which the Puits de Grenelle has penetrated, can scarcely exceed one half of that of the aggregate of the beds at Duckenfield. The rate of increase of temperature in the former case ought, therefore, according to our theory, to be nearly twice as great as in the latter, whereas it is 60 feet for 1° FAHR. at Paris, and only 65 feet at Duckenfield, instead of 110 or 120 feet, as the theory would have led us to suppose.

The instance above mentioned of the Artesian well at Mondorff (art. 12) presents a similar discrepancy between theory and observation, for there can be no doubt but that the mass of rocks through which that well passes, as above stated, is of considerably higher conductivity than the Chalk, though observation there gives a somewhat *more*

* The shaft at Duckenfield is likely, I believe, to attain its ultimate depth of upwards of 2000 feet in a short time. I hope to obtain from its lower portion much more complete evidence than I yet possess.

rapid increase of temperature than in the latter formation, the increase being in this case only 57 feet for 1°. The other cases referred to are more indefinite, on account of the omission of any detailed mention of the nature of the beds through which the heat is transmitted.

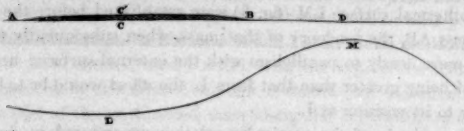
20. There are also other observations to which I have not yet alluded, the correctness of which in their general results may be depended on, though some of them may inevitably, perhaps, have been made under circumstances calculated to diminish their weight when considered separately. I allude to the observations made by Mr. HENWOOD, with great care and labour, and published as an Appendix (No. 1) to his work 'On the Metalliferous Deposits of Cornwall and Devon.' The heat of the mines of that district, situated in slate rock, is well known to be greater than that of the mines situated in granite. From a number of observations Mr. HENWOOD concluded that, on the average, and at depths not less than 600 feet, there was an increase of 1° FAHR. for 51 feet in the granite, and for 37·2 feet in the slate*. I have not been able to ascertain the conducting power of the Cornwall slate, and cannot therefore determine how far these results are consistent with each other, according to theory; but there can be no doubt but that the increase ought to be much more rapid, instead of being slower, in chalk than in either granite or slate; for all the older compact rocks, on which I have experimented, have been found to have comparatively high conducting powers. This increases still further the discrepancy between observation and the theory we are discussing.

21. In the preceding discussion it has been supposed that the isothermal surfaces at depths (such, for instance, as 50 or 60 miles) below the influence of local external causes, are approximately concentric with the surface of the earth itself, in which case it has been shown that the quantity of heat transmitted through the external shell must be very nearly the same in every part of it. It may, however, be conceived that these deeper isothermal surfaces may deviate from concentric spherical forms much more than has been here supposed. It would be difficult, I think, to assign any probable cause for such considerable deviations according to our theory; but admitting, for a moment, the hypothesis, let us examine what consequences will flow from it, and how far it will enable us to account for the general uniformity of the rate of increase of terrestrial temperature as we descend in different localities throughout a region like Western Europe, or for the local variations to which it is subject.

For this purpose let us take the particular example of Art. 17, except that we here suppose the deep isothermal surface LM to be of the form represented in fig. 3, instead of being parallel to the internal surface; and since the conductive power of the unstratified mass CLMD has been supposed three times as great as that of the sedimentary

* Since this paper was read, I have been informed by Mr. FOX, that, generally, those Cornish mines which present the most rapid increase of heat in descending have considerable quantities of water, a portion of which may probably ascend from lower depths, and increase the temperature of the higher portions of the mass. The rate of increase of temperature in the *dry* Cornish mines does not appear to differ much from the general rate of 1° FAHR. for about 60 feet of depth.

Fig. 3.



mass AC'BC, suppose the depth C'L (very nearly the same as CL) to be three times as great as DM. Also let t_1 denote the temperature at any point of LM, t_2 that at the point C, a central point at the lower surface of the sedimentary mass, and t'_2 the temperature at D. The temperature of the surface at C' will be very nearly that of the mean external temperature, and may therefore be considered equal to t'_2, the superficial temperature at D. Consequently, in the particular example we are considering, we shall have

$$t_2 = t'_2 + 48°,$$

and the difference of temperature, at L and C,

$$= 4400° - t_2,$$

$$= 4400° - (t'_2 + 48°);$$

while the difference of temperatures at M and D will

$$= 4400 - t'_2.$$

Hence if Q and Q' be the quantities of heat transmitted along LC and MD respectively in the same time,

$$Q : Q' :: \frac{4400° - (t'_2 + 48)}{LC} : \frac{4400° - t'_2}{MD};$$

and since, by hypothesis, LC = 3.MD, we shall have

$$Q' = 3Q$$

very nearly. Consequently the quantity of heat which will be conducted through MD in a given time, will equal three times that conducted through CC' in the same time; and since the conductive power in the former case is three times that in the latter, the rate of decrease of temperature through CC' and through MD will be very nearly equal.

If, then, we could admit the hypothesis, that the depth of any deep internal surface of equal temperature beneath any point of the earth's external surface is inversely proportional to the conductive power of the superficial stratum at that point, we could thus account for the equality of the rates of increase of temperature in descending through superficial masses of different conductive powers. There does not appear, however, to be any conceivable grounds for the admissibility of this very limited hypothesis, according to the theory of central heat; for though irregularities in the structure and conductivity of the unstratified portion of the earth at such depths might produce considerable irregularities in the forms of the deep isothermal surfaces, it is difficult to imagine the possibility of any such relation as that above mentioned, between these irregularities of form

and the existence of superficial sedimentary masses of lower conductivity. In fact, if the form of the isothermal surface LM (fig. 3) were established before the formation of the sedimentary mass AB, the tendency of that mass, when subsequently formed, would be to reduce LM more nearly to parallelism with the external surface; because the escape of heat from M being greater than that from L, the effect would be to lower the surface at M relatively to its position at L.

But if the hypothesis of the particular relation represented in fig. 3 between the depression of the isothermal surface at L, and the existence of a sedimentary mass like AB, directly above it, be deemed, as I conceive it must be, far too restricted to be admissible, we must consider our hypothesis respecting the non-concentric form of the deep isothermal surfaces independently of this additional restrictive hypothesis. Now if we do not admit this restriction, there is no reason why the sedimentary mass should not be formed above M instead of L (fig. 4). But in such case the quantity of

Fig. 4.

heat to be transmitted through it would be three times as great as at C, while the conductive power at the former point would only be one-third of that at the latter. Consequently the rate of increase of temperature in descending at D would be nine times as great as that at C. Hence the same hypothesis which might enable us to account for a rate of increase of terrestrial temperature, independent of conductive powers, in one region, would lead to the almost necessary conclusion that there must be found in the comparison of these rates of increase in different regions, deviations from the above law far greater than any which have been established by past observations, or are likely to be established by future ones.

I have here supposed (for the sake of speaking of a simple ratio) the conductive power of the unstratified mass to be three times that of the sedimentary mass deposited upon it. This ratio of three to one may be greater than that actually presented to us in the earth's crust. The actual ratio, however, is probably more than two to one, and abundantly sufficient, therefore, to justify the above conclusion.

22. There are other hypotheses, also, which would account for the equality of the rate of increase of temperature in descending beneath the surface, independently of the conductive power of the mass, in the simple case above supposed of a uniform sedimentary mass superimposed on the general mass of unstratified rock. It should be remarked, however, that in this case, as considered in the previous article, the equal rate of increase of temperature along the vertical lines through C' and D respectively, would only hold

to the depth C'C. Beneath C the rate of increase would be only one-third of that above C, on account of the greater conductive power below than above that point. The like equality in the rate of increase in the case of art. 17, in which the isothermal surfaces are parallel to the outer surface, would result from the supposition that the portion of the unstratified mass along the vertical through D should have a conductive power three times as great as that of the unstratified mass along the vertical below C. This would seem absurd; and, besides, all particular and restricted hypotheses of this kind are liable to the objection mentioned in the preceding article. They do not enable us to account for the uniform rate of increase of temperature in different localities, unless we adopt also the hypothesis of certain necessary relations between the interior structure of the unstratified mass beneath, and the deposits of sedimentary beds on its surface—a supposition which must be deemed utterly inadmissible.

With respect to merely local deviations from the prevailing law throughout a region of considerable extent, we may observe that it is difficult to refer them to any deep-seated source of heat. It would seem much more probable that they are referable to some superficial action*.

On the whole then I cannot avoid the conclusion, that the existence of a central heat is not sufficient in itself to account for the phenomena which terrestrial temperatures present to us.

23. Let us now turn to the other case, in which, as expressed in equations (6, 7) (art. 15), the variation of temperature in ascending or descending is independent of the conductive power, and in which the quantity of heat transmitted through a given horizontal area is proportional to that power. This, it must be recollected, involves the supposition that the two strata whose conductive powers are k_r and k'_r, are of great thickness as compared with that of the other portion of the mass. If, however, instead of taking a single stratum of which the conductive powers are respectively k_r and k'_r, we should take groups of strata having mean conductive powers equal to those quantities, the same equations will be applicable. Thus if we take, for example, groups of strata like those penetrated at the Puits de Grenelle, the coal-shaft at Duckenfield, and the other places enumerated above as giving very nearly the same mean variation of temperature along a vertical line, the equations will be applicable, assuming k_r approximately equal to k_r; and it proves that, in such a case, there must be some cause acting to produce and maintain a surface of equal temperature at no great depth beneath that to which the increase of temperature may proceed according to the same law. At the same time, the flow of heat from this isothermal surface must be proportional to the conductive power of the superincumbent strata. I have endeavoured to show that this cannot be the case when all the heat is transmitted from the central portion of the earth, because, in this case, the quantity of heat transmitted must depend on conditions existing at comparatively great depths, where they can have little relation to the more superficial conditions. Not only, therefore, must there be some other cause generating heat to maintain this isothermal surface, but it must also be such as shall generate the

* Or to water rising from a lower level (see note, page 832).

greatest quantity of heat beneath those strata of which the conductive power is the greatest.

24. In this explanation of the conditions required in order that the variations of terrestrial temperature as we descend beneath the earth's surface may be nearly independent of the conductive powers of the masses through which the heat is transmitted, it is assumed that the temperature (t_1) is the same at the lower surfaces of the two groups of strata compared with each other. This, however, is not itself an essential condition. For when the flow of heat through any mass has become *steady*, it is exactly the same at every point along the line of transmission, and, therefore (in the case of the earth), we may consider any cause generating the heat necessary to maintain this steady flow, to act at any depth without thereby changing the temperature of the superincumbent mass. The essential condition is that the quantity generated, at whatever depth it may be produced, must be inversely as the conductive power of the mass through which it has to be propagated*. Thus, in the case of the earth, assuming terrestrial temperature to be due to the generation of heat in the superficial crust of the earth, then, if we should further assume (for the sake of illustration) that this generation of heat only takes place in the non-sedimentary unstratified portion of the crust, a smaller quantity of heat must be generated in those parts of the unstratified mass which are subjacent to sedimentary masses of the lower degrees of conductivity, than in those parts which extend up to the earth's external surface. Nor does this appear difficult to conceive, if, in addition to some such hypothesis as that just mentioned, we suppose the cause producing the heat to extend only to comparatively small depths, for, in such case, a larger portion of the mass capable of having heat generated within it, would be within the operation of the cause producing that effect, in that region in which the unstratified mass should extend up to the external surface.

25. After the preceding investigations, it appears to me extremely difficult, if not impossible, to avoid the conclusion that a part at least of the heat now existing in the superficial crust of our globe is due to superficial and not to central causes. It should be remarked, however, that the argument thus afforded is not directly against the theory of a *primitive* heat, but only against the manifestation of the remains of such heat as the sole cause of the existing terrestrial temperatures at depths beyond the direct influence of solar heat. The argument in favour of the earth's original fluidity (a state only conceivable as the effect of heat), founded on the spheroidal form of the earth, remains unaffected. Whatever cogency it may have been supposed to possess, it possesses still. At the same time, all those collateral arguments derived from the existing temperature of the earth's crust, or the climatal changes which we believe to have taken place on its

* It must be carefully recollected that this conclusion is restricted to those cases in which the generation of heat takes place entirely beneath the mass in which the increase of temperature is to be independent of the conductive power. Any superficial cause would probably generate heat within this mass as well as beneath it, so that its temperature would depend partly on heat thus generated within it, and partly on that transmitted through it from below, in which case the amount of heat generated in a given time need not bear the determinate relation to the conductive power stated in the text.

surface, are deprived, I conceive, of a great part of their weight. Moreover, admitting only a part of existing terrestrial heat to be due to superficial causes, the flow of heat from the earth's central portions must be less by that amount than if the whole flow were due to central heat. Consequently the rate of increase of temperature *due to the flow of central heat* must be proportionally diminished, and the depth at which we should arrive at the temperature of fusion, proportionally increased. The conclusion, therefore, that the earth's solid crust is so thin as many geologists have believed it to be, as well as those theories resting upon that conclusion—whether of volcanic action, or of elevation and depression of the earth's surface, at least in more recent geological times—must be in a great degree invalidated. But on these points I may have some further remarks to make in my next communication. At present it is not my object to carry these speculations beyond the point at which we are now arrived.

§ III. *Description of the Apparatus for determining Conductive Powers, and the mode of conducting the Experiments.*

26. I now proceed to an account of the experiments, the general results of which have been given, and to the description of the apparatus by which they were made.

Fig. 5.

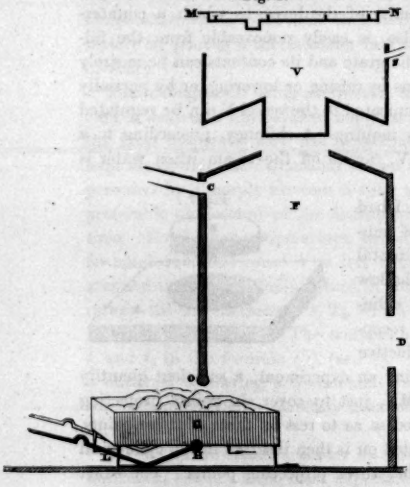


Fig. 6.

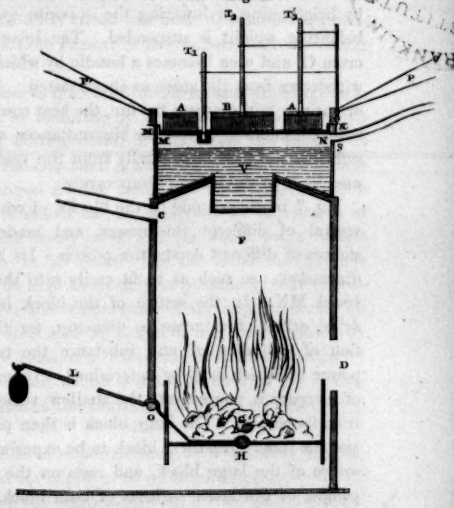


Fig. 5 represents the different parts of the apparatus separated from each other. In fig. 6 they are represented as fitted into each other preparatory to an experiment. F is

the stove, the walls and roof of which are of cast iron. The horizontal section of it below the roof is rectangular; the orifice at the top is circular. C is the commencement of the chimney, which is carried horizontally and then vertically through the roof of the building, the horizontal portion being sufficiently long to prevent any sensible influence on the experiments arising from the heat radiating from the chimney. In the side of the stove is a rectangular opening D, which may be closed at pleasure, and through which fuel is supplied to the grate G. This grate fits the stove, but so as to be capable of moving freely up and down within it. The opening in the lower part of the stove at O is large enough to admit of the grate and the fuel contained in it being withdrawn entirely from the stove when required.

The horizontal section of the upper portion of the vessel V is rectangular; that of the part projecting downwards is circular, and such as to pass freely into the circular orifice at the top of the furnace, to the roof of which also this vessel is made to fit. MN is a shallow vessel, just fitting the top of V, on the upper rim of which it is supported by the projecting edges represented at M and N, when the parts are put together as in fig. 6. L there represents a lever working on a fulcrum at O, and so attached to the grate at the ends of an axis H, perpendicular to the plane of the paper, as to constitute a rough approximation to a parallel motion when the grate is moved vertically by heightening or lowering the opposite extremity of the lever, to which a counterbalancing weight is suspended. The lever, also, is easily removeable from the fulcrum O, and then becomes a handle by which the grate and its contents can be entirely withdrawn from the stove as above stated. Thus by raising or lowering, or by partially or entirely withdrawing the fire, the heat communicated to the vessel V can be regulated with considerable nicety as circumstances may require. A chimney, proceeding to a sufficient distance horizontally from the vessel V, carries off the steam when water is used in V at the boiling temperature.

Fig. 7.

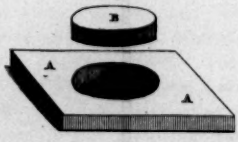


Fig. 7 represents one of the blocks, of which I had several of different thicknesses, and made of substances of different conductive powers. Its horizontal dimensions are such as to fit easily into the shallow vessel MN. In the centre of the block is a cylindrical orifice, five inches in diameter, for the reception of cylinders of any substance the conductive power of which is to be determined. In making an experiment, a sufficient quantity of mercury is poured into the shallow vessel MN, just to cover six points projecting from its bottom. The large block is then placed so as to rest on three of these points, and the inner cylindrical block to be experimented on is then inserted in the cylindrical orifice of the large block, and rests on the other three projecting points. The exact contact of the lower surfaces of both blocks with the mercury is thus secured. The outer block is also surrounded by a square frame of wood, without top or bottom, resting on the rim of the vessel V. The apertures between this frame and the sides of the

large block, and that immediately surrounding the inner block, are filled with cotton-wool to prevent as much as possible the lateral transference of heat, both from the inner block to the outer one, and from the vertical sides of the latter into surrounding space. Also, from the upper edges of the outer block (or rather of the wooden frame surrounding it) two or three wooden floors with intervals between them, were constructed of loose boards, extending several feet on every side (sections of which are represented in fig. 6 at P and P'), and intended to prevent as far as possible the influence of radiation from the heated stove below, and the objects immediately surrounding it. The upper surfaces, both of the outer and inner blocks (fig. 7), are scooped out, leaving a projecting rim round their edges, so as to form shallow vessels for the reception of mercury; or the same object is attained by fixing round the edges a thin, narrow rim of iron. These spaces, in experimenting, are filled with mercury of just sufficient depth to cover the small spherical bulb of a thermometer. In fig. 6, A, B, A, represent the section of the outer and inner blocks, with the interspace between them. One thermometer (T_1) passes down this interspace into the mercury below the blocks; while T_2 gives the temperature of the upper mercury on the block B, the conductive power of which is to be determined; and T_3 gives that of the mercury which covers the surface of the outer block. This latter mercury has sometimes been dispensed with, in which case the temperature of the uncovered surface of the outer block has been determined approximately by placing a thermometer in a small hole formed in the surface, and containing mercury.

This arrangement being completed, the fire is placed in the stove, and the lower mercury is heated to the required temperature, as indicated by T_1. The temperature much the most frequently used has been that of boiling-water, in which case the vessel V is kept nearly full, the exhaustion caused by the continual generation of steam being compensated by a supply through a tube arranged for the purpose. This temperature is preferable on account of the facility with which it can be preserved for any length of time. For higher temperatures, which were rarely used, and for which water cannot be employed, the vessel V is left empty. In all cases when the lower mercury has attained its required temperature, it must be kept as steadily as possible at that temperature, till the thermometer, T_2, which gives the temperature of the upper mercury on B, becomes stationary. The temperatures indicated by T_1 and T_2 are those denoted by t_1 and t_2 in the formula (5), (art. 1). The remaining temperature in that formula (τ) is obtained by waving a thermometer above and within a short distance of the blocks, the instrument being carefully guarded against the influence of direct radiation from surrounding objects.

27. To secure entire accuracy by this mode of experimenting, it is manifest that the horizontal extent of the mass experimented on ought to be so large, compared with its thickness, that no appreciable diminution of the temperature (t_2) of the central portion of the upper surface should be produced by the *lateral* transference of the transmitted heat, the whole of which is supposed in our mathematical formula to be transmitted

vertically. It would have been almost impossible, however, to obtain slabs of any considerable extent from many of the harder and more intractable rocks on which I have experimented. But the fact is, that no great accuracy is required in experiments of this nature; for if the conductive power of any piece of rock were determined with the ultimate degree of accuracy, it would only tell us *approximately*, and not *accurately*, the conductive power of another piece of rock of the same kind, however close might be their resemblance in mineral structure; so that it must be always necessary, when the conductive power of any proposed substance is required with great accuracy, to determine that power by direct experiments on the substance itself. I have endeavoured, however, to take every precaution necessary to secure all the accuracy required with reference to my more immediate objects in these researches.

In order that the error resulting from the observations of temperatures may be as small as possible, it is manifestly desirable that $t_2 - \tau$ and $t_1 - t_2$ should both be as large as possible, and not very different in magnitude. But this condition, if the conductive power and the temperatures used were sufficiently great, would require a thickness of the block which might be undesirable in consequence of its limited extent of horizontal surface. The thickness which I have more generally used for mineral substances, is about 2 inches, and about 1 inch for substances, such as wax and spermaceti, of lower conductive powers, while the outer block has been about 15 inches square, and of very nearly the same thickness as the blocks experimented on, and covered with mercury in the same manner. Having several of these outer blocks made of substances of different conductive powers, I have always been able to select one for each experiment in which the conductivity was nearly the same as that of any proposed substance, taking care, however, that it should never be *greater*; for in such case the temperature of the surface of the outer block, as indicated by T_2, fig. 6, would be higher than the proper temperature of the surface of B, and would, therefore, tend to raise also the observed temperature of B, as given by T_2, above the proper temperature, and thus to give the conductive power of B too great. I have been anxious that, on the contrary, the error should always be in *defect*, and therefore have always selected such a block for each experiment that the temperature given by T_2 should generally be somewhat less than that given by T_1. A small error has, doubtless, been thus superinduced, especially in reference to substances of the highest conductive powers, but I am satisfied that it has in no case been of any importance as regards the conclusions which I have drawn from these results.

28. In all cases in which it has been necessary to compare the conductive powers of two blocks with as much accuracy as possible, I have adopted a somewhat different method from that above described. Instead of a single block (B) placed in the central cylindrical hole of the outer block (A), the two blocks to be compared have been placed in two cylindrical holes pierced through the outer block, as represented in fig. 8, and such that the two blocks

Fig. 8.

(B, B′) placed in them should be as nearly as possible under the same conditions during the experiment. The small orifice between them admits the thermometer T, for the determination of the temperature t_1, while t_2 is determined for each block separately. When this has been done, the two blocks are interchanged, each being thus placed in the hole previously occupied by the other, and the experiment repeated. The mean results of the two experiments have been adopted. The discrepancies, though small, are sufficient to show the necessity of the precaution where considerable accuracy is required.

29. It would be useless, I conceive, to record the details of all the numerous experiments made for the purpose of simply determining the conductive powers of particular substances. They were made in the manner already described (art. 25), and when the apparatus was once arranged, required little but patience to wait for the final indications of the thermometers. Such at least was the case when the steam from boiling water was used to produce the temperature of the lower surface of the block; when any other temperature was employed, constant attention was necessary to preserve it as steady as possible.

It may suffice to give the details of one experiment of this kind made with high temperatures, from which it may be understood how far the steadiness of the temperature t_1 could be maintained. In almost all cases, however, the same temperature, that of boiling water, was used for t_1 in the experiments now spoken of, in order that they might all be made under nearly the same conditions with respect to temperature. The experiments made to compare the conductive powers of two blocks, for the purpose of ascertaining the influence of pressure, moisture, discontinuity, &c., require greater care; and I have therefore thought it right to give all the principal ones of this kind in sufficient detail to enable the reader to judge of the evidence which they afford. The symbols used have the same signification as heretofore (art. 1, &c.). I repeat them here for the convenience of more immediate reference.

t_1 = temperature of the mercury in contact with the lower surface of the cylindrical block B, on which the experiment is made;
t_2 = that of the mercury on its upper surface;
t'_2 = that of the upper surface of the surrounding or outer block A;
r = that of the air immediately above the blocks A and B;
h = the length of the cylindrical block B;
k = conductive power of B;
c = radiating power of mercury.

It will be observed that in some of the following experiments the recorded value of t'_2 is greater than that of t_2, in opposition to the remark in the last paragraph of article 26. In such cases the temperature t'_2 really corresponded to points at somewhat a lower level in the outer block A than the upper surface of B to which t_2 corresponds. I only observed t'_2 to assure myself of there being no amount of *lateral* transference of heat

which could materially affect the results of the experiments. The values of $\frac{k}{c}$ are found by substitution in the formula

$$\frac{k}{c}=\frac{t_2-\tau}{t_1-t_2}h.$$

One foot is adopted as the linear unit.

The blocks were cylindrical, the diameter of their bases being about 3 inches.

The blocks of spermaceti and wax, which were solidified under compression, were formed by pouring the substances in a fluid state into a cylindrical hole in a strong block of iron, and leaving them to cool and solidify under the required pressure, applied by means of a piston exactly fitting the cylindrical hole, and rendered by proper *packing* perfectly fluid-tight. The piston was acted on by a powerful lever. When these substances were compressed after solidification, cylindrical blocks of them were formed of the proper dimensions before the compressing force was applied to them. Blocks of chalk, clay, &c. were compressed in the same manner.

30. *Experiments made for the purpose of determining the effect of pressure on the conductivity of various substances.*

Spermaceti.

I. (*a.*) Spermaceti, solidified under a pressure p, on the square inch. The two cylindrical blocks are denoted by I. and II.

	I. $\begin{cases} h=\cdot086 \text{ ft.} \\ p=900 \text{ lbs.} \end{cases}$		II. $\begin{cases} h=\cdot086 \text{ ft.} \\ p=6500 \text{ lbs.} \end{cases}$		
t_1 91°·3	t_2 68°·2	t_2' 67°	t_2 66°·6	t_2' 67°	τ 45°
	$\frac{k}{c}=\cdot086$		$\frac{k}{c}=\cdot075$		

(*b.*) Blocks interchanged.

	I.	II.	
t_1 90°·5	t_2 69°·3	t_2 68°·6	τ 47°
$\frac{k}{c}=\cdot09$		$\frac{k}{c}=\cdot086$	

Mean values $\begin{cases} \text{I. } \frac{k}{c}=\cdot088 \\ \text{II. } \frac{k}{c}=\cdot08. \end{cases}$

II. Two blocks of spermaceti, one (I.) solidified under pressure, the other (II.) com-

pressed after solidification, the compressing force being in each case =7500 lbs. per square inch.

	I. $\begin{cases} h=\cdot086 \text{ ft.} \\ p=7500 \text{ lbs.} \end{cases}$		II. $\begin{cases} h=\cdot086 \text{ ft.} \\ p=7500 \text{ lbs.} \end{cases}$		
t_1. 94°·2	t_x. 70°·2	t_2.	t_x. 70°·7	t_2.	τ. 47°·2
	$\frac{k}{c}=\cdot089$		$\frac{k}{c}=\cdot086$		

Wax.

III. Block I. uncompressed, and II. compressed.

	I. $\begin{cases} h=\cdot09 \text{ ft.} \\ p=\text{atmosph.} \end{cases}$		II. $\begin{cases} h=\cdot09 \text{ ft.} \\ p=7500 \text{ lbs.} \end{cases}$		
t_1. 132°·5	t_x. 92°·5	t_2. 95°·5	t_x. 94°·6	t_2. 94°	τ. 61°
	$\frac{k}{c}=\cdot072$			$\frac{k}{c}=\cdot079$	

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Chalk.

IV. Block I. uncompressed; II. compressed.

	I. $\begin{cases} h=\cdot13 \text{ ft.} \\ p=\text{atmosph.} \end{cases}$		II. $\begin{cases} h=\cdot11 \text{ ft.} \\ p=7500 \text{ lbs.} \end{cases}$		
t_1. 212°	t_x. 147°·3	t_2. 154°·5	t_x. 148°·5	t_2. 153°·5	τ. 55°
	$\frac{k}{c}=\cdot18$		$\frac{k}{c}=\cdot16$		

The fact of $\frac{k}{c}$ being smaller in the compressed than in the uncompressed block, may probably be attributed to some difference in the quantity of moisture contained in them.

V. (*a.*) Two compressed blocks.

	I. $\begin{cases} h=\cdot11 \text{ ft.} \\ p=4300 \text{ lbs.} \end{cases}$		II. $\begin{cases} h=\cdot11 \text{ ft.} \\ p=7500 \text{ lbs.} \end{cases}$		
t_1. 212°	t_x. 150°	t_2. 159°	t_x. 152°	t_2. 159°	τ. 61°
	$\frac{k}{c}=\cdot157$		$\frac{k}{c}=\cdot165$		

(*b.*) Last experiment repeated, with the blocks interchanged.

	I. $\begin{cases} h=\cdot 11 \text{ ft.} \\ p=4300 \text{ lbs.} \end{cases}$		II. $\begin{cases} h=\cdot 11 \text{ ft.} \\ p=7500 \text{ lbs.} \end{cases}$		
t_1. 212°	t_2. 152°	t_3. 159°	t_2. 152°·4	t_3. 159°	τ. 60°
	$\frac{k}{c}=\cdot 168$		$\frac{k}{c}=\cdot 17$		

Taking the mean of the experiments (*a.*) and (*b.*), we have for

I., $\frac{k}{c}=\cdot 162$; and II., $\frac{k}{c}=\cdot 167$.

Clay.

VI. (*a.*) Block I. uncompressed; II. compressed.

	I. $\begin{cases} h=\cdot 12 \text{ ft.} \\ p=\text{atmosph.} \end{cases}$		II. $\begin{cases} h=\cdot 12 \text{ ft.} \\ p=7500 \text{ lbs.} \end{cases}$		
t_1. 212°	t_2. 163°·1	t_3. 168°	t_2. 171°·3	t_3. 168°	τ. 52°
	$\frac{k}{c}=\cdot 27$		$\frac{k}{c}=\cdot 38$		

(*b.*) Blocks interchanged.

	I.		II.		
t_1. 212°	t_2. 162°	t_3. 168°·5	t_2. 170°	t_3. 170°	τ. 53°
	$\frac{k}{c}=\cdot 25$		$\frac{k}{c}=\cdot 33$		

The means of (*a.*) and (*b.*) give for I. $\frac{k}{c}=\cdot 26$; and for II. $\frac{k}{c}=\cdot 355$.

VII. (*a.*) Two compressed blocks of clay.

	I. $\begin{cases} h=\cdot 12 \text{ ft.} \\ p=4300 \text{ lbs.} \end{cases}$		II. $\begin{cases} h=\cdot 12 \text{ ft.} \\ p=7500 \text{ lbs.} \end{cases}$		
t_1. 212°	t_2. 168°·6	t_3. 169°·2	t_2. 171°·6	t_3. 170°·3	τ. 56°
	$\frac{k}{c}=\cdot 30$		$\frac{k}{c}=\cdot 34$		

(*b*.) Blocks interchanged.

	I.		II.		
t_1. 212°	$t_{2'}$. 165°·6	t_{2}.	$t_{2'}$. 176°·5	t_{2}.	τ. 56°
	$\frac{k}{c}$=·30		$\frac{k}{c}$=·33		

Mixed sand and clay in equal quantities.

VIII. (*a*.) Two compressed blocks.

	I. $\begin{cases} h=\cdot 12 \text{ ft.} \\ p=4300 \text{ lbs.} \end{cases}$		II. $\begin{cases} h=\cdot 12 \text{ ft.} \\ p=7500 \text{ lbs.} \end{cases}$		
t_1. 212°	$t_{2'}$. 174°·3	t_{2}. 167°·5	$t_{2'}$. 175°	t_{2}. 167°·5	τ. 58°·5
	$\frac{k}{c}$=·368		$\frac{k}{c}$=·378		

(*b*.) Blocks interchanged.

	I.		II.		
t_1. 212°	$t_{2'}$. 173°	t_{2}. 166°·5	$t_{2'}$. 175°	t_{2}. 166°5	τ. 58°
	$\frac{k}{c}$=·36		$\frac{k}{c}$=·379		

The means from (*a*.) and (*b*.) give for I., $\frac{k}{c}$=·364; and for II., $\frac{k}{c}$=·378.

31. *Experiments to determine the effect of Discontinuity.*

Two cylindrical blocks of sandstone (freestone) were formed from the same larger block. The experiment IX. was made to test the equality of their conductive powers.

IX. (*a*.) Two blocks of sandstone.

	I. h=·166 ft.		II. h=·166 ft.		
t_1. 211°	$t_{2'}$. 174°·7	t_{2}. 173°·6	$t_{2'}$. 175°	t_{2}. 175°·8	τ. 63°
	$\frac{k}{c}$=·51		$\frac{k}{c}$=·51		

(*b*.) Blocks interchanged.

	I.		II.		
t_1. 210°·9	t_2. 175°·5	t_3. 175°	t_2. 176°	t_3. 176°·8	τ. 65°
	$\frac{k}{c}$=·52		$\frac{k}{c}$=·53		

The means of (*a*.) and (*b*.) give for I., $\frac{k}{c}$=·515; and for II., $\frac{k}{c}$=·52.

X. One block was broken transversely, and the two parts again fitted accurately together without any interposing substance.

	Broken Block.		Whole Block.		
t_1. 210°·9	t_2. 166°·3	t_3. 175°·2	t_2. 176°	t_3. 175°·5	τ. 65°

XI. The two parts of the broken block cemented together by plaster of Paris applied wet and allowed to dry and harden. The length of the block was thus increased one-twentieth of an inch.

	Broken Block.		Whole Block.		
t_1. 210°·2	t_2. 170°	t_3. 172°·6	t_2. 174°	t_3. 172°	τ. 57°

XII. Finely powdered dry clay placed between the two parts of the broken block. The length was increased by one-tenth of an inch.

	Broken Block.		Whole Block.		
t_1. 210°·2	t_2. 165°·6	t_3. 173°	t_2. 174°	t_3. 173°·2	τ. 59°

XIII. Without removing the blocks the temperature was reduced.

	Broken Block.		Whole Block.		
t_1. 154°·6	t_2. 126°·3	t_3. 131°·3	t_2. 131°	t_3. 129°·2	τ. 56°

XIV. (*a*.) The two portions of the broken block cemented by moist clay, and pressed

together with a pressure of 140 lbs. per square inch for twelve hours. The length was =·18 foot.

	Broken Block.		Whole Block.		
t_1. 211°	t_2. 168°·2	t_3. 173°·4	t_4. 173°·4	t_5. 171°·7	τ. 56°

(*b.*) Blocks interchanged.

	Broken Block.		Whole Block.		
t_1. 211°	t_2. 167°·9	t_3. 172°	t_4. 175°	t_5. 175°·5	τ. 59°

XV. The last experiment repeated at a lower temperature.

	Broken Block.		Whole Block.		
t_1. 149°·8	t_2. 124°·3	t_3. 125°·7	t_4. 128°·6	t_5. 128°	τ. 57°

32. *Experiments to ascertain the influence of Moisture on the conductive powers of Rocks.*

XVI. (*a.*) A block of chalk (from the Lower Chalk near Cambridge), very moist.

l=·16 feet.

t_1.	t_2.	τ.
211°·5	162°	65°

$\frac{k}{c}$=·3.

(*b.*) The same block saturated with moisture, tried at a lower temperature.

t_1.	t_2.	τ.
123°·5	101°·4	60°

$\frac{k}{c}$=·3.

(*c.*) The same block well dried.

t_1.	t_2.	τ.
211°·5	147°	70°

$\frac{k}{c}$=·19.

Another similar block gave almost identical results.

XVII. (*a.*) Sandstone block, dry.

l=·17 feet.

t_1.	t_2.	τ.
211°	173°	63°

$\frac{k}{c}$=·49.

(*b.*) Same block, saturated with moisture.

t_1.	t_2.	τ.
129°	114°·4	60°

$\frac{k}{c}$=·62.

XVIII. (*a.*) New Red Sandstone block, dry.

l=·16 feet.

t_1.	t_2.	τ.
126°·5	100°·8	59°

$\frac{k}{c}$=·25.

(*b.*) Same block, saturated.

t_1.	t_2.	τ.
210°·4	179°·5	61°·5

$\frac{k}{c}$=·6.

XIX. (*a.*) Millstone Grit, very hard and compact, dry.

l=·166 feet.

t_1.	t_2.	τ.
210°·25	182°	60°

$\frac{k}{c}$=·71.

(*b.*) Same block, saturated with moisture.

t_1.	t_2.	τ.
210°·5	181°·2	59°

$\frac{k}{c}$=·69.

33. Other similar experiments were made, of which it seems unnecessary to repeat the details. All the results obtained have been already discussed (art. 10). I shall select an example of the experiments for simply determining the conductive power of any proposed substance. In such cases a single block (B) was generally used with an outer block (A) having a single cylindrical hole. The base of the block experimented on (B) was usually between 4 and 5 inches in diameter. In this example the temperature t_1 is much higher than that of boiling water. The reader will be able to judge from it the degree of approximation to stationary temperatures which I was able to obtain, when not availing myself of the constant temperature of boiling water. For low temperatures there was no difficulty in maintaining the temperatures very nearly stationary.

XX. Block of Sandstone.

h=·17 feet.

Time.	t_1.	t_2.	t_2'.	τ.
h m				
3 30	437°·0	.....	.....	79°
3 45	448·0			
3 50	447·5			
3 55	447·5	314°·8		
4 0	447·5			
4 5	448·2			
4 10	448·2	320·8		
4 15	447·8			
4 20	447·6			
4 25	447·1	323·8		
4 30	447·8	324·6		
4 35	447·7	325·3		
4 40	447·6	326·0		
4 45	448·1	326·4		
4 50	447·0	326·3	.....	81·5
4 55	446·8			
5 0	447·8	326·8	330°·5	
5 5	447·4			
5 10	448·0	327·3	.....	81·0
5 30	.....	Continued slow oscillations through 1°·5.	.....	81·0

Taking the mean of all the values of t_1, from $3^h\ 45^m$ to $5^h\ 10^m$ inclusive, we have t_1=447°·65; and taking the mean of the values of t_2, from $4^h\ 40^m$ to $5^h\ 10^m$ inclusive, we have t_2=326°·5 nearly. Also taking τ=81°, we obtain

$$\frac{k}{c}=\frac{245\cdot5}{121\cdot15}(\cdot17)=\cdot34.$$

XXXVIII. *Experimental Researches on the Strength of Pillars of Cast Iron from various Parts of the Kingdom.* By Eaton Hodgkinson, *Esq., F.R.S., F.G.S., Hon. Mem. Inst. Civ. E., Hon. Mem. Soc. Ing. Civ. Paris, and Professor of the Mechanical Principles of Engineering in University College, London.*

Received November 20, 1856,—Read January 8, 1857.

Having in the year 1840 offered to the Royal Society an extensive research upon this subject, which was honoured with the kindest notice of the Society, I felt grateful for the reception it had met with; and though in its preparation it had occupied my leisure time for some years, and contained the results of as many as 277 experiments, which I had made to prove the conclusions arrived at in it, I was still very anxious to improve and extend it. Indeed the importance of the subject would seem to justify every effort I could make for the purpose, when it is considered that a large portion of the houses, warehouses and shops in London, Manchester, Liverpool and throughout the country, depend for their principal supports upon iron pillars, which frequently appear very thin for the weight they have to bear, and being hollow do not allow us to judge from their appearance how small a quantity of metal they have in them, or in other words, whether the building is abundantly strong, or is ready to fall down and crush the persons within it, as has frequently happened to warehouses and other buildings dependent on iron supports. Some of the pillars are made to pass through more than one story, or even are based on the foundation, and support an intermediate floor and the roof.

The importance of the subject, in a practical point of view at least, rendered it desirable that a number of pillars of large size should be broken, to obtain data for the application of the principles established in the preceding research; but this was impracticable at that time, notwithstanding the liberality of Mr. Fairbairn, who bore the expense of that inquiry. For by Mr. Fairbairn's lever then used, more than 18 tons could not be safely applied, and the iron box or frame in which the pillars were broken did not admit pillars of greater length than 7½ feet; but the laborious inquiry in which I was afterwards engaged by Mr. Stephenson, for investigating the properties of the Menai and Conway tubular bridges (that over the Conway in particular), required larger and more powerful apparatus than the preceding, and I can now apply more than three times the pressure formerly used, and break pillars of 10 feet long, and any shorter lengths, with even more accuracy than before.

A drawing of the former machine is given with the paper (Philosophical Transactions, 1840), and of the latter, with my experiments in the Report of the Commissioners appointed to inquire into the application of Iron to Railway Structures, 1849. The principle of both machines is the same; the present is a single lever, about 17 feet,

6 inches long, acting (usually) with a leverage of 8·4 upon the upper end of a strong bolt, which moves in a cylinder, just fitting it and keeping it always in the straight line of its pressure. The top of the bolt is rounded, so that the pressure may act through the axis, and the bottom is flat, and made horizontal, that it may press on the top of a vertical pillar, enclosed in a strong box, as before. See Plate XXXI.

The present experiments, and others in progress, were undertaken at the suggestion of Mr. Robert Stephenson, M.P., F.R.S., who knew that I had several experimental researches unfinished. He has generously contributed £200 towards the expense of the inquiry, and the Royal Society has added £200*. By the kind permission of the Council of University College, London, my apparatus has been set up at the College, and the experiments made there.

I have in the present research broken only larger pillars than before, having had them cast 10 feet long, and from 2½ to 4 inches in diameter, solid and hollow. The first series includes twenty pillars (twelve of 10 feet long, and eight shorter ones cut out of them); the whole of these were supplied by the proprietors of the Low Moor Iron-works, for which no charge has yet been made†. The iron of which these, with five exceptions, were made, was Low Moor iron, No. 2. The former research of 1840 was from Low Moor iron, No. 3; and the difference of the two is not very great, though the No. 2, as now used, is somewhat stronger than the other to bear a strain as a long pillar (p. 864), though softer than it and more easily crushed. The results of these experiments have been applied to the formulæ in my former research, which have undergone no change, except in the constants.

The former researches, and the formulæ to which they were reduced, had been applied to pillars of Low Moor iron only, with one exception; but though we could compute by them the strength for Low Moor iron, we had no evidence as to what were the relative strengths of pillars of the irons of the kingdom in general; and, at my request, Messrs. Easton and Amos have had cast for this purpose twenty-two solid pillars, each 10 feet long and 2½ inches in diameter, of eleven kinds of iron, making thirteen kinds with two I have had from Low Moor. The comparison of the breaking weights of these pillars shows that the strengths of the irons vary at least as 2 to 3, and their crushing strengths as 5 to 9 nearly‡.

In carrying out the experiments, in taking the dimensions, and applying the results to the formulæ in my research above mentioned, I have had the assistance of Mr. John Bridge, M.A., of University College, my health and eyesight not being sufficient to enable me to bear the labour alone.

* Since finishing the experiments in the present research, except those contained in the short Appendix, another £100 has been contributed by the Royal Society, and £100 more by Mr. Stephenson, with which I am making experiments on pillars of timber, &c.

† A good balk of Memel timber was likewise sent gratuitously for experiment by the late Mr. Thomas Cubitt, of Thames Bank, but his death prevented me having the power to thank him. His executors have liberally sent other timber for the purpose.

‡ The pillars were cast 12 feet long, to increase their soundness, and afterwards cut to 10 feet long.

1. Before referring to the experiments which form the basis of the present memoir, I will give some notice of the principal matters contained in the former, but not sufficiently dilated upon; especially as they have an important bearing upon what will follow, and will themselves receive further illustration, or perhaps be capable of demonstration.

The profound and beautiful researches of EULER on the strength of pillars* are entirely theoretical; they proceed on the supposition that the strength of a pillar is bounded by its power of resisting being bent out of a straight line, the resistance to incipient flexure being supposed to be the measure of its strength, which in long pillars, whose length is l and external and internal diameters d and d_1, is $\frac{d^4}{l^2}$ or $\frac{d^4-d_1^4}{l^2}$, according as they are solid or hollow†.

2. In commencing experiments, in my former research, on this subject, and keeping in view the theory of EULER, I sought with great care for the weight which would produce incipient flexure in columns, and more particularly in those of cast iron. In this metal flexure commenced with very small weights, much smaller than would be useful to load pillars with in practice; and I became convinced that no such point existed in cast iron, or at any rate none that would be useful to the engineer; and my subsequent experiments upon wrought-iron pillars have been attended with very little more success, in seeking for the weight producing incipient flexure. Having been unsuccessful in seeking for the weight which would first produce flexure in columns, and being convinced that if it were found it would be of little or no use in practice, I sought in future for the weights necessary to *break* the pillars tried, and in most instances for the deflection and decrement of length produced by the weights laid on.

3. The pillars broken were placed in a vertical position during the experiments, and their ends were pressed between two horizontal plates of hardened steel, which, from the nature of the machine, were kept perfectly parallel. It was found that in cylindrical pillars of cast iron, whose ends were turned perfectly flat, parallel and perpendicular to the axis, the breaking weight varied as the 3·55th power of the diameter nearly, and inversely as the 1·7th power of the length, or as $\frac{d^{3\cdot55}}{l^{1\cdot7}}$, instead of $\frac{d^4}{l^2}$, as in the theory of EULER. In hollow pillars it is as $\frac{d^{3\cdot55}-d_1^{3\cdot55}}{l^{1\cdot7}}$. The fractional indices above were found to apply to all solid pillars of cast iron, whose lengths varied from 120 times the diameter to as low as 30 times the diameter, or thereabouts. For pillars shorter than these it will be necessary to correct the results obtained, by considerations dependent on the crushing strength of the material. See pp. 403 to 406 of the former research‡.

4. The first experiments were made upon long uniform pillars, with their ends rounded in such a manner that the pressure applied to them would act in the direction of their axis, to meet the requirements of EULER's theory of the strength of pillars; but

* Berlin Memoirs, 1757.

† POISSON, Mécanique, 2nd edition.

‡ Philosophical Transactions, 1840, Part II.

it soon became evident that pillars of the same size with their ends flat, and well supported or bedded throughout, would require a much greater weight to break them than those with rounded ends; a large number of both kinds were made and broken by experiment to determine their relative strength to resist fracture, and it was found that so long as the pressure required had not been more than about one-fourth of that necessary to crush short specimens of the same material, the strength of a solid pillar with flat ends was a little more than three times that of one with rounded ends, the mean being 3·167, as shown in the abstract, page 387 of that research, but in other experiments it is still nearer to 3.

5. If the pressure necessary to break the pillar were much greater than about one-fourth of that which would crush short specimens of the same material, then the ratio 1 to 3 would become 1 to 2, or even less.

6. In the following abstract, pages 856 and 857, which includes experiments on hollow pillars with rounded and with flat ends, as above, and upon solid pillars of wrought iron and timber, the ratio of the strengths of the two kinds of pillars is nearly as 1 to 3, varying not widely from the former. From this abstract we see that pillars which have been loaded with more than about one-fourth of the crushing weight are in some degree injured, and the ratio in them differs from 1 : 3, becoming 1 : 2·6, 1 : 2·4, &c.

In the last column of the abstract, page 857, denominated $\frac{B}{C}$, it may be seen at a glance what portion of the crushing weight was required to break each pillar; and in those where it amounted to ·282, ·272, or above ·25*, they were generally injured by the pressure.

7. In the abstract, p. 393 of the former research, the relative strengths of pillars—with both ends rounded, one end rounded and one end flat, and both ends flat,—are determined in a great number of experiments, and found to be nearly as 1, 2, 3, from experiments on cast iron, wrought iron, steel and timber.

8. In the abstract, page 391 of the former research, pillars 60½ inches long, with their ends turned flat, are compared, as to their strengths, with pillars of the same size with discs on their ends turned flat: the strengths of these two classes of pillars differed but little, but the pillars with discs on the ends are rather the stronger. In this abstract it was shown that pillars one-half the length of the preceding, and of the same diameter but with the ends rounded, were of the same strength as the preceding ones nearly, as shown by many experiments.

Attempt to account for the principal Results in the preceding description of the abstracts.

1. Of the general conclusion arrived at in the abstract last referred to, I would offer the following as an explanation. Suppose a long uniform bar of cast iron were bent by a pressure at its ends so as to take the form A*bcdef*B, where all the curves A*bc*, *cde*, *ef*B, separated by the straight line A*ce*B, would be equal, since the bar was supposed to

* The pillars of the "Second London Mixture," 6 feet 3 inches long, gave the same result when $\frac{B}{C} = ·26$.

be uniform. The curve having taken this form, suppose it to be rendered immoveable at the points *b* and *f*, by some firm fixings at those points. This done, it is evident we may remove the parts near to A and B, without at all altering the curve *bcdef* of the part of the pillar between *b* and *f*, and consider only that part. The part *bf*, which alone we shall have to consider, will be equally bent at all the points *b*, *d*, *f*. The points *c* and *e* too are points of contrary flexure, consequently the pillar is not bent in them. These points are unconstrained except by the pressure which forces them together, and the pillar might be reduced to any degree in them, provided they were not crushed or detruded by the compressing force. These points may then be conceived as acting like the rounded ends of the pillars in Table I. in the former research, and the part *cde* of the pillar, with its ends *c* and *e* rounded, will be bearing the same weight as the whole pillar *bcdef* of double the length with its ends *b*, *f* firmly fixed.

2. That the strain at all the points *b*, *d*, *f* is nearly equal, is shown by subsequent experiments, in Tables V. and VI., upon pillars varying in length from 6 feet 3 inches to 5 feet and 2 feet 6 inches; that fracture in some cases took place at all three points at the same time, and in others at two of them, showing the fracture to have commenced on opposite sides of the middle and of the ends, as might be inferred from the foregoing figure, or the last in Plate XXXII.

3. That the strength of a pillar, with moveable or rounded ends, is equal to that of a pillar of the same diameter, and double the length, the ends being flat or firmly fixed, may be inferred from the reasoning above; but it is *proved* experimentally from various results in the abstract p. 391 of the former research, or as below.

4. The strengths of pillars of the same diameter, but of different lengths, vary inversely as the 1·7th or 1·6th power of the length, or in this case as $2^{1·7}$ or $2^{1·6}$ nearly, according to my experiments. Hence the strength of a pillar one-half the length of another is 3·249 or 3·031 times greater than it; or in other words, the strength of a pillar with flat ends is somewhat more than three times as great as that of one of the same length and diameter with rounded ends.

5. This is shown in the abstract, p. 387 of the former research, from numerous experiments on *solid* pillars, with both ends rounded and both ends flat, the mean ratio of their strengths being . . . . . 1:3·167

In the abstract, pages 856 and 857 of the present research, the mean ratio from the comparison of the strengths of four *hollow* pillars, with rounded ends and of the like number with flat ends, was . . . . . 1:3·077

In the same abstract, solid pillars of the two forms, of wrought iron and of timber, gave from several experiments a mean ratio of . . . . . 1:3·076

} Mean ratio 1:3·107.

Abstract from Tables VIII., IX., XII., XIII., &c. of the former Research on the results obtained from two hollow

	Pillars with rounded ends.				Pillars
	Length of pillar.	Diameter of section.	Number of experiments deduced from.	Breaking weight.	Length of pillar.
Hollow columns of Low Moor iron, No. 3.	7 ft. 6½ in., or 90¾ in.	External diameter 1·78 in. Internal diameter 1·21 in.	1	5585	7 ft. 4½ in., or 88½ in.
	90¼ in.	External diameter 1·74 in. Internal diameter 1·187 in.	1	5711	88¼ in.
	90¼ in.	External diameter 2·01 in. Internal diameter 1·415 in.	1	8357	88¼ in.
	90¼ in.	External diameter 2·23 in. Internal diameter 1·54 in.	1	12389	88¼ in.
Hollow columns of Low Moor iron, No. 2.	10 ft. 2 in., or 122 in., including caps on the ends.	External diameter 3·555 in. Internal diameter 2·693 in.	1	42331 lbs., or 43537 lbs. for 10 ft. or 120 in.	120 in.
	122 in., including caps on the ends.	External diameter 3·532 in. Internal diameter 2·659 in.	1	44683 lbs., or 45956 lbs. for 10 ft. or 120 in.	120 in.
Solid pillars. Wrought iron.	90¾ in.	Diameter 1·017 in.	2	1808	90¾ in.
	60½ in.	Diameter 1·015 in.	2	3938	60½ in.
Timber.	60½ in.	Side of square 1·75 in.	2	3197 The second pillar was capped with iron at the ends, to prevent their being crushed.	60½ in.

Strength of Pillars of Low Moor Iron, No. 3, and of other Materials, with the pillars in the present research.

with flat ends, and sometimes with discs to give them a larger bearing at the ends.

Diameter of section.	Number of experiments deduced from.	Breaking weight (B).	Weight which would crush the specimen (C).	Ratio of the strengths of pillars with both ends rounded and with both flat, the diameters being considered equal.	$\frac{B}{C}$
			lbs.		
External diameter 1·78 in. Internal diameter 1·21 in. Area of section 1·339 sq. in.	1	17840 lbs., or 17177 lbs. for length 90¾ in.	146975	1 : 3·075	·1237
External diameter 1·74 in. Internal diameter 1·187 in. Area of section 1·2712 sq. in.	1	16705 lbs., or 16084 lbs. for length 90¾ in.	139584	1 : 2·816	·115
External diameter 2·01 in. Internal diameter 1·415 in. Area of section 1·6005 sq. in.	1	28353 lbs., or 27299 lbs. for length 90¾ in.	175742	1 : 3·266	·155
External diameter 2·23 in. Internal diameter 1·54 in. Area of section 2·0431 sq. in.	1	40569 lbs., or 39061 lbs. for length 90¾ in.	224330	1 : 3·153	·174
		Crushing weight per square inch of section 109601 lbs. = 49·018 tons.		Mean ratio 1 : 3·077	
External diameter 3·555 in. Internal diameter 2·693 in. Area of section 4·230 sq. in.	1	114479 lbs. = 51·11 tons.	405774	1 : 2·629	·282
External diameter 3·532 in. Internal diameter 2·659 in. Area of section 4·245 sq. in. Weight of pillar 130 lbs.	1	110649 lbs. = 49·39 tons.	407203	1 : 2·408	·272 This pillar had been split at one end in the first experiment, and was hooped.
		Crushing weight per square inch 95938 lbs. = 42·825 tons.			
Diameter 1·02	1	5280	.....	1 : 2·920	
Diameter 1·02	1	12990	.....	1 : 3·298	
Side of square 1·75 in.	4	9625	.....	1 : 3·011	
				Mean ratio 1 : 3·076	

6. The great difference between the strength of a pillar with both ends rounded and both ends flat, is rendered obvious by their different modes of fracture, the former breaking in one place only, the middle; and the latter in three, at the middle and at each end.

7. In pillars with one end rounded and one end flat, fracture takes place at about one-third of the distance from the rounded end; and if the base of the pillar be enlarged with a disc, the distance of the place of fracture from the top of a pillar 30½ inches long is but ·28 inch from one-third of the whole length from the top, deduced from a mean of two experiments, in which one pillar broke at 10·42 inches, and the other at 10·30 inches (mean, 10·36 inches) from the rounded end (see p. 439 of former research).

8. If the pillar had had both ends rounded, then the centre of fracture would have been at the middle, at *b* (first pillar, Plate XXXII.), or at half the height: in the cases before us, fracture took place at two-thirds of the height nearly, or at *b* (second pillar). Hence, comparing the distances A*b*, A*b* of the points of fracture from the rounded ends of two pillars of equal diameter and height, one with both ends rounded, and the other with one end rounded and one end flat, we have those distances as ½ to ⅓, or as 3 to 2 nearly.

9. The relative strengths of the two pillars should therefore be inversely as the 1·7th or 1·6th powers of their lengths, A*b*, A*b*, or as 3·249:6·473, or 3·031:5·8, differing not widely from 1 to 2, as is shown by various experiments in Tables I., II., V., XII., XIII. of the former research.

10. Moreover, it will appear that in the curve indicating the form of flexure to the third pillar, Plate XXXII., the distance A*bc* will apply to the first pillar, A*bcd* to the second pillar, and *bcdef* to the third pillar; these distances being inversely as 2, 3, 4, the 1·7th powers of which are 3·249, 6·473, 10·556, and the 1·6th powers 3·031, 5·8, 9·19, or as 1, 2, 3, nearly.

11. Hence, of three pillars of the same length and diameter, one with both ends rounded, one with one end rounded and one end flat, and the other with both ends flat, the strengths are as 1, 2, 3.

The following facts, from the subsequent experiments, illustrate the last of the preceding figures. See also forms of fracture, Plate XXXIII.

In pillars of "Second London Mixture," 2 feet 6 inches long and 1½ inch diameter, fracture took place near to the middle and by breaking across near each end.

A pillar of Old Hill iron, 5 feet long and 2½ inches diameter, broke in the middle, and broke across near each end.

In pillars 6 feet 3 inches long, fracture took place in the middle, and each end became split. This was the case in all these pillars.

In pillars 7 feet 6 inches long and 2½ inches in diameter—that of Derwent iron broke in the middle and split off at each end; one of Old Park and one of London Mixture broke in three pieces, and split off at one end: there would appear to have been more points of contrary flexure in each of these two pillars than in the other cases.

General Remarks on the following Experiments.

The experiments forming the basis of this research are on the strength of cast-iron pillars in two series, the first being on Low Moor Iron, No. 2, each 10 feet long, varying in diameter from $2\frac{1}{2}$ to 4 inches, solid and hollow; of these there are fourteen, including two on Blaenavon iron, No. 3; the second series are on solid pillars of cast iron, 10 feet long and $2\frac{1}{2}$ inches diameter, from various parts of the kingdom, making twenty-four pillars, with two of a smaller diameter; other pillars cut out of the preceding ones, and made 7 feet 6 inches, 6 feet 3 inches, 5 feet, &c. long, amount with the former ones to seventy-two.

The experiments on these are distributed in Tables I. to VI. In two other Tables (VII. and VIII.) other experiments are given, to determine the transverse strengths of some of the irons, in order to compare these with their direct strength as pillars, besides numerous experiments to obtain the crushing strengths of the pillars, and the comparative hardness of the irons in different parts of their sections. The forms of the fractures of pillars in the first six Tables are given in Plate XXXIII.

It has been seen that the strength of long pillars of cast iron varies as a constant power n of the diameter nearly; the pillars being solid and uniform in texture, and nearly as the difference of the same powers of the external and internal diameters, when the pillars are hollow; or as d^n in solid pillars, and $D^n - d^n$ in hollow ones, D and d being the external and internal diameters. This power n was found in my former researches* to be 3·55, from the mean result of many experiments on the strength of pillars flat at the ends. In the following experiments, which are on pillars of a larger kind than before, and the iron therefore somewhat softer than iron cast in smaller masses, the average value of n is much more near to 3·50 than to 3·55, the mean from nineteen comparisons being 3·513 (page 865). The strength of pillars, whose diameter is the same, varied inversely as the 1·7th power of the length nearly, according to the average result from pillars with rounded ends and with flat ends, in my former experiments. But in the present ones, on flat-ended pillars only, the power of the length from the results of twenty-five comparisons (page 866) is 1·63 nearly, instead of 1·7, as before found.

The preceding results apply only to pillars whose length is thirty times the diameter or upwards; for pillars shorter than this, the formula for their strength would need correcting by the introduction of the crushing weight of the pillar†.

Tables of the powers above for pillars likely to occur in practice are as below.

* Philosophical Transactions, 1840. † Ibid. Part II. pp. 404, 405.

TABLE I.—Powers of Diameters, or $d^{2\cdot5}$.

$1\cdot0^{2\cdot5} = 1\cdot0000$	$3\cdot7^{2\cdot5} = 97\cdot433$	$5\cdot7^{2\cdot5} = 442\cdot14$	$7\cdot7^{2\cdot5} = 1266\cdot83$
$1\cdot25^{2\cdot5} = 2\cdot1837$	$3\cdot75^{2\cdot5} = 102\cdot12$	$5\cdot75^{2\cdot5} = 455\cdot87$	$7\cdot75^{2\cdot5} = 1295\cdot85$
$1\cdot5^{2\cdot5} = 4\cdot1335$	$3\cdot8^{2\cdot5} = 106\cdot965$	$5\cdot8^{2\cdot5} = 469\cdot89$	$7\cdot8^{2\cdot5} = 1325\cdot35$
$1\cdot75^{2\cdot5} = 7\cdot0898$	$3\cdot9^{2\cdot5} = 117\cdot15$	$5\cdot9^{2\cdot5} = 498\cdot86$	$7\cdot9^{2\cdot5} = 1385\cdot78$
$2\cdot0^{2\cdot5} = 11\cdot314$	$4\cdot0^{2\cdot5} = 128\cdot00$	$6\cdot0^{2\cdot5} = 529\cdot09$	$8\cdot0^{2\cdot5} = 1448\cdot15$
$2\cdot1^{2\cdot5} = 13\cdot4205$	$4\cdot1^{2\cdot5} = 139\cdot55$	$6\cdot1^{2\cdot5} = 560\cdot60$	$8\cdot25^{2\cdot5} = 1612\cdot83$
$2\cdot2^{2\cdot5} = 15\cdot7935$	$4\cdot2^{2\cdot5} = 151\cdot835$	$6\cdot2^{2\cdot5} = 593\cdot43$	$8\cdot5^{2\cdot5} = 1790\cdot47$
$2\cdot25^{2\cdot5} = 17\cdot086$	$4\cdot25^{2\cdot5} = 158\cdot26$	$6\cdot25^{2\cdot5} = 610\cdot35$	$8\cdot75^{2\cdot5} = 1981\cdot66$
$2\cdot3^{2\cdot5} = 18\cdot452$	$4\cdot3^{2\cdot5} = 164\cdot87$	$6\cdot3^{2\cdot5} = 627\cdot61$	$9\cdot0^{2\cdot5} = 2187\cdot00$
$2\cdot4^{2\cdot5} = 21\cdot416$	$4\cdot4^{2\cdot5} = 178\cdot68$	$6\cdot4^{2\cdot5} = 663\cdot18$	$9\cdot25^{2\cdot5} = 2407\cdot11$
$2\cdot5^{2\cdot5} = 24\cdot705$	$4\cdot5^{2\cdot5} = 193\cdot305$	$6\cdot5^{2\cdot5} = 700\cdot16$	$9\cdot5^{2\cdot5} = 2642\cdot61$
$2\cdot6^{2\cdot5} = 28\cdot340$	$4\cdot6^{2\cdot5} = 208\cdot76$	$6\cdot6^{2\cdot5} = 738\cdot59$	$9\cdot75^{2\cdot5} = 2894\cdot12$
$2\cdot7^{2\cdot5} = 32\cdot3425$	$4\cdot7^{2\cdot5} = 225\cdot08$	$6\cdot7^{2\cdot5} = 778\cdot51$	$10\cdot0^{2\cdot5} = 3162\cdot28$
$2\cdot75^{2\cdot5} = 34\cdot488$	$4\cdot75^{2\cdot5} = 233\cdot58$	$6\cdot75^{2\cdot5} = 799\cdot03$	$10\cdot25^{2\cdot5} = 3447\cdot73$
$2\cdot8^{2\cdot5} = 36\cdot733$	$4\cdot8^{2\cdot5} = 242\cdot295$	$6\cdot8^{2\cdot5} = 819\cdot94$	$10\cdot5^{2\cdot5} = 3751\cdot13$
$2\cdot9^{2\cdot5} = 41\cdot533$	$4\cdot9^{2\cdot5} = 260\cdot43$	$6\cdot9^{2\cdot5} = 862\cdot92$	$10\cdot75^{2\cdot5} = 4073\cdot14$
$3\cdot0^{2\cdot5} = 46\cdot765$	$5\cdot0^{2\cdot5} = 279\cdot51$	$7\cdot0^{2\cdot5} = 907\cdot49$	$11\cdot0^{2\cdot5} = 4414\cdot43$
$3\cdot1^{2\cdot5} = 52\cdot4525$	$5\cdot1^{2\cdot5} = 299\cdot57$	$7\cdot1^{2\cdot5} = 953\cdot68$	$11\cdot25^{2\cdot5} = 4775\cdot66$
$3\cdot2^{2\cdot5} = 58\cdot617$	$5\cdot2^{2\cdot5} = 320\cdot635$	$7\cdot2^{2\cdot5} = 1001\cdot53$	$11\cdot5^{2\cdot5} = 5157\cdot54$
$3\cdot25^{2\cdot5} = 61\cdot886$	$5\cdot25^{2\cdot5} = 331\cdot56$	$7\cdot25^{2\cdot5} = 1026\cdot08$	$11\cdot75^{2\cdot5} = 5560\cdot74$
$3\cdot3^{2\cdot5} = 65\cdot283$	$5\cdot3^{2\cdot5} = 342\cdot74$	$7\cdot3^{2\cdot5} = 1051\cdot07$	$12\cdot0^{2\cdot5} = 5985\cdot96$
$3\cdot4^{2\cdot5} = 72\cdot473$	$5\cdot4^{2\cdot5} = 365\cdot91$	$7\cdot4^{2\cdot5} = 1102\cdot33$	
$3\cdot5^{2\cdot5} = 80\cdot212$	$5\cdot5^{2\cdot5} = 390\cdot18$	$7\cdot5^{2\cdot5} = 1155\cdot35$	
$3\cdot6^{2\cdot5} = 88\cdot5235$	$5\cdot6^{2\cdot5} = 415\cdot58$	$7\cdot6^{2\cdot5} = 1210\cdot17$	

TABLE II.—Powers of Lengths, or $l^{1\cdot69}$.

$1^{1\cdot69} = 1\cdot$	$6\frac{1}{4}^{1\cdot69} = 19\cdot8282$	$12^{1\cdot69} = 57\cdot4203$	$19^{1\cdot69} = 121\cdot442$
$2^{1\cdot69} = 3\cdot0951$	$7^{1\cdot69} = 23\cdot8512$	$13^{1\cdot69} = 65\cdot4226$	$20^{1\cdot69} = 132\cdot032$
$2\frac{1}{4}^{1\cdot69} = 4\cdot4529$	$7\frac{1}{4}^{1\cdot69} = 26\cdot6901$	$14^{1\cdot69} = 73\cdot8225$	$21^{1\cdot69} = 142\cdot961$
$3^{1\cdot69} = 5\cdot9939$	$8^{1\cdot69} = 29\cdot6508$	$15^{1\cdot69} = 82\cdot6093$	$22^{1\cdot69} = 154\cdot223$
$4^{1\cdot69} = 9\cdot5798$	$9^{1\cdot69} = 35\cdot9265$	$16^{1\cdot69} = 91\cdot7731$	$23^{1\cdot69} = 165\cdot812$
$5^{1\cdot69} = 13\cdot7823$	$10^{1\cdot69} = 42\cdot6580$	$17^{1\cdot69} = 101\cdot305$	$24^{1\cdot69} = 177\cdot723$
$6^{1\cdot69} = 18\cdot5518$	$11^{1\cdot69} = 49\cdot8276$	$18^{1\cdot69} = 111\cdot197$	

Further Remarks on the Experiments on the Strength of Pillars (Tables I. and II.).

If the strength of a pillar be represented by $W = m \times \frac{D^n - d^n}{l^p}$, where D, d are the external and internal diameters in inches, l the length in feet, m, n, p constant quantities, the following experiments (1st Summary) will furnish us with the value of

$\frac{W}{D^n - d^n} = \frac{n}{p}$, of which expressions the first is represented by x, and is calculated on two suppositions for the value of n; the other depends on the value of p. From my former experiments, n was taken as 3·55, and p as 1·7; and from the experiments in the present research, we have $n = 3{\cdot}5$, and $p = 1{\cdot}63$, as mentioned before. To assist in computing the strength of pillars, according to this last supposition, the two Tables of powers of the diameters and lengths of pillars previously given (page 860) have been formed. The experiments give the mean value of x, or $\frac{W}{D^n - d^n}$ as 2085·2 or 2223·7 lbs. for the strength of a pillar 10 feet long and 1 inch diameter; but for a pillar of 1 foot long and 1 inch diameter it would be $x \times 10^p$, where $10^p = 50{\cdot}119$, or 42·658.

First Summary of Results.

Results from the Experiments in Tables I. and II. upon pillars of Low Moor Iron, No. 2, and Blaenavon Iron, No. 3, the length of each being 10 feet, and the ends turned flat and perpendicular to the axis.

No. of experiment.		Value of x, or the breaking weight of a solid pillar 10 feet long and 1 inch diameter, flat at the ends, from the formula $x = \frac{W}{D^{3{\cdot}55} - d^{3{\cdot}55}}$		Value of x from the formula $x = \frac{W}{D^{3{\cdot}5} - d^{3{\cdot}5}}$		Error from using the mean from power 3·55, in terms of the value of x.	Error from using the mean from power 3·5, in terms of the value of x.
		lbs.		lbs.			
Hollow pillars, from 2½ to 4 inches external diameter ...	1.	2201·1	Low Moor, No. 2. Mean 2085·2	2324·9	Mean 2223·7	$-\frac{1}{19{\cdot}3}$	$-\frac{1}{20}$
	2.	2332·9		2463·7		$-\frac{1}{9{\cdot}4}$	$-\frac{1}{10{\cdot}3}$
	3.	2226·7		2374·2		$-\frac{1}{15{\cdot}7}$	$-\frac{1}{15{\cdot}9}$
	4.	2285·8		2471·3		$-\frac{1}{11{\cdot}4}$	$-\frac{1}{10}$
	5.	1975·6		2121·7		$+\frac{1}{19{\cdot}02}$	$+\frac{1}{20{\cdot}9}$
	6.	2023·3		2173·9		$+\frac{1}{29{\cdot}7}$	$+\frac{1}{43{\cdot}7}$
	7.	1963·8		2127·6		$+\frac{1}{18{\cdot}3}$	$+\frac{1}{29{\cdot}1}$
	8.	1785·3		1928·8		$+\frac{1}{8{\cdot}96}$	$+\frac{1}{8{\cdot}54}$
Solid pillars, 2½ inches in diameter.....	11.	2008·9		2105·2		$+\frac{1}{26{\cdot}3}$	$+\frac{1}{17{\cdot}9}$
	12.	2048·5		2146·1		$+\frac{1}{55{\cdot}6}$	$+\frac{1}{37{\cdot}7}$
	13.	2034·1	Blaenavon, No. 3. Mean 2065·4	2131·8	Mean 2164·4	$+\frac{1}{65}$	$+\frac{1}{65{\cdot}4}$
	14.	2096·7		2197·0		$-\frac{1}{67}$	$-\frac{1}{67{\cdot}4}$

In the first two experiments of the preceding summary, the results are perhaps somewhat too high, though great care was taken to avoid error. In the columns (Experiments 7 and 8) they are too low. These had been weakened by being turned on the outside to reduce their strength, as they had been cast stronger than it was supposed the

machinery would break without injury to it. The great reduction in strength produced by taking away the external crust, shows that to ornament a pillar it would not be prudent to plane it. It may, however, be mentioned that the eighth pillar had many flaws in it.

In experiments upon hollow pillars, it is frequently found that the metal on one side is much thinner than that on the other, but this does not produce so great a diminution in the strength as might be expected, for the thinner part of a casting is much harder than the thicker (Table VIII.), and this usually becomes the compressed side.

The strength of a solid pillar, 10 feet long and 1 inch diameter, being, from the first mean above, 2085·2 lbs., that of a solid pillar, 1 foot long and 1 inch diameter, will, since $10^{1·7}=50·119$, be $2085·2\times50·119=104508$ lbs., or 46·65 tons; the strength being supposed to vary directly as the 3·55 power of the diameter, and inversely as the 1·7 power of the length, as obtained from the experiments in my former paper on this subject*.

The larger and more varied pillars in the present research, have, by experiments deduced from most of the irons in the kingdom, given more precise results for the constants on which the strength depends. They show that it varies directly as the 3·5 power of the diameter, much more nearly than as the 3·55 power, and inversely as the 1·63 power of the length, from a mean of many experiments in each case (see pages 865 and 866).

From the second mean value of x in the preceding summary, that depending on the 3·5 power of the diameter, the strength of a pillar 10 feet long and 1 inch diameter is 2223·7 lbs., and this multiplied by $10^{1·63}$ or 42·658 (see Table II. page 860), gives 94858·59 lbs. = 42·347 tons, for the strength of a pillar 1 foot long and 1 inch diameter.

Whence w being the breaking weight in tons of a pillar whose length is l in feet, and external and internal diameters D and d in inches, the ends being flat and well-bedded, we have, in Low Moor iron, No. 2—

$$w=46·65\times\frac{D^{3·55}-d^{3·55}}{l^{1·7}},\text{ from formulæ in Philosophical Transactions, 1840;}$$

$$w=42·347\times\frac{D^{3·5}-d^{3·5}}{l^{1·63}},\text{ from formulæ in the present paper.}$$

The preceding chapter contains the results of experiments on the strength of pillars 10 feet long, solid as well as hollow, the hollow ones varying from $2\frac{1}{2}$ to 4 inches external diameter, and the solid ones cast from models $2\frac{1}{2}$ inches diameter. They are all of Low Moor iron, No. 2, except two, Nos. 13 and 14, of Blaenavon iron. In the previous research† the iron used was that of Low Moor, No. 3, and there appears to be very little difference in their strengths, though the No. 2 seems to be rather the stronger. The experiments in Table IX. of the previous paper are on hollow pillars 1·75 inch external diameter and upwards; and on solid pillars of the same diameter nearly, the

* Philosophical Transactions, 1840, Part II. † Ibid.

length being 7 feet 6¼ inches. To compare the results of this latter series of pillars with those of a larger kind in the present research, we will select three hollow pillars, among the least objectionable in the series, and two solid ones, and reduce their results to what they would have been if the pillars had been 1 inch diameter and 10 feet long.

Pillars of Low Moor Iron, No. 3.

Length.		Diameter.	Breaking weight.	Value of x, or strength of a pillar 1 inch diameter and 10 feet long, from formula $x=\frac{W}{D^{3\cdot55}-d^{3\cdot55}}\times\frac{l^{1\cdot7}}{10^{1\cdot7}}$.		Value of x, or breaking weight of a pillar 1 inch diameter and 10 feet long, from formula $x=\frac{W}{D^{3\cdot5}-d^{3\cdot5}}\times\frac{l^{1\cdot63}}{10^{1\cdot63}}$.	
	in.		lbs.	lbs.		lbs.	
Hollow pillars...	90¼	External diameter 1·75 in. Internal diameter 1·11 in.	20957	2230·7	Mean 2166·5	2352·8	Mean 2338·2
	88¼	External diameter 2·04 in. Internal diameter 1·46 in.	32413	2222·6		2369·7	
	90¼	External diameter 2·01 in. Internal diameter 1·368 in.	30789	2046·3		2292·2	
Solid pillars ...	90¼	Diameter 1·76 in.	23179	1937·6	Mean 1966·2	2032·5	Mean 2061·4
	90¼	Diameter 1·72 in.	21995	1994·9		2090·3	
The length l is taken in feet = 7·5625 feet.							

Comparing the results of the pillars in the abstract above, which are on Low Moor iron, No. 3, with those in the first Summary, which are on the same iron, No. 2, and taking the value of x as a mean from the first six hollow pillars, we have as below:—

On the first supposition, that is, using the power 3·55 of the diameter, and 1·7 of the length,—

From the first summary, 2028·7 lbs. for solid, and 2174·2 for hollow pillars.
From the above abstract, 1966·2 lbs. for solid, and 2166·5 for hollow pillars.

On the second supposition, that is, using the power 3·5 of the diameter, and 1·63 of the length—

From the first summary, 2125·6 lbs. for solid, and 2321·6 for hollow pillars.
From the above abstract, 2061·4 lbs. for solid, and 2338·2 for hollow pillars.

These results are nearly in accordance with one another, and they show in both series of experiments, and according to both suppositions, that the hollow cylinders are somewhat stronger than the solid ones; which is attributable, no doubt, to the superior hard-

ness of the metal in the outer ring of the hollow castings: on this subject we shall see more further on (page 867).

According to the first supposition, the values of x for a hollow pillar and for a solid pillar are nearly as 22 to 20, and on the second supposition, nearly as 23 to 21. Of the two irons, the No. 2 seems to be somewhat the stronger, as appears also from the next article.

Relative Strengths of solid Pillars of Low Moor Iron, No. 3, and those of No. 2 of the same Iron.

From the experimental researches* we have for solid cylindrical pillars of Low Moor iron, No. 3, as a mean from ten experiments on small pillars, $w = 98922 \times \frac{d^{3\cdot55}}{l^{1\cdot7}}$, where w is the breaking weight in lbs., d the diameter in inches, l the length in feet, 3·55 the power of the diameter, and 1·7 the inverse power of the length on which the strength depends, nearly, and 98922 lbs. the strength of a pillar 1 foot long and 1 inch diameter. To make the comparison above, we shall adduce the results of the pillars in Table II. Exp. 11, 12, and Table IV. Exp. 11 of this research, from which we obtain, by the same method of computation, for the strength of a pillar 1 foot long and 1 inch diameter of Low Moor iron, No. 2,—

	lbs.
From Table II. Exp. 11	100684
From Table II. Exp. 12	102668
From Table IV. Exp. 11	100518
Mean	101290

Whence the strengths of Low Moor iron, No. 3 and No. 2, are as 98922 to 101290, or as 1 to 1·02394, by which the strengths of the pillars of Low Moor iron, No. 3, must be multiplied to obtain those of No. 2.

Two other small solid pillars in the preceding abstract, compared with the three in the present, give 98544 to 101290, or 1 to 1·02786.

Strength of Pillars dependent on their Lateral Dimensions.

In long pillars with their ends flat we have†, in cylinders 60½ inches long, comparing the strength of that of

·77	inch diameter with that of ·997	inch diameter, $n = 3\cdot606$
·77	inch diameter with that of 1·29	inch diameter, $n = 3\cdot641$
·77	inch diameter with that of 1·56	inch diameter, $n = 3\cdot495$
·997	inch diameter with that of 1·29	inch diameter, $n = 3\cdot670$
·997	inch diameter with that of 1·56	inch diameter, $n = 3\cdot428$
	Mean . . .	$n = 3\cdot568$

* Philosophical Transactions, 1840, Part II. p. 402.

† Ibid. p. 397.

In pillars of the same length with discs at the ends, comparing

·51 inch diameter with 1·00 inch diameter, $n=3·922$
·51 inch diameter with 1·28 inch diameter, $n=3·820$
·51 inch diameter with 1·53 inch diameter, $n=3·775$
·775 inch diameter with 1·53 inch diameter, $n=3·568$
·775 inch diameter with 1·28 inch diameter, $n=3·578$
1·00 inch diameter with 1·53 inch diameter, $n=3·412$
Mean . . . $n=3·679$

In the larger pillars of the present research, each 10 feet long, and cast from models varying from $1\frac{1}{2}$ to $2\frac{1}{2}$ inches diameter, the iron being that termed "Second London Mixture" (see Table III. Exp. 21 to 24), comparing the strengths of pillars of the following diameters, with their ends turned flat, we have

1·530 inch diameter with 2·511 inches diameter, $n=3·5016$
1·541 inch diameter with 2·511 inches diameter, $n=3·6154$
1·530 inch diameter with 2·496 inches diameter, $n=3·3708$
1·541 inch diameter with 2·496 inches diameter, $n=3·4840$
Mean . . . $n=3·4929$

In the same pillars rendered quite straight, and cut to be 6 feet 3 inches long, the ends being turned flat (see Table V. Exp. 5 to 9), we have, by comparing, as before,

1·530 inch diameter with 2·511 inches diameter, $n=3·3068$
1·541 inch diameter with 2·511 inches diameter, $n=3·2603$
1·530 inch diameter with 2·496 inches diameter, $n=3·1670$
1·541 inch diameter with 2·496 inches diameter, $n=3·1179$
Mean . . . $n=3·2130$

Mean from all the above comparisons, nineteen in number, $=3·513$.

Strength of Pillars dependent on their Length.

To find the inverse power of the length on which the strength of solid pillars of cast iron, cast from the same model, depends, the pillars being of various lengths and having their ends turned flat and perpendicular to the axis. From my former experiments* we have as below, x being the inverse power, and the lengths of the pillars varying from eighty to twenty times the diameter nearly.

Diameters considered equal.	Lengths of which the strengths are compared.	Value of x.	Mean value of x.
in. in.	ins. ins.		
·77 and ·77	60·5 and 30·25	1·843	
·77 and ·777	60·5 and 20·1666	1·682	
·77 and ·775	60·5 and 15·125	1·550	1·650
·997 and 1·01	60·5 and 30·25	1·691	
·997 and 1·022	60·5 and 20·1666	1·483	

* Philosophical Transactions, 1840, Part II. p. 399.

From the present experiments on pillars whose models were $2\frac{1}{2}$ inches diameter in most cases, Tables I. to VI.

Diameters of pillars compared.	Lengths of pillars compared.	Values of x.	Mean values of x.	Remarks.
in. in.	ft. ft.			
$2\frac{1}{2}$ and $2\frac{1}{2}$	10 $7\frac{1}{2}$	1·6303[a]	1·6303	[a] This result is a mean from ten comparisons on different kinds of iron. Tables II., III., IV.
$2\frac{1}{2}$ and $2\frac{1}{2}$	10 $6\frac{1}{2}$	1·8864 [b]		
$2\frac{1}{2}$ and $2\frac{1}{2}$	10 $6\frac{1}{2}$	1·5737	1·6063	
$2\frac{1}{2}$ and $2\frac{1}{2}$	10 $6\frac{1}{2}$	1·5514		[b] From four kinds of iron, Level, London Mixture, Old Hill and Derwent. Tables III. and V.
$2\frac{1}{2}$ and $2\frac{1}{2}$	10 $6\frac{1}{2}$	1·4137		
$2\frac{1}{2}$ and $2\frac{1}{2}$	10 $6\frac{1}{2}$	1·482 [c]		
$2\frac{1}{2}$ and $2\frac{1}{2}$	10 $6\frac{1}{2}$	1·4752	1·6724	
$1\frac{1}{2}$ and $1\frac{1}{2}$	10 $6\frac{1}{2}$	1·8338		[c] From the Second London Mixture. Tables III. and V.
$1\frac{1}{2}$ and $1\frac{1}{2}$	10 $6\frac{1}{2}$	1·8986		
$2\frac{1}{2}$ and $2\frac{1}{2}$	10 5	1·4523 [d]	1·5232	[d] From Old Hill Iron. Tables III. and VI.
$2\frac{1}{2}$ and $2\frac{1}{2}$	10 5	1·5942		
Mean value of x from all the comparisons, twenty-five in number, 1·6285.				

In the former experimental paper, mentioned above, I assumed 1·7 as the inverse power which the strength of pillars of the same diameter and of different lengths followed nearly, this applying both to pillars with rounded ends and to those with flat ends; but it appears from above, that in flat-ended pillars, representing the great mass of those used in practice, the strength would be better represented by some number intermediate between 1·6 and 1·7: the results of the experiments here detailed give 1·63 as a mean nearly, and this number may be used for all pillars whose length is more than about thirty times the diameter. In pillars shorter than this the formula used for calculating their strength would require correcting, as previously mentioned in page 859.

Strength of Pillars, the Iron in the section of which is not uniform.

The iron of all the pillars in this research is assumed to be uniform and of equal resistance throughout the surface of fracture, but this is perhaps seldom or never the case in cast iron, except possibly in very large castings. My experiments on the crushing of cast iron, made for the British Association, to determine the relative values of hot and cold blast iron*, of which an abstract is given in my 'Experimental Researches on the Strength and other properties of Cast Iron, 1846†,' showed that small cylinders of cast iron, No. 2, from the Carron Iron-works, turned to the exact dimensions from castings made but little larger than the size intended, required to crush them 130163 lbs. per square inch when they were $\frac{1}{4}$ inch diameter, and 132319 lbs. per square inch when they were $\frac{3}{8}$ inch diameter, from means of several experiments. In cylinders and prisms cut out of larger masses, the crushing weight per square inch was reduced to 119730 lbs., 103012 lbs., 102884 lbs., 108315 lbs., 96764 lbs., &c. In the experiments made to determine the crushing strength of the iron in my former experiments on pillars‡, cylinders ·52 inch diameter and nearly of the size of the original casting, required 115910 lbs.

* Seventh Report, 1837.

† Weale, London.

‡ Philosophical Transactions, 1840, Part II. p. 419.

per square inch to crush them, and small rectangles cut out of masses 1 inch square required 103692 lbs. only.

In some experiments made by Captain (now Colonel) JAMES, as a Member of the Royal Commission for inquiring into the application of Iron to Railway Structures*, it was found that the central part of bars of iron planed was much weaker to bear a transverse strain than bars cast of the same size. He states that "it was found by planing out $\frac{3}{4}$-inch bars from the centre of 2-inch square and 3-inch square bars, that the central portion was little more than half the strength of that from an inch bar, the relation being as 7 to 12" (there may be some doubt whether the iron was sound at the centre). In page 111 of the same Report, I showed that rectangular bars of cast iron, cast 1, 2, and 3 inches square, laid upon supports $4\frac{1}{2}$ feet, 9 feet, and $13\frac{1}{2}$ feet asunder, were broken by weights of 447 lbs., 1394 lbs., and 3043 lbs. respectively. These weights, divided by the squares of the lengths, should give equal results; the quotients, however, were as 447, 349, and 338 respectively. I attributed this falling off and deviation from theory partly to the defect of elasticity, which I had always found in cast iron, but principally to the superior hardness of the smaller castings.

From the experiments on the resistance to a crushing force of the various kinds of iron used in this research, it was found that short cylinders, $\frac{3}{4}$ inch in diameter and $1\frac{1}{2}$ inch high, cut out of pieces cast $2\frac{1}{2}$ inches diameter, resisted with more force when cut out of the part near to the surface than when cut out of that near to the centre. To try this, many small cylinders, from four or five kinds of iron, were cut from the centre, and from the part intermediate between the centre and the circumference (Table VIII.), and it was found that the cylinders from the latter part were always stronger than those from the centre; thus, in cylinders from pillars of Old Park iron, No. 1, cast as above mentioned $2\frac{1}{2}$ inches diameter, the mean crushing force of the iron near to the circumference was 39·32 tons per square inch, and that from the centre 33·33 tons per square inch, the difference being 5·99 tons. Whence $\frac{5{\cdot}99}{39{\cdot}32} = {\cdot}1523$, the ratio of the falling off of the resistance of the central part to the resistance of the external part.

Cylinders from other irons used gave the results below (see Table VIII.).

Low Moor Iron, No. 2.—Crushing force per square inch.

Centre, 29·65 tons. Intermediate part, 34·59 tons.

External ring of a hollow pillar, 4 inches diameter, of which the outer crust had been removed, 39·06 tons.

External ring of a hollow pillar, about $3\frac{1}{2}$ inch. diameter, 51·78 tons. From another pillar, of about the same diameter, 49·20 tons; the iron of both of these being very thin.

Blaenavon Iron, No. 3.

Centre, 34·22 tons. Intermediate part, 37·28 tons.

* See Report, 1849, p. 250.

Second London Mixture.

Centre, 35·02 tons. Intermediate part, 42·56 tons.

Second London Mixture.—Specimens cut from pillars $1\frac{1}{2}$ inch diameter.

Centre, 46·46 tons. Intermediate part, 49·59 tons.

By comparing the results of the experiments on the resisting powers of the different irons in the kingdom to a crushing force, with those obtained from the direct strengths of pillars of the same irons, 10 feet long and $2\frac{1}{2}$ inches diameter, it appears that the crushing force varies rather widely from being as the direct strength of pillars from the same model. Though the ratio assumed is merely an approximation, it will afford means of approaching to the falling off in the strength of pillars, when the resistance of the different parts to a crushing force is known.

Taking the results from the breaking weights of three of the strongest iron pillars, each 10 feet long and $2\frac{1}{2}$ inches diameter, and the crushing weights of the irons from those pillars, we have from the strongest pillars—

Breaking weights of the pillars.		Crushing weights of the iron in their sections.	
66086	Mean 63342 lbs.	437350	Mean 464070 lbs.
62794		485750	
61147		469110	

and from those of the weakest irons—

52680	Mean 48995 lbs.	338450	Mean 319345 lbs.
49387		346130	
44918		273455	

Comparing the breaking weights of the pillars with the crushing weights of the irons, we have

$$48995 : 63342 :: 319345 : 412857.$$

The last number, 412857, ought to have been 464070 if the breaking weights had been as the crushing weights. But the breaking weight of pillars depends on the tensile and crushing weights conjointly, though principally upon the latter; the assumption of that proportion must be taken as an approximation only, as previously mentioned.

Let r, r' be the resistances to crushing per square inch of the parts near to the circumference and to the centre respectively; D, d the external diameter, and that which includes the softer part of the metal near to the centre. Then $r-r'=$ the falling off per square inch in the resistance of the softer part below that of the external part. Whence $\frac{r-r'}{r}=$ the defect of resistance of the central part, the resistance of the external part being represented by unity.

If then w be the breaking weight of a pillar; m, n constants, and as before assumed for solid pillars of uniform texture,

$$w=mD^n, \quad \ldots \quad (A.)$$

the length being constant, then

$$w=m\left(\mathrm{D}^n-\frac{r-r'}{r}d^n\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{B.})$$

for pillars of variable texture, as above. If the pillar be hollow $r'=0$, and

$$w=m(\mathrm{D}^n-d^n), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{C.})$$

as before found.

The falling off in the strength of solid pillars from that of hollow ones of the same external diameter, and of the same kind of iron, may be seen very evidently from the results of the experiments in Tables I. and II., and made the subject of further investigation, if the principles assumed be admitted.

Example 1. If in a hollow pillar, n, in equation C, be 3·5, and d be taken of different values as below, we shall have $w=m(\mathrm{D}^{3\cdot5}-d^{3\cdot5})$; and

$$\text{if } d=\frac{\mathrm{D}}{4},\ w=m(\mathrm{D}^{3\cdot5}-\cdot0078\,\mathrm{D}^{3\cdot5})=m\times\cdot9922\,\mathrm{D}^{3\cdot5};$$

$$\text{if } d=\frac{\mathrm{D}}{2},\ w=m(\mathrm{D}^{3\cdot5}-\cdot0884\,\mathrm{D}^{3\cdot5})=m\times\cdot9116\,\mathrm{D}^{3\cdot5};$$

$$\text{if } d=\frac{2\mathrm{D}}{3},\ w=m(\mathrm{D}^{3\cdot5}-\cdot242\ \mathrm{D}^{3\cdot5})=m\times\cdot758\ \mathrm{D}^{3\cdot5};$$

$$\text{if } d=\frac{3}{4}\mathrm{D},\ w=m(\mathrm{D}^{3\cdot5}-\cdot365\ \mathrm{D}^{3\cdot5})=m\times\cdot635\ \mathrm{D}^{3\cdot5};$$

$$\text{if } d=\frac{4}{5}\mathrm{D},\ w=m(\mathrm{D}^{3\cdot5}-\cdot458\ \mathrm{D}^{3\cdot5})=m\times\cdot542\ \mathrm{D}^{3\cdot5};$$

$$\text{if } d=\frac{5}{6}\mathrm{D},\ w=m(\mathrm{D}^{3\cdot5}-\cdot528\ \mathrm{D}^{3\cdot5})=m\times\cdot472\ \mathrm{D}^{3\cdot5}.$$

Example 2. In the formula (B.), for metals of variable texture, $w=m\left(\mathrm{D}^n-\frac{r-r'}{r}d^n\right)$; and supposing, for example, as in Low Moor iron, $\frac{r-r'}{r}=\frac{39-29\frac{1}{2}}{39}=\frac{1}{4}$ nearly, or $w=m\left(\mathrm{D}^{3\cdot5}-\frac{1}{4}d^{3\cdot5}\right)$,

$$\left.\begin{aligned}
&\text{if } d=\frac{\mathrm{D}}{4},\ w=m\left(\mathrm{D}^{3\cdot5}-\tfrac{1}{4}\times\cdot00781\,\mathrm{D}^{3\cdot5}\right)=m(\mathrm{D}^{3\cdot5}-\cdot0019\,\mathrm{D}^{3\cdot5})=m\times\cdot998\ \ \mathrm{D}^{3\cdot5}=22\cdot954\,\mathrm{D}^{3\cdot5};\\
&\text{if } d=\frac{\mathrm{D}}{2},\ w=m\left(\mathrm{D}^{3\cdot5}-\tfrac{1}{4}\times\cdot0884\ \ \mathrm{D}^{3\cdot5}\right)=m(\mathrm{D}^{3\cdot5}-\cdot0221\,\mathrm{D}^{3\cdot5})=m\times\cdot9779\,\mathrm{D}^{3\cdot5}=22\cdot492\,\mathrm{D}^{3\cdot5};\\
&\text{if } d=\frac{2\mathrm{D}}{3},\ w=m\left(\mathrm{D}^{3\cdot5}-\tfrac{1}{4}\times\cdot242\ \ \mathrm{D}^{3\cdot5}\right)=m(\mathrm{D}^{3\cdot5}-\cdot0605\,\mathrm{D}^{3\cdot5})=m\times\cdot9395\,\mathrm{D}^{3\cdot5}=21\cdot608\,\mathrm{D}^{3\cdot5};\\
&\text{if } d=\frac{3\mathrm{D}}{4},\ w=m\left(\mathrm{D}^{3\cdot5}-\tfrac{1}{4}\times\cdot365\ \ \mathrm{D}^{3\cdot5}\right)=m(\mathrm{D}^{3\cdot5}-\cdot091\ \ \mathrm{D}^{3\cdot5})=m\times\cdot909\ \ \mathrm{D}^{3\cdot5}=20\cdot907\,\mathrm{D}^{3\cdot5};\\
&\text{if } d=\frac{4\mathrm{D}}{5},\ w=m\left(\mathrm{D}^{3\cdot5}-\tfrac{1}{4}\times\cdot458\ \ \mathrm{D}^{3\cdot5}\right)=m(\mathrm{D}^{3\cdot5}-\cdot1145\,\mathrm{D}^{3\cdot5})=m\times\cdot8855\,\mathrm{D}^{3\cdot5}=20\cdot366\,\mathrm{D}^{3\cdot5};
\end{aligned}\right\}\ m \text{ being assumed as 23, as below.}$$

The mean value of x obtained from the first six pillars in the summary (page 861) is 2174 lbs., the power being 3·55; and for solid pillars it is 2029 lbs. (Exp. 11 and 12). These values for a power 3·5, as assumed in the problem above, are 2322 and 2126 lbs.

respectively, or as 23 to 21 nearly; if then m be assumed as 23, the falling off for different values of d will be, as in the example above, becoming successively 22·954, 22·492, 21·608, 20·907, 20·366; and the value of d, which gives 21, is about $\frac{3}{4}$D.

Example 3. Again, suppose $\frac{r-r'}{r}=\frac{1}{5}$, and the rest as before,

if $d=\frac{2}{3}$D, $w=m\left(D^{2\cdot5}-\frac{1}{5}\times\cdot242\ D^{2\cdot5}\right)=m(D^{2\cdot5}-\cdot048\ D^{2\cdot5})=m\times\cdot952\ D^{2\cdot5}=21\cdot896\ D^{2\cdot5}$;

if $d=\frac{3}{4}$D, $w=m\left(D^{2\cdot5}-\frac{1}{5}\times\cdot365\ D^{2\cdot5}\right)=m(D^{2\cdot5}-\cdot073\ D^{2\cdot5})=m\times\cdot927\ D^{2\cdot5}=21\cdot321\ D^{2\cdot5}$;

if $d=\frac{4}{5}$D, $w=m\left(D^{2\cdot5}-\frac{1}{5}\times\cdot458\ D^{2\cdot5}\right)=m(D^{2\cdot5}-\cdot0916\ D^{2\cdot5})=m\times\cdot9084\ D^{2\cdot5}=20\cdot893\ D^{2\cdot5}$;

} if m be assumed as 23, as in the last example;

and the value of d, which gives 21, is about $\frac{4}{5}$D.

The pillars in the following summary of results were all cast solid from models $2\frac{1}{2}$ inches diameter, and made 10 feet long, with their ends flat. Those of iron, denominated No. 1, were chosen for comparison from some of the leading irons in the kingdom. They vary in strength from 29·50 tons to 20·05 tons, or nearly as 3 to 2. Formulæ for the strength of pillars of all the irons tried, thirteen in number, are given, both on the supposition used in the former research, and also on that arrived at in the present. The weights which would crush short pillars of the same diameter are likewise given.

Second Summary.

Results of all the Experiments in Tables II. and III. on Solid Pillars of Cast Iron, 10 feet long, cast from models 2½ inches diameter, and supplied from Low Moor, and from Messrs. Easton and Amos; with applications to the formulæ for their strength on different suppositions; and also the weights which would crush the pillars.

Description of Iron of which the pillar was formed.	Breaking weight of pillar.		Value of x from formula $x=\frac{W}{D^{3\cdot55}}$, where W is the breaking weight, D the diameter, and x the breaking weight of a pillar 1 inch diameter and 10 feet long.		Value of x from formula $x=\frac{W}{D^{3\cdot5}}$, where W is the breaking weight, D the diameter, and x the breaking weight of a pillar 1 inch diameter and 10 feet long.		Weight which would crush the pillar, calculated from Table VIII.	
	lbs.	Mean.	lbs.		lbs.		lbs.	
Old Park Iron, No. 1.	66792	66086	2571·5	2504·1	2692·2	2622·2	433350	437350
Stourbridge. Cold blast.	65381		2436·8		2552·3		441350	
Derwent Iron, No. 1.	57855	62794	2147·2	2351·9	2249·2	2463·3	488060	485750
Durham. Hot blast.	67733		2556·6		2677·4		483440	
Portland Iron, No. 1.	59736	61147	2332·8	2330·5	2441·8	2440·3	462760	469110
Tovine, Scotland. Hot blast.	62559		2328·3		2438·8		475460	
Calder Iron, No. 1.	60677	60677	2379·7	2331·7	2490·7	2441·2	412200	417025
Lanarkshire. Hot blast.	60677		2283·8		2391·8		421850	
London Mixture.	54562	56914	2013·7	2077·8	2109·4	2176·9	407790	410210
One-half old plate iron and one-half Calder Iron.	59266		2141·9		2244·4		412630	
Level Iron, No. 1.	54091	55267	2038·8	2109·0	2135·2	2208·3	339700	337415
Staffordshire. Hot blast.	56443		2179·2		2281·5		335130	
Coltness Iron, No. 1.	56443	52680	2133·5	2017·8	2234·2	2112·6	313460	310980
Edinburgh. Hot blast.	48917		1902·1		1991·1		308505	
Carron Iron, No. 1.	52210	52680	2001·6	2013·8	2095·7	2108·6	337910	338450
County of Stirling. Hot blast.	53151		2026·1		2121·6		338990	
Blaenavon Iron, No. 1.	48917	49387	1926·7	1928·6	2016·5	2018·7	344460	346130
South Wales. Cold blast.	49858		1930·5		2020·9		347800	
Old Hill Iron. No. 1.	44683	44918	1679·5	1684·7	1759·0	1764·5	273130	273455
Staffordshire. Cold blast.	45154		1690·0		1770·0		273780	
Second London Mixture.	63499	60912	2417·2	2342·6	2531·1	2452·6	472120	469305
One-third No. 1 best Scotch pig iron, and two-thirds old metal.	58325		2268·0		2374·1		466490	
Low Moor Iron, No. 2.	55973	55973	2008·9	2028·7	2105·2	2125·6	397590	395415
Yorkshire. Cold blast.	55973		2048·5		2146·1		393240	
Blaenavon Iron, No. 3.	56914	57384	2034·1	2065·4	2131·8	2164·4	428540	426865
South Wales. Cold blast.	57855		2096·7		2197·0		425190	

Formulæ for computing the Breaking Weights of Solid Pillars of the Irons in the preceding summary; where W is the breaking weight in lbs. or in tons, D the diameter in inches, and l the length in feet.

Description of Iron in the Pillar.	Strength of a pillar from formula $W=m.\frac{D^{3\cdot55}}{l^{1\cdot7}}$ where m is as below, according as the weight is in lbs or tons.	Strength of a pillar from formula $W=m.\frac{D^{3\cdot5}}{l^{1\cdot63}}$ where m is as below, according as the weight is in lbs. or tons.
	lbs. tons.	lbs. tons.
Old Park Iron, No. 1	125502=56·03	111858=49·94
Derwent Iron, No. 1	117872=52·62	105079=46·91
Portland Iron, No. 1	116802=52·14	104098=46·47
Calder Iron, No. 1	116862=52·17	104137=46·49
London Mixture	104137=46·49	92862=41·46
Level Iron, No. 1	105700=47·19	94202=42·05
Coltness Iron, No. 1	101130=45·15	90119=40·23
Carron Iron, No. 1	100929=45·06	89949=40·16
Blaenavon Iron, No. 1	96659=43·15	86114=38·44
Old Hill Iron, No. 1	84435=37·69	75270=33·60
Second London Mixture	117408=52·41	104623=46·21
Low Moor Iron, No. 2	101676=45·39	90674=40·48
Blaenavon Iron, No. 3	103515=46·21	92329=41·22

The numbers above represent the strength of a pillar 1 foot long and 1 inch diameter.

To compare the direct strength of long pillars of cast iron with their transverse strength.—If the pillars be cylindrical, and have their ends turned flat and perpendicular to their axes, the ultimate strength of a solid pillar will be represented by $w=m\frac{D^n}{l^p}$, where D is the diameter, l the length, and m, n, p constants. My former experiments* gave $n=3{\cdot}55$ and $p=1{\cdot}7$, but the present experiments on larger pillars give 3·5 and 1·63 for their values much more nearly.

The transverse strength of a cylinder, laid with its ends upon supports, and broken by a weight at its middle, is

$$w'=b.\frac{D^3}{l}.$$

Whence the ratio of the direct strength of a pillar to its transverse strength is

$$\frac{w}{w'}=\frac{m}{b}.\frac{D^{n-3}}{l^{p-1}}.$$

If n be taken as 3·55 and p as 1·7, we shall have

$$\frac{w}{w'}=\frac{m}{b}.\frac{D^{\cdot55}}{l^{\cdot7}};$$

and if n be taken as 3·5 and p as 1·63,

$$\frac{w}{w'}=\frac{m}{b}.\frac{D^{\cdot5}}{l^{\cdot63}}.$$

* Philosophical Transactions, 1840.

Tables III., IV. and VII. give both the direct and transverse strength of the pillars to which the preceding remarks are applicable.

The following summary contains the results of the experiments in Tables II. to VI. upon solid pillars varying in length from 10 feet to 7 feet 6 inches, 6 feet 3 inches, and 5 feet, or as 8, 6, 5, 4, all from models $2\frac{1}{2}$ inches diameter; the results are from thirteen kinds of cast iron, as before, or from twenty-six pillars 10 feet long, eleven pillars 7 feet 6 inches long, six 6 feet 3 inches long, and three 5 feet long, forty-six pillars in all. This does not include pillars round at the ends, and others cast $1\frac{1}{2}$ inch diameter.

The mean crushing strength of short pillars of the irons, cast $2\frac{1}{2}$ inches diameter, is 175·7 tons. The other mean values are as below:—

Length.	Breaking weight.	Ultimate decrement of length.	Ultimate deflection.	Ratio of breaking weight to crushing weight.
	tons.	in.	in.	
10 ft.	23·27	·176	·75	·146
7 ft. 6 in.	40·57	·235	·66	·231
6 ft. 3 in.	51·77	·248	...	·294
5 ft.	57·69*	·296*	·57*	·472*

whence it appears that the ultimate decrement of length is inversely as the length nearly. Another object of these experiments, and of others on pillars of smaller diameter in Tables III. and V., was to determine the powers of the diameter and length on which the strength of pillars depends (see pp. 865 and 866).

* From two experiments only.

Third Summary.

Third Summary.—Comparison of the Crushing Weights, Breaking Weights, ultimate diameter, the lengths being different, and the

Description of Iron in the Pillar.	Weight which would crush a short pillar of the iron used, and of the same diameter, 2½ inches (from Table VIII.). (C.)	Pillars 10 feet long, from Tables II. and III.			
		Breaking weight. (B.)	$\frac{B}{C}$	Ultimate decrement of length.	Ultimate deflection.
	lbs.	lbs. Mean.		in.	in.
Old Park Iron, No. 1. Cold blast	437350	65381 } 66086 lbs. = 29·50 tons.	·151	·181 } ·176	·31 } ·38
		66792 }		·171 }	·45 }
Derwent Iron, No. 1. Hot blast	485750	57855 } 63794 lbs. = 28·03 tons.	·129	·164 } ·172	·58 } ·63
		67733 }		·180 }	·69 }
Portland Iron, No. 1. Hot blast	469110	59736 } 61147 lbs. = 27·30 tons.	·130	·175 } ·177	·84 } ·88
		62559 }		·179 }	·92 }
Calder Iron, No. 1. Hot blast	417095	60677 } 60677 lbs. = 27·09 tons.	·145	·177 } ·184	·45 } ·79
		60677 }		·192 }	1·13 }
London Mixture. One-half old plate iron and one-half Calder Iron	410210	59266 } 56914 lbs. = 25·41 tons.	·139	·190 } ·189	1·06 } 1·08
		54562 }		·188 }	1·10 }
Level Iron, No. 1. Hot blast	337415	54091 } 55967 lbs. = 24·67 tons.	·164	·199 } ·189	1·14 } 1·03
		56443 }		·179 }	·91 }
Coltness Iron, No. 1. Hot blast	310980	56443 } 52680 lbs. = 23·52 tons.	·169	·163 } ·159	·57 } ·51
		48917 }		·155 }	·44 }
Carron Iron, No. 1. Hot blast	338450	53151 } 52680 lbs. = 23·52 tons.	·156	·166 } ·169	·38 } ·42
		52210 }		·173 }	·45 }
Blaenavon Iron, No. 1. Cold blast	346130	48917 } 49387 lbs. = 22·05 tons.	·143	·169 } ·164	·83 } ·74
		49858 }		·159 }	·65 }
Old Hill Iron, No. 1. Cold blast	273455	44683 } 44918 lbs. = 20·05 tons.	·164	·191 } ·186	·66 } ·64
		45154 }		·181 }	·61 }
Second London Mixture. One-third No. 1 best Scotch pig-iron, and two-thirds of old metal	469305	63499 } 60912 lbs. = 27·19 tons.	·130	·189 } ·174	·66 } ·68
		58325 }		·159 }	·70 }
Low Moor Iron, No. 2. Cold blast	395415	55973 } 55973 lbs. = 24·99 tons.	·141	.....	1·23 } 1·01
		55973 }			·79 }
Blaenavon Iron, No. 3. Cold blast	426865	56914 } 57384 lbs. = 25·62 tons.	·134	.....	1·00 } ·95
		57855 }			·90 }
Means from all the results in each vertical column	393650 = 175·74 tons.	52832 = 23·27 tons.	·146	·176	·75

Deflections, and ultimate Decrements of Length of the Cast Iron Pillars, $2\frac{1}{2}$ inches ends turned flat and perpendicular to the axis.

Pillars 7 ft. 6 in. long, from Table IV.				Pillars 6 feet 9 inches long, from Table V.				Pillars 5 feet long, from Table VI.			
Breaking weight. (B.)	$\frac{B}{C}$.	Ultimate decrement of length.	Ultimate deflection.	Breaking weight. (B.)	$\frac{B}{C}$.	Ultimate decrement of length.	Ultimate deflection.	Breaking weight. (B.)	$\frac{B}{C}$.	Ultimate decrement of length.	Ultimate deflection.
		in.	in.			in.					
100191 lbs. = 44·73 tons.	·229	·228	·56								
105365 lbs. = 47·04 tons.	·217	·225	·51	119438 lbs.* = 50·20 tons.	·231	·221					
102543 lbs. = 45·78 tons.	·219	·245	·69								
93135 lbs. = 41·58 tons.	·223	·231	·68								
93605 lbs. = 41·79 tons.	·228	·234	·67	114390 lbs. = 51·04 tons.	·279	·277					
98309 lbs. = 43·89 tons.	·291	·244	·65	131275 lbs. = 58·60 tons.	·389	·272					
87019 lbs. = 38·85 tons.	·280	·254	·92								
73437 lbs. = 32·34 tons.	·214	·235	·76								
71967 lbs. = 32·13 tons.	·208	·222	·59					117612 lbs.† = 52·51 tons.	·339	·253	
				93622 lbs. = 41·79 tons.	·342	·264		135592 122863 Mean 129227 lbs. = 57·69 tons.	·496 ·449 } ·472	·285 ·308 } ·296	·50 ·64 } ·57
				127432 116672 Mean 122052 lbs. = 54·49 tons.	·272 ·249 } ·260	·244 ·213 } ·228					
89371 lbs. = 39·9 tons.	·236	.....	·63								
85608 lbs. = 38·22 tons.	·201	.....	·59								
90868 lbs. = 40·57 tons.	·231	·235	·66	115960 lbs. = 51·77 tons.	·294	·248					

* This bar had probably been reduced in strength by straightening after a former experiment upon it.

† This result is probably less than it ought to be, as too large a weight was laid on at once.

The results hitherto given are principally deduced from the experiments in the eight following Tables.

TABLE I.—Experiments to obtain the Strength of Hollow Uniform Pillars of Low Moor Iron, No. 2, each pillar being 10 feet long, and having its ends turned flat and perpendicular to the axis.

No. of Experiment.	Description of Pillar. 10 feet long, and as above.	Deflection.	Weight producing the deflection.	Breaking weight, or that with which the pillar sank.	Value of x from formula $x = \frac{W}{D^{3\cdot55} - d^{3\cdot55}}$ where W is the breaking weight, D and d the external and internal diameters, the length of the pillar being 10 feet.	Value of x from formula $x = \frac{W}{D^{3\cdot1} - d^{3\cdot1}}$	Remarks.
	inches.	inch.	lbs.	lbs.	lbs.	lbs.	
1.	Hollow uniform cylinder.	·35	20339				†It bore this nearly, but broke with 33197 lbs. laid on again.
	External diameter by various measurements ... 2·513	·40	22635				
		·46	24931				
	Internal diameter by various measurements ... 1·941	·54	27227				
		·68	29523				
	Weight of pillar 60¼ lbs.	·78	31819				
		1·00	34115	36411†, or 33197 Mean 34804 = 15·54 tons.	2201·1	2324·9	For the form of fracture of this and the next pillar see Plate XXXIII.
2.	Hollow uniform cylinder.	·18	20339				
	External diameter by six measurements ..... 2·590	·25	24931				
		·34	27227				
	Internal diameter by eight measurements ..... 1·903	·39	29523				
		·48	31819				
	Weight of pillar 64½ lbs.	·61	34115				
		·75	36411				
		1·09	38707				
		1·16*	39166	39166 = 17·48 tons.	2332·9	2463·7	*The deflections marked thus in this and the following experiments were computed from the two preceding results.
3.	Hollow uniform cylinder.	·04	20339				†With this weight it broke, after bearing it several minutes. The thickness of the ring of metal on the convex side was ·55 and on the concave ·17. The fracture took place nearly in the middle.
	External diameter from six measurements ..... 3·021	·06	24931				
		·08	29523				
	Internal diameter from ten measurements ..... 2·354	·13	38707				
		·16	43299				
	Weight of pillar 89½ lbs.	·22	47891				
		·30	52483				
		·38	57075				
		·55	61667				
		·93	66259	66259† = 29·58 tons.	2226·7	2374·2	
4.	Hollow uniform cylinder.	·06	20339				†It bore this nearly, but broke with 67636 lbs. laid on again. The pillar broke nearly in the middle, and split at both ends; the metal differed in thickness on the opposite sides as 2 to 1 nearly, the thicker part being on the concave side and therefore compressed. See Plate XXXIII.
	External diameter from nine measurements ... 3·035	·10	24931				
		·15	38707				
	Internal diameter from ten measurements ..... 2·354	·17	43299				
		·23	47891				
	Weight of pillar 88 lbs.	·31	61667				
		·55	66259				
		·91	68555	72228†, or 67636 Mean 69932 = 31·22 tons.	2285·8	2471·3	
5.	Hollow uniform cylinders.	·18	48917				With the weight 110649 this pillar sank down, but was not allowed to bend so as to break. It was preserved in order that an experiment might be made upon it in another way. When unloaded it became nearly straight, and was easily rendered so.
	External diameter from six measurements ..... 3·532	·22	58325				
		·32	77141				
	Internal diameter from eight measurements ... 2·646	·42	86549				
		·55	95957				
	Internal diameter from computation ..... 2·673	·84	105365				
		1·00*	110649	110649 = 49·39 tons.	1975·6	2121·7	
	Mean internal diameter ... 2·659						
	Weight of pillar 130 lbs.						
6.	Hollow uniform cylinder.	·11	58325				The remark in Experiment 5 applies also to the weight 114479 with which this pillar sank.
	External diameter from eight measurements ... 3·555	·15	67733				
		·19	77141				
	Internal diameter from eight measurements ... 2·664	·23	86549				
		·31	95957				
	By computation ..... 2·722	·45	105365				
		·63	110069				
	Mean internal diameter ... 2·693	·80*	114479	114479 = 51·11 tons.	2023·3	2173·9	
	Weight of pillar 128 lbs. 7 oz.						

TABLE I. (continued).

No. of Experiment.	Description of Pillar, 10 feet long, and as above.	Deflection.	Weight producing the deflection.	Breaking weight, or that with which the pillar sank.	Value of x from formula $x=\frac{W}{D^3-d^3}$, where W is the breaking weight, D and d the external and internal diameters, the length of the pillar being 10 feet.	Value of x from formula $x=\frac{W}{D^3-d^3}$	Remarks.
		inch.	lbs.		lbs.		
7.	Hollow uniform cylinder 9 ft. 9 in. long. inches. External diameter from twelve measurements... 3·804 Internal diameter from eight measurements ... 3·17 Weight of pillar 106 lbs. 10 oz.	·03	20693				This and the following pillar were cast to be 4 inches external diameter, and were found so strong as to require weights which would injure the machinery. They were therefore turned outside to reduce their strength, and this, by taking away the hard external surface of the metal, seems to have produced that effect in a very great degree. They were cast 9 feet 9 inches long instead of 10 feet, so that they might be broken with rounded caps on the ends, which would only have required one-third of the weight required by pillars with flat ends. Ratio of thicknesses of metal on opposite sides in middle ·37 to ·25.
		·04	46271				
		·07	74495				
		·10	93311				
		·29	102719				
		·48*	112127	112127, or 107403 lbs. = 47·95 tons, if the pillar had been 10 feet long.	1963·8	2127·6	
8.	Hollow uniform cylinder 9 ft. 9 in. long. External diameter from twelve measurements... 3·819 Internal diameter at middle by eight measurements ..... 3·211 Weight of pillar 99 lbs. 14 oz.	·03	20693				Ratio of thicknesses of metal on opposite sides in middle ·21 to ·39, the thin side being the concave one. This pillar had many flaws in it.
		·05	39509				
		·09	58325				
		·18	77141				
		·24	86549				
			99720	99720, or 95519 lbs. = 42·64 tons, if the pillar had been 10 feet long.	1785·3	1928·8	
9.	Pillars with rounded caps on the ends. Hollow pillar, the same as in Experiment 5. Length, including caps on the ends, 10 ft. 2 in.	·12	20693				
		·18	25397				
		·26	30101				
		·30	32453				
		·36	34805				
		·42	37157				
		·55	39509				
		·68	41861				
		1·16 to 1·37	44213	44683, or 45956 lbs. = 20·51 tons, the breaking weight of the pillar if 10 feet long.			
10.	Hollow pillar, the same as in Experiment 6. Length, including caps on the ends, 10 ft. 2 in.	·14	20693				
		·21	25397				
		·31	30101				
		·47	34805				
		·63	37157				
		·85	39509				
		1·40	41861	42331 lbs., or 43537 lbs. = 19·44 tons, if the pillar had been 10 feet long.			

The hollow pillars in Experiments 5 and 6 had been prevented from being broken by a prop, which was made to act on their middle immediately after they had been bent *beyond* their utmost power of resisting. They were then unloaded and reserved for another experiment, as below; after being rendered straight and turned afresh at the ends, rounded caps were affixed to the ends, to enable the straining force to be applied

in the direction of the axis, as in Experiments 9 and 10. The strength in these cases should have been only one-third of what it was in the former ones, according to the results in pages 855 to 857; but it is greater than that, in the ratio of 20 to 17 nearly, arising in a great measure from crushing of the plates and the ends of the pillars, through the weight.

TABLE II. (TABLE I. continued).—Experiments to obtain the Strength of Solid Uniform Pillars of Low Moor Iron, No. 2, and of Blaenavon Iron, No. 3, the pillars being 10 feet long and 2½ inches diameter nearly, with the ends turned flat and perpendicular to the axis.

No. of Experiment.	Description of Pillar, 10 feet long, and as above.	Deflection.	Weight producing the deflection.	Breaking weight, or weight with which the pillar sank.	Value of s from formula $s=\frac{W}{D^{3.63}}$, where W is the breaking weight and D the diameter, the length of the pillar being 10 feet.	Value of s from formula $s=\frac{W}{D^{3.7}}$.	Remarks.
		inches.	inch.	lbs.		lbs.		
11.	Solid uniform cylinder of Low Moor Iron, No. 2. Diameter from eighteen measurements ..... 2·553 Weight of pillar 159 lbs. 9 oz.		·10	20693			
			·15	25397			
			·20	34805			
			·25	37157			
			·29	39509			
			·33	41861			
			·38	44213			
			·43	46565			
			·51	48917			
			·61	51269			
			·78	53621			
			1·23	55973	55973 = 24·99 tons.	2008·9	2105·2	
12.	Solid uniform cylinder of Low Moor Iron, No. 2. Diameter from eighteen measurements ..... 2·539 Weight of pillar 158 lbs. 13 oz.		·10	20693			
			·13	25397			
			·16	30101			
			·19	34805			
			·22	37157			
			·25	39509			
			·28	41861			
			·35	44213			
			·39	46565			
			·45	48917			
			·55	51269			
			·67	53621			
			·79*	55973	55973† = 24·99 tons.	2048·5	2146·1	* See note to Experiment 2, which applies to the cases below marked with an asterisk. † After bearing the weight 55973 lbs. for half a minute or more, it sank.
13.	Solid uniform cylinder of Blaenavon Iron, No. 3. Diameter from eighteen measurements ..... 2·556 Weight of pillar 160 lbs. 3 oz.		·10	20693			
			·13	25397			
			·16	30101			
			·20	34805			
			·25	39509			
			·34	44213			
			·42	48917			
			·55	51269			
			·66	53621			
			·90	55973			
			1·00*	56914	56914 = 25·41 tons.	2034·1	2131·8	
14.	Solid uniform cylinder of Blaenavon Iron, No. 3. Diameter from eighteen measurements ..... 2·546 Weight of pillar 159 lbs. 6 oz.		·07	20693			
			·11	25397			
			·14	30101			
			·17	34805			
			·21	39509			
			·28	44213			
			·37	48917			
			·43	51269			
			·54	53621			
			·74	55973			
			·90*	57855	57855 = 24·83 tons.	2096·7	2197·0	

TABLE III.—Experiments to determine the Strength of Solid Uniform Pillars, 10 feet long and 2½ inches diameter nearly, from various parts of the kingdom. The pillars were all cast erect and in dry sand, from the same iron model, nearly 12 feet long; they were afterwards cut to 10 feet long, and their ends turned flat and perpendicular to the axis. Two pillars were cast from each kind of iron. The irons are arranged in the order of their strength.

No. of Experiment.	Description of Pillar; each being cast solid, 2½ inches diameter nearly, and turned, 10 feet long, and flat at the ends.	Weights laid on.	Decrement of length.	Deflection of middle.	Breaking weight, or that with which the pillar sank.	Value of x from formula $x = \frac{W}{D^{3\cdot 5}}$ where W is the breaking weight, D the diameter, and x the strength of a pillar 1 inch diameter and 10 ft. long.	Value of x from formula $x = \frac{W}{D^{3\cdot 5}}$	Remarks.
	inches.	lbs.	inch.	inch.	lbs.	lbs.	lbs.	
1.	Old Park Iron, No. 1.	20693	·065	·04				
	Stourbridge. Cold blast.	25397	·076					
	Mean diameter from nine measurements ..... 2·503	30101	·087	·06				
		34805	·097					
	Weight of pillar 151½ lbs.	39509	·106	·08				
		44213	·116					
		48917	·127	·10				
		53621	·138					
		58325	·149	·18				
		63029	·161	·33				
		66792	·171*	·45*	66792	2571·5	2692·2	
2.	Old Park Iron, No. 1.	20693	·072					
	Stourbridge. Cold blast.	25397	·084					
	Mean diameter from eight measurements ..... 2·536	30101	·095					
		34805	·105					
	Weight of pillar 154 lbs.	39509	·116					
		44213	·127					
		48917	·139					
		53621	·150	·07				
		58325	·161	·14				
		63029	·174	·27				
		65381	·181*	·31*	65381	2436·8	2552·3	
3.	Derwent Iron, No. 1.	20693	·066	·08				
	Durham. Hot Blast.	25397	·077					
	Mean diameter from eight measurements ..... 2·529	30101	·089	·10				
		34805	·100					
	Weight of pillar 152 lbs.	39509	·112	·16				
		44213	·123	·20				
		48917	·137	·27				
		53621	·151	·43				
		57855	·164*	·58*	57855	2147·2	2249·2	
4.	Derwent Iron, No. 1.	20693	·060	·02				
	Durham. Hot blast.	25397	·071					
	Mean diameter from eight measurements ..... 2·517	30101	·081					
		34805	·091					
	Weight of pillar 151½ lbs.	39509	·102					
		44213	·113					
		48917	·124					
		53621	·135	·06				
		58325	·147	·13				
		60677	·153	·19				
		63029	·159	·23				
		65381	·166	·35				
		66733	·180	·69	66733	2556·6	2677·4	
5.	Portland Iron, No. 1.	20693	·068	·09				
	Tovine, Scotland. Hot blast.	25397	·079					
	Mean diameter from eight measurements ..... 2·493	30101	·090	·11				
		34805	·101					
	Weight of pillar 150½ lbs.	39509	·112	·15				
		44213	·124	·19				
		48917	·136	·24				
		53621	·150	·37				
		55973		·46				
		58325	·169	·68 to ·73				
		59736	·175*	·84*	59736	2339·8	2441·8	

* The quantities marked with an asterisk were computed in each case from the two preceding results, as before.

TABLE III. (continued).

No. of Experiment.	Description of Pillar: each being cast solid, 2½ inches diameter nearly, and turned, 10 feet long, and flat at the ends.	Weights laid on.	Decrement of length.	Deflection of middle.	Breaking weight, or that with which the pillar sank.	Value of x from formula $x=\frac{W}{D^{3.55}}$, where W is the breaking weight, D the diameter, and x the strength of a pillar 1 inch diameter and 10 ft. long.	Value of x from formula $x=\frac{W}{D^{3.6}}$.	Remarks.
	inches.	lbs.	inch.	inch.	lbs.	lbs.	lbs.	
6.	Portland Iron, No. 1. Tovine, Scotland. Hot blast. Mean diameter from eight measurements ..... 2·527. Weight of pillar 156¼ lbs.	20693	·063	·09				
		25397	·075					
		30101	·086	·11				
		34805	·097					
		39509	·107	·17				
		44213	·118					
		48917	·130	·27				
		53621	·145	·41				
		58325	·159	·58				
		60677	·170	·77				
		62559	·179*	·92*	62559	2328·3	2438·8	
7.	Calder Iron, No. 1. Lanarkshire. Hot blast. Mean diameter from eight measurements ..... 2·490. Weight of pillar 149 lbs. 10 oz.	20693	·069	·04				
		25397	·081					
		30101	·093	·05				
		34805	·105					
		39509	·116	·08				
		44213	·128					
		48917	·140	·15				
		53621	·153	·25				
		58325	·169	·38				
		60677	·177*	·45*	60677	2379·7	2490·7	
8.	Calder Iron, No. 1. Lanarkshire. Hot blast. Mean diameter from eight measurements ..... 2·519. Weight of pillar 151¼ lbs.	20693	·069	·06				
		25397	·081					
		30101	·092	·08				
		34805	·104					
		39509	·115	·14				
		44213	·127					
		48917	·140	·28				
		53621	·154	·43				
		58325	·179	·83				
		60677	·192*	1·13	60677	2283·8	2391·8	
9.	London Mixture. One-half old plate iron and one-half Calder iron. Mean diameter from eight measurements ..... 2·533. Weight of pillar 152¼ lbs.	20693	·069	·04				
		25397	·082					
		30101	·095	·06				
		34805	·107					
		39509	·121	·14				
		44213	·134	·20				
		48917	·149	·35				
		53621	·183	·94				
		54562	·190*	1·06*	54562	2013·7	2109·4	
10.	London Mixture. One-half old plate iron, and one-half Calder iron. Mean diameter from eight measurements ..... 2·548. Weight of pillar 156 lbs. 14 oz.	20693	·067	·09				
		25397	·079					
		30101	·091	·12				
		34805	·103					
		39509	·114	·20				
		44213	·126	·23				
		48917	·139	·33				
		53621	·153	·48				
		58325	·180 to ·182	·91 to 1·00 in 2 minutes.				
		59266	·188*	1·10*	59266	2141·9	2244·4	
11.	Level Iron, No. 1. Staffordshire. Hot blast. Mean diameter from eight measurements ..... 2·518. Weight of pillar 151 lbs. 2 oz.	20693	·070	·08				
		25397	·082					
		30101	·094	·13				
		34805	·107					
		39509	·122	·23				
		44213	·137	·34				
		48917	·155	·53				
		53621	·195	1·08				
		54091	·199*	1·14*	54091	2038·8	2135·2	
12.	Level Iron, No. 1. Staffordshire. Hot blast. Mean diameter from eight measurements ..... 2·501. Weight of pillar 150½ lbs.	20693	·069	·08				
		25397	·082					
		30101	·094	·13				
		34805	·107					
		39509	·121	·24				
		44213	·132	·31				
		48917	·147	·45				
		53621	·167	·74				
		56443	·179*	·91*	56443	2179·2	2281·5	

TABLE III. (continued).

No. of Experiment.	Description of Pillar; each being cast solid, 2½ inches diameter nearly, and turned, 10 ft. long, and flat at the ends.	Weights laid on.	Decrement of length.	Deflection of middle.	Breaking weight, or that with which the pillar sank.	Value of x from formula $x=\frac{W}{D^{3.6}}$, where W is the breaking weight, D the diameter, and x the strength of a pillar 1 inch diameter and 10 ft. long.	Value of x from formula $x=\frac{W}{D^{3.5}}$.	Remarks.
	inches.	lbs.	inch.	inch.	lbs.	lbs.	lbs.	
13.	Coltness Iron, No. 1. Edinburgh, Hot blast. Mean diameter from eight measurements ..... 2·516 Weight of pillar 151½ lbs.	20693	·065	·05				
		25397	·077					
		30101	·089	·06				
		34805	·100					
		39509	·112	·10				
		44213	·125	·15				
		48917	·139	·23				
		53621	·154	·44				
		56443	·163*	·57*	56443	2133·5	2234·2	
14.	Coltness Iron, No. 1. Edinburgh. Hot blast. Mean diameter from eight measurements ..... 2·496 Weight of pillar 149 lbs. 10 oz.	20693	·068	·04				
		25397	·081					
		30101	·094	·06				
		34805	·108					
		39509	·123	·16				
		44213	·139	·30				
		48917	·155*	·44*	48917	1902·1	1991·1	
15.	Carron Iron, No. 1. County of Stirling. Hot blast. Mean diameter from eight measurements ..... 2·506 Weight of pillar 149½ lbs.	20693	·070	·03				
		25397	·083					
		30101	·097	·04				
		34805	·110					
		39509	·124	·06				
		44213	·137	·12				
		48917	·154	·27				
		52210	·166	·38*	52210	2001·6	2093·7	
16.	Carron Iron, No. 1. County of Stirling. Hot blast. Mean diameter from eight measurements ..... 2·510 Weight of pillar 149½ lbs.	20693	·076	·04				
		25397	·091					
		30101	·104	·06				
		34805	·116					
		39509	·130	·11				
		44213	·144	·18				
		48917	·159	·32				
		53151	·173*	·45*	53151	2026·1	2121·6	
17.	Blaenavon Iron, No. 1. South Wales. Cold blast. Mean diameter from eight measurements ..... 2·487 Weight of pillar 150 lbs. 14 oz.	20693	·070					
		25397	·082	·08				
		30101	·096	·10				
		34805	·109					
		39509	·122	·21				
		44213	·137					
		48917	·169	·83	48917	1926·7	2016·5	
18.	Blaenavon Iron, No. 1. South Wales. Cold blast. Mean diameter from eight measurements ..... 2·499 Weight of pillar 151½ lbs.	20693	·064	·03				
		25397	·076	·05				
		30101	·088	·07				
		34805	·100					
		39509	·117	·16				
		44213	·130					
		48917	·154	·60				
		49858	·159*	·65*	49858	1930·5	2020·9	
19.	Old Hill Iron, No. 1. Staffordshire. Cold blast. Mean diameter from eight measurements ..... 2·520 Weight of pillar 151½ lbs.	20693	·088	·07				
		25397	·106					
		30101	·123	·09				
		34805	·140					
		39509	·159	·13				
		44213	·188	·61				
		44683	·191*	·66*	44683	1679·5	1759·0	
20.	Old Hill Iron, No. 1. Staffordshire. Cold blast. Mean diameter from eight measurements ..... 2·523 Weight of pillar 151 lbs. 10 oz.	20693	·079	·04				
		25397	·094					
		30101	·111	·08				
		34805	·126					
		39509	·144	·30				
		44213	·175	·56				
		45154	·181*	·61*	45154	1690·0	1770·0	

TABLE III. (continued).

No. of Experiment.	Description of Pillar; each being cast solid, 3¼ inches diameter nearly, and turned, 10 feet long, and flat at the ends.	Weights laid on.	Decrement of length.	Deflection of middle.	Breaking weight, or that with which the pillar sank.	Value of s from formula $s = \frac{W}{D^{3.55}}$ where W is the breaking weight, D the diameter, and s the strength of a pillar 1 inch diameter and 10 ft. long.	Value of s from formula $s = \frac{W}{1583}$	Remarks.
	inches.	lbs.	inch.	inch.	lbs.	lbs.	lbs.	
21.	Second London Mixture.	20693	·061					
	Two-thirds of old metal and one-third of No. 1 best Scotch pig-iron.	25397	·071					
		30101	·081					
		34805	·092					
	Mean diameter from eight measurements ..... 3·511	39509	·103	·05				
		44213	·113	·08				
	Weight of pillar 153 lbs.	48917	·128	·16				
		53621	·141	·29				
		58325	·156	·46				
		63029	·186					
		63499	·189*	·66*	63499	2417·2	2531·1	
22.	Second London Mixture.	20693	·060					
	Two-thirds of old metal and one-third of No. 1 best Scotch pig-iron.	25397	·071					
		30101	·081	·05				
		34805	·091					
	Mean diameter from eight measurements ..... 3·496	39509	·102	·10				
		44213	·114	·18				
	Weight of pillar 151¼ lbs.	48917	·127	·30				
		53621	·143	·53				
		58325	·159*	·70*	58325	2268·0	2374·1	
23.	Second London Mixture.	7284	.....	·02				
	Two-thirds of old metal and one-third of No. 1 best Scotch pig-iron.	9521	.....	·13				
		10980	.....	·70	11204			
	Mean diameter from eight measurements ..... 1·530							
	Weight of pillar 56 lbs. 2 oz.							
24.	Second London Mixture.	7284	.....	·12				
	Two-thirds of old metal and one-third of No. 1 best Scotch pig-iron.	9534	.....	·22				
		9972	.....	·30				
		10490	.....	·42				
	Mean diameter from eight measurements ..... 1·541	10532	.....	·48	10868			
	Weight of pillar 56 lbs. 2 oz.							

TABLE IV.—Solid Pillars cut out of the 10 feet pillars, 2¼ inches diameter, previously experimented upon, and now made 7 feet 6 inches long, quite straight, and with their ends turned flat and perpendicular to the axis, the iron being obtained from various parts of the United Kingdom (Tables III. and II.).

No. of Experiment.	Description of Pillar, 7 feet 6 inches long, and as above.	Weights laid on.	Decrement.	Deflection.	Breaking weight, or that with which the pillar sank.	Remarks.
		lbs.	inch.	inch.		
1.	Old Park Iron, No. 1.	20693	·058			Broke into four pieces.
	Stourbridge. Cold blast.	30101	·078			
	Cut out of the pillar in Table III.	39509	·098			
	Exp. 2.	48917	·118			
	Diameter 2·526 inches.	58325	·138			
		67733	·157	·10		
		77141	·176	·17		
		86549	·196	·25		
		91253	·207	·32		
		95957	·221	·44		
		100191	·234*	·55*	100191 lbs. = 44·73 tons.	
2.	Old Park Iron, No. 1.	20693	·059	·02		The low result of the breaking weight in this experiment renders it probable that the pillar had sustained some injury by straightening after the former experiment.
	From Exp. 1, Table III.	30101	·078			
	Diameter 2·503 inches.	39509	·096	·05		
		48917	·113	·08		
		58325	·132	·12		
		67733	·151	·15		
		77141	·171	·23		
		86549	·195	·36		
		91253	·209	·47		
		95487	·232*	·57*	95487 lbs. = 42·63 tons.	This pillar broke in three pieces of nearly equal length, splitting at one end. See Plate XXXIII.
3.	Derwent Iron, No. 1.	20693	·059			
	Durham. Hot blast.	30101	·076			
	From Exp. 4, Table III.	39509	·094	·02		
	Diameter 2·517 inches.	48917	·112			
		58325	·129	·04		
		67733	·146	·06		
		77141	·165	·10		
		86549	·182	·15		
		95957	·205	·29		
		100661	·215	·40		
		105365	·225*	·51*	105365 lbs. = 47·04 tons.	
4.	Portland Iron, No. 1.	20693	·059			
	Tovine, Scotland. Hot blast.	30101	·077			
	From Exp. 6, Table III.	39509	·095			
	Diameter 2·527 inches.	48917	·112	·06		
		58325	·131	·08		
		67733	·149	·10		
		77141	·168	·14		
		86549	·188	·19		
		95957	·210	·33		
		100661	·235	·45 to ·59		
		102543	·245*	·69*	102543 lbs. = 45·78 tons.	
5.	Calder Iron, No. 1.	20693	·063			
	Lanarkshire. Hot blast.	30101	·082			
	From Exp. 7, Table III.	39509	·102			
	Diameter 2·490 inches.	48917	·121	·05		
		58325	·143	·08		
		67733	·164	·13		
		72437	·175	·17		
		77141	·187	·22		
		81845	·200	·31		
		86549	·213	·40		
		91253		·60		
		93135	·231*	·68	93135 lbs. = 41·58 tons.	
6.	London Mixture.	20693	·057	·02		This pillar broke in three pieces of nearly equal length, splitting at one end.
	One-half old plate iron and one-half Calder Iron.	30101	·077			
	From Exp. 10, Table III.	39509	·097	·06		
	Diameter 2·548 inches.	48917	·116	·08		
		58325	·137	·10		
		67733	·157	·16		
		77141	·179	·24		
		81845	·191	·29		
		86549	·205	·41		
		91253	·224	·58		
		93605	·234*	·67*	93605 lbs. = 41·79 tons.	It would appear that in this pillar, and in that of Exp. 2, there were more points of contrary flexure than in others. Plate XXXIII.

TABLE IV. (continued).

No. of Experiment	Description of Pillar, 7 feet 6 inches long, and as above.	Weights laid on.	Decrement.	Deflection.	Breaking weight, or that with which the pillar sank.	Remarks.
		lbs.	inch.	inch.		
7.	Level Iron, No. 1.	20693	·061			
	Staffordshire. Hot blast.	30101	·083			
	From Experiment 12, Table III.	39509	·102			
	Diameter 2·501 inches.	48917	·124			
		58325	·144			
		67733	·165			
		77141	·185	·08		
		81845	·196	·17		
		86549	·207	·22		
		91253	·218	·32		
		95957	·235	·54		
		98309	·244*	·65*	98309 lbs. = 43·89 tons.	
8.	Coltness Iron, No. 1.	20693	·068			
	Edinburgh. Hot blast.	30101	·088			
	From Experiment 13, Table III.	39509	·110	·08		
	Diameter 2·516 inches.	48917	·132	·10		
		58325	·153	·14		
		67733	·177	·23		
		72437	·189	·29		
		77141	·201	·35		
		79493	·209	·48		
		81845	·218			
		86549	·251	·72 to ·81		
		87019	·254*	·92*	87019 lbs. = 38·85 tons.	
9.	Carron Iron, No. 1.	20693	·078	·10		Comparing the result from this pillar with that from the preceding one on Coltness Iron, the strengths of which should have been equal, as shown by the 10-feet pillars of these irons, I conceive this experiment to be in some degree defective, the pillar having probably been injured by the former experiment upon it.
	County of Stirling. Hot blast.	30101	·101	·12		
	From Experiment 15, Table III.	39509	·124	·16		
	Diameter 2·506 inches.	48917	·149	·22		
		58325	·175	·32		
		67733	·209	·56		
		70085	·222	·66		
		72437	·233*	·76*	72437 lbs. = 32·34 tons.	
10.	Blaenavon Iron, No. 1.	20693	·083	·07		
	South Wales. Cold blast.	30101	·106	·11		
	From Experiment 17 or 18, Table III.	39509	·128	·13		
	Diameter 2·5 inches nearly.	48917	·152	·17		
		58325	·176	·27		
		67733	·208	·49		
		71967	·222*	·59*	71967 lbs. = 32·13 tons.	
11.	Low Moor Iron, No. 2.	20693	.....	·04		
	Cut from the pillar in Experiment 12, Table II.	48917	.....	·07		
		67733	.....	·14		
	Diameter 2·539 inches.	72437	.....	·16		
		77141	.....	·21		
		81845	.....	·28		
		86549	.....	·50		
		89371	.....	·63*	89371 lbs. = 39·90 tons.	
12.	Blaenavon Iron, No. 3.	48917	.....	·08		
	Cut out of the pillar in Experiment 13, Table II.	53621	.....	·10		
		58325	.....	·13		
	Diameter 2·556 inches.	63029	.....	·17		
		67733	.....	·22		
		72437	.....	·27		
		77141	.....	·36		
		81845	.....	·49		
		85608	.....	·59*	85608 lbs. = 38·22 tons.	
13.	Low Moor Iron, No. 2.	20693	.....	·09		The strength was probably reduced by being experimented upon before, and afterwards rendered straight; the breaking weight would otherwise have been 63871 lbs. nearly, according to the inverse 1·7th power of the length. Ratio of the thicknesses of the sides ·20 to ·35, the thin side being compressed.——For form of fracture see Plate XXXIII.
	This pillar differs from the preceding ones in being *hollow*; it was cut out of the pillar in Experiment 2, Table I.	25397	.....	·13		
		30101	.....	·18		
		34805	.....	·21		
	Length 7 feet 6 inches.	39509	.....	·26		
	External diameter 2·520 inches.	44213	.....	·29		
	Internal diameter 1·903 inch.	48917	.....	·36		
	Ends flat.	51969	.....	·40		
		53621	.....	·55		
		55973	.....	·70*	55973 lbs. = 24·99 tons.	

TABLE V.—Solid Pillars cut out of the 10-feet pillars, $2\frac{1}{2}$ inches diameter, previously experimented on, and now made 6 feet 3 inches long, quite straight, and with their ends turned flat and perpendicular to the axis, the irons being obtained from various parts of the United Kingdom (Table III.).

No. of Experiment.	Description of Pillar, 6 feet 3 inches long, and as above.	Weights laid on.	Decrement.	Deflection.	Breaking weight, or that with which the pillar sank.	Remarks.
		lbs.	inch.	inch.		
1.	Derwent Iron, No. 1.	20693	·024			The pillar broke where it had been beaten with a hammer to straighten it after the former experiment, by which its strength was probably reduced. The situation of the neutral line in this and the following experiment is strongly marked. The pillar broke at the middle, and was cracked at each end. See Plate XXXIII.
	Durham. Hot blast.	30101	·043			
	From Exp. 3, Table III.	39509	·062			
	Diameter 2·599 inches.	48917	·083	·02		
		58325	·100			
		67733	·119			
		77141	·137	·10		
		86549	·158			
		95957	·177			
		105365	·202	·32		
		112438	·221*	.....	112438 lbs. = 50·20 tons.	
2.	London Mixture.	33881	·085			The pillar cracked at both ends, and broke in the middle. For form of fracture see Plate XXXIII.
	One-half old plate iron,	43289	·105			
	and one-half Calder iron.	52697	·124	·10		
		62105	·142			
	From Exp. 9, Table III.	71513	·161			
	Diameter 2·533 inches.	80921	·181	·12		
		90329	·200			
		99737	·218	·18		
		104441	·232	·27		
		109145	·247	·32		
		113849	·274	·52		
		114320	·277*	.....	114320 lbs. = 51·04 tons.	
3.	Level Iron, No. 1.	33881	·085			It broke into a number of pieces. See figure, Plate as above.
	Staffordshire. Hot blast.	43289	·103			
	From Exp. 11, Table III.	52697	·123			
	Diameter 2·518 inches.	62105	·142			
		71513	·161			
		80921	·179			
		90329	·186			
		99737	·213			
		109145	·232			
		118553	·252			
		129137	·269			
		131275	·272*	.....	131275 lbs. = 58·60 tons.	
4.	Old Hill Iron, No. 1.	20693	·065			The pillar broke across in the middle, and cracked at both ends. See form of fracture, Plate as above.
	Staffordshire. Cold blast.	33881	·103			
	From Exp. 20, Table III.	43289	·122	·03		
	Diameter 2·523 inches.	52697	·146			
		62105	·164			
		71513	·188			
		80921	·214	·22		
		85625	·230	·28		
		90329	·250	·41		
		93629	·264*	.....	93629 lbs. = 41·79 tons.	
5.	Second London Mixture.	33881	·068			Broke by splitting in the middle and at the ends, one being broken off. See Plate as above.
	About two-thirds of old	43289	·086			
	metal, and one-third of	52697	·102			
	best Scotch pig-iron.	62105	·117			
	From Exp. 21, Table III.	71513	·133			
	Diameter 2·511 inches.	80921	·149			
		90329	·164			
		99737	·181			
		109145	·197	·04		
		118553	·218	·14		
		123601	·233	·26		
		127432	·244*	.....	127432 lbs. = 56·89 tons.	

TABLE V. (continued).

No. of Experiment.	Description of Pillar, 6 feet 3 inches long, and as above.	Weights laid on.	Decrement.	Deflection.	Breaking weight, or that with which the pillar sank.	Remarks.
		lbs.	inch.	inch.		
6.	Second London Mixture. About two-thirds of old metal, and one-third of best Scotch pig-iron. From Exp. 22, Table III. Diameter 2·496 inches.	33881	·060			Broke in the middle and at the ends. See Plate XXXIII.
		43289	·077			
		52697	·094			
		62105	·110			
		71513	·126			
		80921	·143			
		90329	·160			
		99737	·177			
		109145	·197			
		116672	·213*	.....	116672 lbs. = 52·09 tons.	
7.	Second London Mixture. Mean diameter from eight measurements 1·530 in.	7358	·014			For the form of fracture in this and the two following pillars, see Plate as above.
		9710	·024			
		12062	·032			
		14414	·040			
		16766	·048			
		19118	·057			
		21470	·066			
		23822	·075			
		26174	·091			
		26527	·093*	.....	26527 lbs.	
8.	Second London Mixture. From Exp. 23, Table III. Diameter 1·530 inch.	7358	·027			
		9710	·036			
		12062	·044			
		14414	·051	·06		
		16766	·059			
		19118	·068	·12		
		21470	·077	·28		
		23822	·090	·38		
		24763	·095*	.....	24763 lbs.	
9.	Second London Mixture. From Exp. 24, Table III. Diameter 1·541 inch.	7358	·027			
		9710	·034			
		12062	·043			
		14414	·051			
		16766	·060			
		19118	·068	·07		
		21470	·077			
		23822	·088	·19		
		25939	·098*	.....	25939 lbs.	

Additions to the preceding experiments in Table V., the pillars being cut from the $1\frac{1}{2}$ inch bars used in Experiments 7, 8 and 9, their lengths being as below, and their ends turned flat.

No. of Exp.	Description of Pillar.	Breaking weight.	Remarks.
		lbs.	
10.	Length 3 feet. Diameter 1·528 inch. From Experiment 7.	57384	
11.	Length 3 feet. Diameter 1·544 inch. From Experiment 8.	70555	
12.	Length 3 feet. Diameter 1·527 inch. From Experiment 9.	57856	
13.	Length 2 feet 6 inches. Diameter 1·531 inch. From Experiment 8.	85138	These pillars broke in the middle, and broke off at each end. For form of fracture see figures in Plate XXXIII.
14.	Length 2 feet 6 inches. Diameter 1·530 inch. From Experiment 9.	72907	
15.	Length 2 feet 6 inches. Diameter 1·515 inch. From Experiment 7.	70555	

TABLE VI.—Solid Pillars cut out of the 10-feet pillars, $2\frac{1}{4}$ in. diameter, from various parts of the kingdom, the pillars being now rendered straight and cut to 5 feet long nearly, and of which the first three have their ends turned flat and perpendicular to the axis, and the others have their ends rounded, so that the force is applied along the axis of the pillar.

No. of Experiment.	Description of Pillar, 5 feet long nearly, and as above.	Weights laid on.	Decrement of length.	Deflection of middle.	Breaking weight, or that with which the pillar sank.	Remarks.
		lbs.	inch.	inch.		
1.	Old Hill Iron, No. 1. Staffordshire. Cold blast. From the pillar in Exp. 19, Table III. Length 59·9 inches.	20693	·038			Supposing that the strength of pillars of the same diameter varies inversely as the 1·7th power of the length, the breaking weight of this, if 5 feet long, would be 135592 lbs., and of the following pillar 122863 lbs.; mean 129227 lbs. = 57·69 tons. For form of fracture see figure in Plate XXXIII.
		38644	·074			
		48052	·090			
		57460	·109			
		66868	·128			
		76276	·146			
		85684	·165			
		95092	·184	·09		
		104500	·202			
		113908	·223	·16		
		117612	·234	·20		
		123212	·247	·27		
		128812	·264	·37		
		135977	·285*	·50*	135977 lbs. = 60·70 tons.	
2.	Old Hill Iron, No. 1. From the same pillar as the preceding experiment, Table III., Exp. 19. Length 59·9 inches.	38644	·097	·05		One of these two pillars broke across in the middle, and near to each end.
		48052	·118			
		57460	·137	·07		
		66868	·157			
		76276	·178	·10		
		85684	·198	·13		
		95092	·220	·17		
		104500	·243	·21		
		113908	·270	·34		
		117612	·285	·46		
		123212	·308*	·64*	123212 lbs. = 55·01 tons.	
3.	Blaenavon Iron, No. 1. From the pillar in Exp. 17 or 18, Table III. Length 5 feet.	20693	·052			
		38644	·080			
		48052	·096			
		57460	·111			
		66868	·127			
		76276	·145			
		85684	·161			
		95092	·179			
		104500	·202			
		113908	·239			
		117612	·253*	.....	117612 lbs. = 52·51 tons.	
4.	Pillars with their ends rounded (see figure). Low Moor Iron, No. 2. From the pillar in Exp. 11, Table II. Length 59·5 inches. Diameter 2·553 ins.	48917	.....	·04		
		53691	.....	·07		
		58325	.....	·12		
		63029	.....	·26		
		65381	.....	·60	65381 lbs., or 64458 lbs. for a length of 5 feet.	The ends of the pillar were crushed by the pressure, the diameter of the crushed part being ·73 inch, and the crushing had been perceptible, and apparently one-third to half an inch when two-thirds of the breaking weight was laid on. There were several cracks at the ends of the pillar through the pressure. The strength had probably been much increased by the flattening of the ends. See Plate XXXIII. for the forms of fracture in this and the two following pillars.
5.	Blaenavon Iron, No. 3. From the pillar in Exp. 14, Table II. Length 59·5 inches. Diameter 2·546 ins.	53691	.....	·05		With a pressure of three-fourths of the breaking weight, the diameter of the compressed circle at the ends was about ·6 inch, and the diameter after fracture was ·79 inch nearly.
		58325	.....	·07		
		63029	.....	·10		
		65381	.....	·13		
		67733	.....	·18	71967 lbs., or 70950 lbs. if the length had been 5 feet.	
		70085	.....	·26		

TABLE VI. (continued).

No. of Experiment.	Description of Pillar, 5 feet long nearly, and as above.	Weights laid on.	Decrement of length.	Deflection of middle.	Breaking weight, or that with which the pillar sank.	Remarks.
		lbs.	inch.			
6.	Blaenavon Iron, No. 3.	58325	.....	·14		Diameter of crushed end of pillar ·63
	From the pillar in Exp. 14,	67733	.....	·22		with weight 30101 lbs., and after
	Table II., same as last.	72437	.....	·30	73378 lbs., or 72342 lbs.	fracture ·79.
	Length 59·5 inches.				for a length of 5 feet.	Many small cracks in the end.
	Diameter 2·546 inches.					

The strength of the three last pillars had probably been much increased by the flattening at the ends. If the ends of the pillar had been small and not crushed, the strength would have been about one-third of that of a pillar of the same diameter and length, with its ends turned flat and well-bedded.

The crushing at the ends was not only shown in the pillar, but in the hard steel plates against which it was pressed. The case is analogous to that of a wheel sinking into a hard surface upon which it should roll.

TABLE VII.—Transverse strength of the 10-feet Pillars, cast $2\frac{1}{2}$ inches diameter, and now cut to 5 feet long, laid on supports 4 feet 6 inches asunder, and broken by weights laid on in the middle.

Portland Iron, No. 1. Diameter 2·493 inches. Distance between the supports 4 feet 6 inches.

First half of bar.		Second half of bar.		
Weights laid on.	Deflections.	Weights laid on.	Deflections.	
lbs.	inch.	lbs.	inch.	
560	·06	560	·063	
1120	·145	1120	·113	
1680	·22	1680	·215	Breaking weight of first bar ..... 4324 lbs.
2240	·31	2240	·31	Breaking weight of second bar ..... 4940 } Mean 4632 lbs.; whence
2800	·40	2800	·40	Breaking weight for 10-ft. bar = 2084
3360	·51	3360	·50	Breaking weight for 7-ft. 6-in. bar = 2779
3920	·66	3920	·63	
4324	·75*	4480	·77	
broke.	ult. defl.	4940	·87*	
		broke.	ult. defl.	

Calder Iron, No. 1. Diameter 2·519 inches. Distance between the supports 4 feet 6 inches.

560	·05	560	·06	
1120	·14	1120	·14	
1680	·22	1680	·21	
2240	·30	2240	·30	Breaking weight of first bar ..... 4324 lbs.
2800	·38	2800	·395	Breaking weight of second bar ..... 3484 } Mean 3904 lbs.; whence
3360	·49	3360	·49	Breaking weight for 10-ft. bar = 1757
3640	·55	3484	·51*	Breaking weight for 7-ft. 6-in. bar = 2342
3920	·59	broke.	ult. defl.	
4324	·64*			
broke.	ult. defl.			

Coltness Iron, No. 1. Diameter 2·496 inches. Distance between the supports 4 feet 6 inches.

1120	·18	560	·09	
1680	·295	1120	·19	
2240	·41	1680	·31	
2800	·56	2240	·43	Breaking weight of first bar ..... 4044 lbs.
3360	·75	2800	·59	Breaking weight of second bar ..... 3652 } Mean 3848 lbs.; whence
4044	·96*	3360	·77	Breaking weight for 10-ft. bar = 1732
broke.	ult. defl.	3652	·84*	Breaking weight for 7-ft. 6-in. bar = 2309
		broke.	ult. defl.	

TABLE VII. (continued).

Carron Iron, No. 1. Diameter 2·510 inches. Distance between the supports 4 feet 6 inches.

First half of bar.		Second half of bar.	
Weights laid on.	Deflections.	Weights laid on.	Deflections.
lbs.	inch.	lbs.	inch.
560	·07	560	·08
1120	·155	1120	·16
1680	·25	1680	·26
2240	·35	2240	·37
2800	·46	2800	·48
3080	·54	3080	·55
3428	·62*	3428	·63*
broke.	ult. defl.	broke.	ult. defl.

Breaking weight of first bar ..... 3428 lbs.
Breaking weight of second bar ..... 3428 lbs. } Mean 3428 lbs.; whence
Breaking weight for 10-ft. bar = 1543
Breaking weight for 7-ft. 6-in. bar = 2057

TABLE VIII.—Crushing weights of short cylinders of different kinds of Cast Iron, cut from the bars, $2\frac{1}{2}$ inches diameter previously used, and now turned to be $\frac{3}{4}$ inch diameter nearly, and $1\frac{1}{2}$ inch high. The results are means from three or four experiments on each kind of iron. The specimens were usually cut out of the iron between the centre and the circumference of the bar, denominated the medium part. In several cases they were cut out of the centre of the bar, and sometimes out of the circumference.

Description of Iron.	Diameter of specimen.	Crushing weight of specimen.	Crushing weight per square inch of section.
	inch.	lbs.	lbs.
Old Park Iron, No. 1. Medium.	·747	38333	87466 } 88070 lbs. = 39·32 tons.
	·747	38862	88674 }
Old Park Iron, No. 1. Centre.	·747	32923	75122 } 74853 lbs. = 33·33 tons.
	·747	32453	74050 }
	·747	32923	75122 }
	·747	32571	74319 }
Derwent Iron, No. 1. Medium.	·747	43096	98334 } 97160 lbs. = 43·37 tons.
	·747	42684	97394 }
	·747	42684	97394 }
	·747	41861	95517 }
Coltness Iron, No. 1. Medium.	·747	27749	63316 } 63048 lbs. = 28·14 tons.
	·747	27925	63718 }
	·747	27455	62646 }
	·747	27396	62511 }
Blaenavon Iron, No. 1. Medium.	·748	31336	71310 } 70909 lbs. = 31·66 tons.
	·748	31395	71444 }
	·748	30748	69972 }
Level Iron, No. 1. Medium.	·749	30277	68716 } 68217 lbs. = 30·45 tons.
	·748	30101	68500 }
	·750	30101	68135 }
	·749	29748	67516 }
Carron Iron, No. 1. Medium.	·750	31747	71860 } 68509 lbs. = 30·58 tons.
	·750	29278	66272 }
	·751	30101	67953 }
	·751	30101	67953 }
London Mixture. Medium.	·749	36157	82061 } 80923 lbs. = 36·08 tons.
	·750	35452	80247 }
	·749	35452	80461 }
Calder Iron, No. 1. Medium.	·750	38039	86103 } 84848 lbs. = 37·79 tons.
	·749	35040	79596 }
	·750	38980	88233 }
	·749	37333	84730 }

TABLE VIII. (continued).

Description of Iron.	Diameter of specimen.	Crushing weight of specimen.	Crushing weight per square inch of section.	
	inch.	lbs.	lbs.	
Portland Iron, No. 1.	·747	41567	94848	
Medium.	·748	41273	93923	94809 lbs. = 42·32 tons.
	·750	42743	96750	
	·750	41391	93690	
Old Hill Iron, No. 1.	·749	24255	55049	
Medium.	·747	24221	55266	54761 lbs. = 24·45 tons.
	·750	23843	53969	
Low Moor Iron, No. 2.	·748	34570	78669	
Medium.	·748	33511	76260	77489 lbs. = 34·59 tons.
	·747	33982	77539	
Low Moor Iron, No. 2.	·742	29631	68525	
Centre.	·741	27749	64346	66407 lbs. = 29·65 tons.
	·742	28690	66349	
Blaenavon Iron, No. 3.	·737	35511	83941	
Medium.	·737	35746	83792	83517 lbs. = 37·28 tons.
	·737	35629	83517	
Blaenavon Iron, No. 3.	·747	38568	88003	
Centre.	·747	34805	79416	76643 lbs. = 34·22 tons.
	·747	27396	62511	
Second London Mixture.	·747	41626	94980	
Medium. From 2½-inch. pillar, as	·747	41861	95517	95338 lbs. = 42·56 tons.
all above have been.	·747	41861	95517	
Second London Mixture.	·747	34805	79411	
Centre. From 2½-inch. pillar, as	·747	34099	77806	78451 lbs. = 35·02 tons.
all above have been.	·748	34335	78135	
Second London Mixture.	·750	49858	112855	
Medium. From 1¼-inch pillar.	·750	48917	110725	111080 lbs. = 49·59 tons.
	·750	48447	109661	
Second London Mixture.	·750	41861	94754	
Centre. From 1¼-inch pillar.	·750	48094	108863	104071 lbs. = 46·46 tons.
	·750	47976	108595	
Low Moor Iron, No. 2.	·420	19415	89610	
From a hollow pillar 4 inches dia-	·422	11944	85395	87302 lbs. = 39·06 tons.
meter and ½ inch thick.	·421	11599	83273	
The height of the first two of these was ·72 inch, and of the last 1·502 inch.				
Low Moor Iron, No. 2.	·298	8010	114845	
From the thin ring of a hollow	·300	8255	116785	115993 lbs. = 51·78 tons.
pillar about 3¼ inches dia-	·298	8115	116350	
meter.				
Height of specimens ·53 inch.				
Low Moor Iron, No. 2.	·297	7905	114105	
From the thin ring of a hollow	·296	7555	109790	110212 lbs. = 49·20 tons.
pillar about 3¼ inches dia-	·296	7345	106740	
meter.				
Height of specimens ·53 inch.				

Of the different irons tried in the preceding experiments on pillars, whether solid or hollow, the external part of the casting was always harder than that near to the centre, and the iron of the external ring of a hollow casting was very hard, the hardness increasing with the thinness. Thus, in solid pillars 2½ inches diameter of Low Moor iron, No. 2, the crushing force per square inch of the central part was 29·65 tons, and that of the intermediate part near to the surface was 34·59 tons, whilst the external

ring, $\frac{1}{8}$ inch thick, of a hollow cylinder 4 inches diameter, of which the outer crust had been removed, was crushed with 39·06 tons per square inch; and external rings of the same iron thinner than half an inch, required from 49·2 to 51·78 tons per square inch to crush them. These facts show the great superiority of hollow pillars over solid ones of the same weight and length.

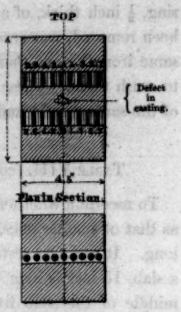
TABLE VIII. (continued).—Casting of larger dimensions than before used.

To ascertain whether the internal strength of larger pillars varied in the same manner as that of smaller ones, a cylindrical casting was made 5 inches diameter and 15 inches long. It was cast vertically, from Blaenavon iron. Through the axis of this cylinder, a slab, 15 inches long, 5 inches broad, and about 1 inch thick, was taken. Across the middle of this slab five cylinders, $1\frac{1}{2}$ inch long and $\frac{3}{4}$ inch diameter, were obtained at equal distances from each other, the middle one being in the centre, and the outer ones as near as possible to the sides. Three other small cylinders of the same size were also obtained, the positions of which were as in the figure.

Mark on Cylinder.	Diameter of Cylinder.	Crushing weight of Cylinder.	Crushing weight per square inch.	
	inch.	lbs.	lbs.	
A4	·749	27749	62979	Mean. 65739
B4	·748	30101	68500	
A2	·750	27749	62811	62444
B2	·748	27279	62078	
A	·748	27749	63147	64218
B	·748	28690	65289	
A1	·748	28690	65289	
A3	·748	29160	66358	

Comparing the results of the experiments above, it appears that the external part of the casting was somewhat stronger than the internal. But the variation was only from 62 to 66, and therefore much less than was obtained from the smaller masses.

To obtain some additional information as to the distribution of the metal in a large casting, one was formed 9 inches square and 12 inches long, or upwards. It was cast erect from Derwent iron, No. 1, and was then placed upon a slotting machine, and cut directly across in the middle. It was found, however, that there was a hole in the centre, about $\frac{1}{4}$ inch or 1 inch in diameter; transverse slices were therefore cut, one from above and one from below the central defective piece; these two slices, each 9 inches square and 1 inch thick nearly, were divided from one side to the other, passing through the centre of each, into nine equal parts, and from each part cylinders were cut, all of the same diameter, $\frac{3}{4}$ inch, and length $1\frac{1}{2}$ inch, four on each side of the central one. See figure annexed.



Upper Slice of Mass.

Mark on Cylinder.	Diameter of Ditto.	Crushing weight of Ditto.
	inch.	lbs.
8	·748	31512
6	·747	32923
4	·748	31042
2	·747	37157
A	·748	37627
1	·748	39980
3	·748	39039
5	·748	35875
7	·748	31983

Lower Slice of Mass.

Mark on Cylinder.	Diameter of Ditto.	Crushing weight of Ditto.
	inch.	lbs.
8	·748	30571
6	·748	31512
4	·749	32924
2	·748	30571
B	·748	29160
1	·748	30101
3	·748	32924
5	·749	32924
7	·749	33394

The iron in this mass seems to have been extremely irregular, particularly in the upper part of the casting. In the part marked B, which was cast under the greater head of metal, the iron from three divisions near the centre, or one-third of the distance across, was crushed with somewhat less weight than that from the external part, the greatest difference being from 29160 at the centre, to 33394 at the side. The mean resistance of the three central cylinders in this slice is 29944 lbs., and of the six external ones 32375 lbs., the difference being much less than in the iron of the smaller castings tried.

The iron in the upper part, marked A in the figure, being cast under a less head of metal than the former, and floating at the top of the softer metal below, was considerably stronger than the other. The central portion of this was, contrary to all our previous experience, much harder than the rest, and varied from 31512 and 31983 at the sides, to 37627, 39039 and 39980 near to the middle. To explain this strange anomaly in the absence of other knowledge, it seems not improbable that some additional heavier metal had been poured into the top part of the mould, when that in the lower part had in some degree solidified.

From the experiments upon the two last castings, which are on a larger scale than those which precede them, it appears that the difference of hardness between the external and internal parts of a large casting is much less than in a small one, and may frequently be neglected.

Appendix.

The following experiments were made after the others had been offered to the Royal Society, or during the months of October and November 1857, and as the former solid pillars were from various parts of the kingdom, and usually of the iron termed No. 1, I was desirous of adding the results from a few solid pillars of the quality No. 2. These latter pillars were mostly cast circular, and were made 10 feet long and $2\frac{1}{2}$ inches diameter, with their ends turned flat as before. To these were added pillars of nearly the same weight, of a square and a triangular form of section, to ascertain the comparative strength of pillars whose sections were circular, square, and triangular.

The cylindrical pillars, in the experiments of which an abstract is given below, though of different irons, were all cast from the same model; and the square and the triangular pillars were of the same kind of iron as two of the circular ones (the Calder, No. 2), and were intended to be of the same weights as these. All the pillars were made uniform throughout their lengths, 10 feet long each, and turned flat at the ends.

As the weights of the different pillars did not differ widely from 150 lbs. each, the breaking weight of each is reduced to what it would have been if the weight had been 150 lbs.

The reduced results are as below.

Pillars broken.	Second quality of iron, or that of No. 2.	Assumed weight of Pillars.	Breaking weights and forms of section.	
		lbs.	lbs.	
1st pillar.....	Coltness Iron, No. 2	150	○ 50328	
2nd pillar	Old Park Iron, No. 2	150	○ 54736	
3rd pillar	Blaenavon Iron, No. 2	150	○ 55611	
4th pillar	Calder Iron, No. 2	150	55658	Mean.
			○	55299 lbs.
5th pillar	Calder Iron, second specimen	150	54940	
6th pillar	Calder Iron, No. 2	150	49185	
			□	51537 lbs.
7th pillar	Calder Iron, second specimen	150	53890	
8th pillar	Calder Iron, No. 2	150	57332	
			△	61056 lbs.
9th pillar	Calder Iron, second specimen	150	64780	

From the three last mean results it appears that the strengths of circular, square, and triangular solid pillars of the same quality, weight and length, vary as 55299, 51537, and 61056, the last being the strongest. This last conclusion, respecting the strength of triangular solid pillars, may not be considered void of importance. See Notes to experiments 8 and 9, Table IX. following.

Whence it appears that solid triangular pillars with flat ends are stronger than those with either circular or square sections; but this seems not to be the case in pillars with rounded ends, for from my former experiments*, a pillar, whose section is +, as in the connecting rod of a steam-engine, the ends being moveable, is very weak to bear a strain

* Philosophical Transactions, 1840, pages 413 and 455.

as a pillar; and indeed less than half the strength of a hollow cylindrical pillar of the same weight and length, rounded at the ends.

Comparing the results of the cylindrical pillars tried, it appears that those of the second quality, or No. 2, are usually weaker than those of No. 1 of the same iron. In the irons below we have the breaking weights of pillars, 10 feet long and $2\frac{1}{2}$ inches diameter, of the first and second quality, as follow:—

	lbs.		lbs.
Old Park Iron, No. 1 . .	66086	No. 2, or second quality . .	55032
Calder Iron, No. 1 . . .	60677	No. 2, or second quality . .	55032
Coltness Iron, No. 1 . .	52680	No. 2, or second quality . .	50328

The properties of the irons of the second quality are deduced from the following Table of experiments; those of irons termed No. 1 may be obtained from the second summary of results, page 871.

TABLE IX.

Cast-iron pillars of the quality No. 2, supplied by Messrs. EASTON and AMOS, October and November 1857. The pillars were cast to be $2\frac{1}{2}$ inches diameter, and each was cut 10 feet long, the ends being turned flat and parallel.

Experiment 1.

Cylindrical Pillar, Coltness Iron, No. 2.
Mean diameter from eight admeasurements 2·494 inches.
Weight of Pillar 150 lbs.

Weights laid on. lbs.	Deflections. inch.	Decrements of length. inch.
20693	·17	·057
25397	·20	·078
30101	·24	·091
34805	·31	·105
39509	·42	·120
44213	·62	·135
48917	·98	·163
50328	Broke with this in two places near the middle.	

As the weights of the following pillars differed generally but little from that above, and a leading object being to compare the results together, the breaking weight has been reduced, as before mentioned, in each case, to what it would have been for a pillar weighing exactly 150 lbs.

Experiment 2.

Cylindrical Pillar, Old Park Iron, No. 2.

Mean diameter from eight admeasurements 2·503 inches.

Weight of Pillar 150 lbs. 13 oz. = 150·812 lbs.

Weights laid on. lbs.	Deflections. inch.	Decrements of length. inch.
20693	—	·084
25397	·07	·098
30101	—	·111
34805	·09	·124
39509	·12	·138
44213	·18	·152
48917	·30	·166
53621	·78	·194
55032 broke it.	It broke exactly in the middle.	—

lbs. lbs. lbs. lbs.

Whence 150·812 : 150 :: 55032 : 54736 = reduced breaking weight.

Experiment 3.

Cylindrical Pillar, Blaenavon Iron, No. 2.

Mean diameter from eight admeasurements 2·486 inches.

Weight of Pillar 148 lbs. 7 oz. = 148·437 lbs.

Weights laid on. lbs.	Deflections. inch.	Decrements of length. inch.
20693	—	·046
25397	—	·059
30101	—	·073
34805	·02	·086
39509	·09	·098
44213	·15	·112
48917	·30	·128
53621	·72	·156
55032 sank.	Not broken.	—	—

lbs. lbs. lbs. lbs.

Whence 148·437 : 150 :: 55032 : 55611 = reduced breaking weight.

Experiment 4.

Cylindrical Pillar, Calder Iron, No. 2.

Mean diameter from eight admeasurements 2·469 inches.

Weight of Pillar 148 lbs. 5 oz.=148·3125 lbs.

Weights laid on. lbs.	Deflections. inch.	Decrements of length. inch.
20693	·11	·070
25397	·13	·084
30101	·16	·097
34805	·19	·110
39509	·24	·122
44213	·30	·130
48917	·45	·147
53621	·72	·165
55032 broke it.	Broke $4\frac{1}{2}$ inches from the middle.	—

lbs. lbs. lbs. lbs.

Whence 148·312 : 150 :: 55032 : 55658=reduced breaking weight.

Experiment 5.

Cylindrical Pillar, Calder Iron, No. 2.

Mean diameter from six admeasurements 2·497 inches.

Weight of Pillar 150·25 lbs.

Weights laid on. lbs.	Deflections. inch.	Decrements of length. inch.
20693	·17	·077
25397	·21	·089
30101	·25	·102
34805	·29	·117
39509	·34	·131
44213	·38	·144
48917	·55	·158
53621	·94	·182
55032	Sank without fracture.	—

lbs. lbs. lbs. lbs.

Whence 150·25 : 150 :: 55032 : 54940=reduced breaking weight.

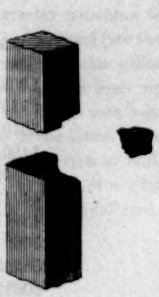
lbs.

$$\therefore \frac{55658+54940}{2}=55299 \text{ lbs.},$$

the mean strength of a cylindrical pillar of Calder Iron, No. 2, 10 feet long and $2\frac{1}{2}$ inches diameter nearly.

Experiment 6.—Square Pillar, Calder Iron, No. 2.
Side of square from eight admeasurements 2·194 inches.
Weight of Pillar 146 lbs. 5 oz. = 146·312 lbs.

Weights laid on. lbs.	Deflections. inch.	Decrements of length. inch.
20693	·19	·068
25397	·23	·081
30101	·27	·094
34805	·35	·108
39509	·47	·122
44213	·69	·142
47976 Broke.	—	—



Broke about 5 inches from the middle by bending in the diagonal.

lbs. lbs. lbs. lbs.
Whence 146·312 : 150 :: 47976 : 49185 = reduced breaking weight.

The figures standing over each other represent the pillar, at the place of fracture, near the middle; and the small additional piece, that of the part broken out from one corner by compression.

Experiment 7.—Square Pillar, Calder Iron, No. 2.
Side of square from eight admeasurements 2·221 inches.
Weight of Pillar 149·25 lbs.

Weights laid on. lbs.	Deflections. inch.	Decrements of length. inch.
20693	·14	·077
25397	·17	·090
30101	·21	·103
34805	·27	·115
39509	·34	·128
44213	·45	·143
48917	·64	·160
53621	1·14	·195

Broke in about a minute with this weight, bending in the direction of the side.

lbs. lbs. lbs. lbs.
Whence 149·25 : 150 :: 53621 : 53890 = reduced breaking weight.

$$\therefore \frac{\overset{\text{lbs.}}{53890 + 49185}}{2} = 51537 \text{ lbs.},$$

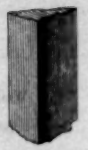
the mean strength of a square pillar 150 lbs. in weight, its length being 10 feet.

The vertical figures represent the pillar at the place of fracture, which in this case was different in form from the preceding one, and along the side; these and the smaller figure, representing the transverse section, show the small size of the compressed part.

Experiment 8.—Triangular Pillar, Calder Iron, No. 2.

Mean from six admeasurements of sides of triangle near the middle 3·545 inches.

Weight of Pillar 158·75 lbs.



Weights laid on. lbs.	Deflections. inch.	Decrements of length. inch.
20693	·23	·080
25397	·27	·092
30101	·33	·107
34805	·39	·121
39509	·46	·134
44213	·53	·147
48917	·64	·161
53621	·79	·177
58325	1·06	·203
60677	Broke, having bent parallel to one side.	—

lbs. lbs. lbs. lbs.

Whence 158·75 : 150 :: 60677 : 57332 = reduced breaking weight.

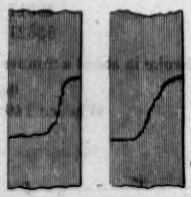
This triangular pillar differed but little, in the form of its fracture, from the preceding rectangular one, as shown by the figure of the compressed part along its side; its strength, however, is greater than that of any other yet tried.

Experiment 9.—Triangular Pillar, Calder Iron, No. 2.

Mean breadth of the sides from nine admeasurements, in three parts, near to the middle 3·47 inches.

Weight of Pillar 155·75 lbs.

Weights laid on. lbs.	Deflections. inch.	Decrements of length. inch.
20693	·08	·070
25397	·09	·082
30101	·11	·093
34805	·14	·105
39509	·18	·117
44213	·22	·130
48917	·27	·141
53621	·36	·154
58325	·48	·169
63029	·69	·185
67263	Broke. See form of fracture.	—



lbs. lbs. lbs. lbs.

Whence 155·75 : 150 :: 67263 : 64780 = reduced breaking weight.

lbs.

$\therefore \frac{57332 + 64780}{2} = 61056$ lbs.,

the mean strength of a triangular pillar, 10 feet long, if its weight were 150 lbs.

This pillar broke, different from the last, by compression of the edge, showing great additional strength.

Note. The last pillar broke by the compression of one of the edges, and the iron in these, being harder than that in the thicker parts, would offer greater resistance to crushing than in them; and hence the strength of the pillar would be increased (see the experiments in Table VIII.). The part subjected to crushing would in this pillar be at a greater distance from the centre of the section than in the others; and from my numerous experiments on the resistances to crushing and to tearing asunder in cast iron, it appears that the former is, on the average, about six times as great as the latter. See my experiments in the Report of the Commissioners on the Application of Iron to Railway Structures, page 15. From these and other causes the greater strength of a solid triangular pillar of cast iron, above one of a circular or square section, may be inferred; a conclusion not without importance in some applications of the metal.

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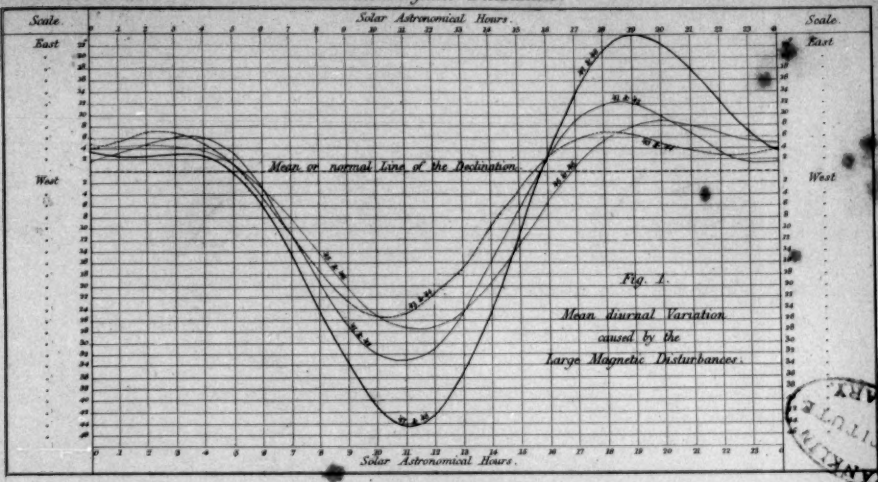

Fig. 1.

Mean diurnal Variation caused by the large Magnetic Disturbances.

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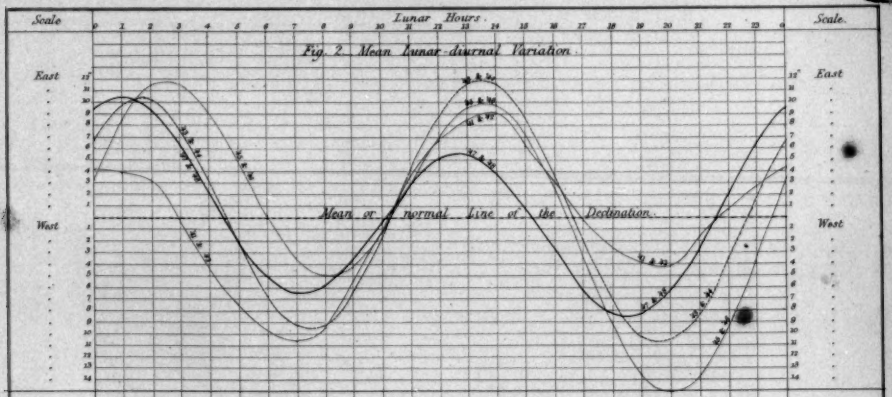

Fig. 2. Mean Lunar diurnal Variation.

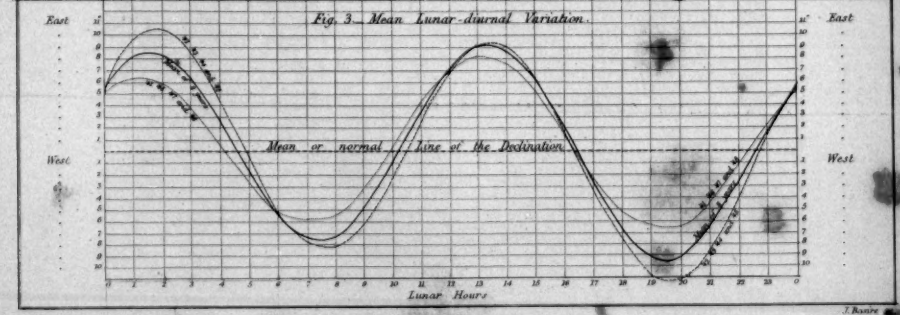

Fig. 3. Mean Lunar diurnal Variation.

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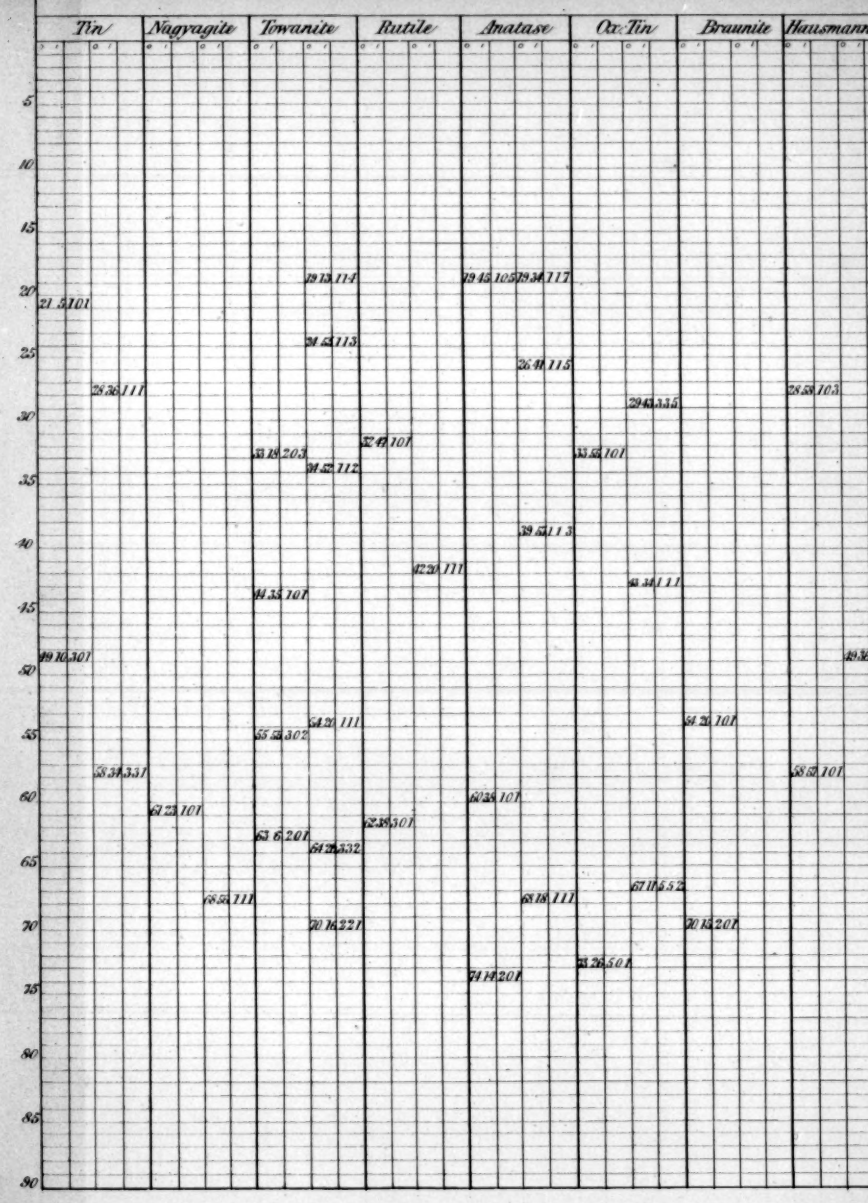
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54 21. 101
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60 25. 101
61 23. 101
62 38. 501
63 6. 201
64 21. 332
67 17. 5. 52
68 50. 111
68 18. 111
70 16. 321
70 15. 201
73 26. 501
74 14. 201

nnite	Idocrase	Zircon	Somervillite	Sarcolite	Scapolite	Apophyllite	Faujasite	Edingtonite	Scheelite
	34 10 113					14 5 105			16 32 102
	2 44 112					19 30 115			
				22 41 113					
					23 46 101				
								25 26 102	
	28 9 101								
		32 38 101	32 9 101		31 57 111	32 2 102			
									34 58 112
	31 7 111								35 38 102
				41 35 101					
		42 10 111							
								43 40 101	
	46 57 201								46 22 112
48 36 112									
				52 22 111		51 21 101	52 45 101		
								53 28 111	
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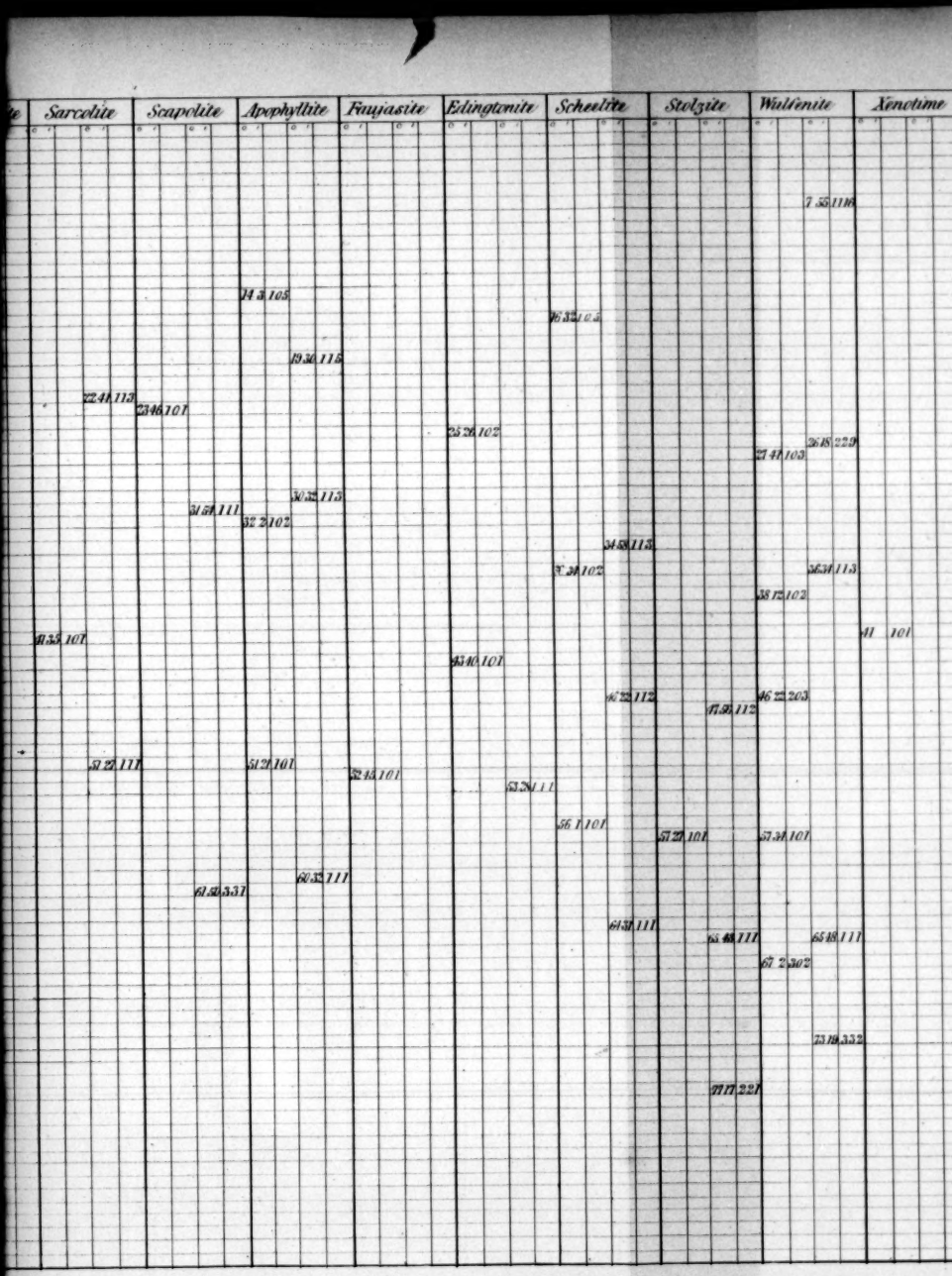
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Faujasite
Edingtonite
Scheelite
Stolzite
Wulfenite
Xenotime
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76 32 102
19 30 115
22 41 113
23 46 101
25 26 102
27 47 103
26 18 229
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30 32 115
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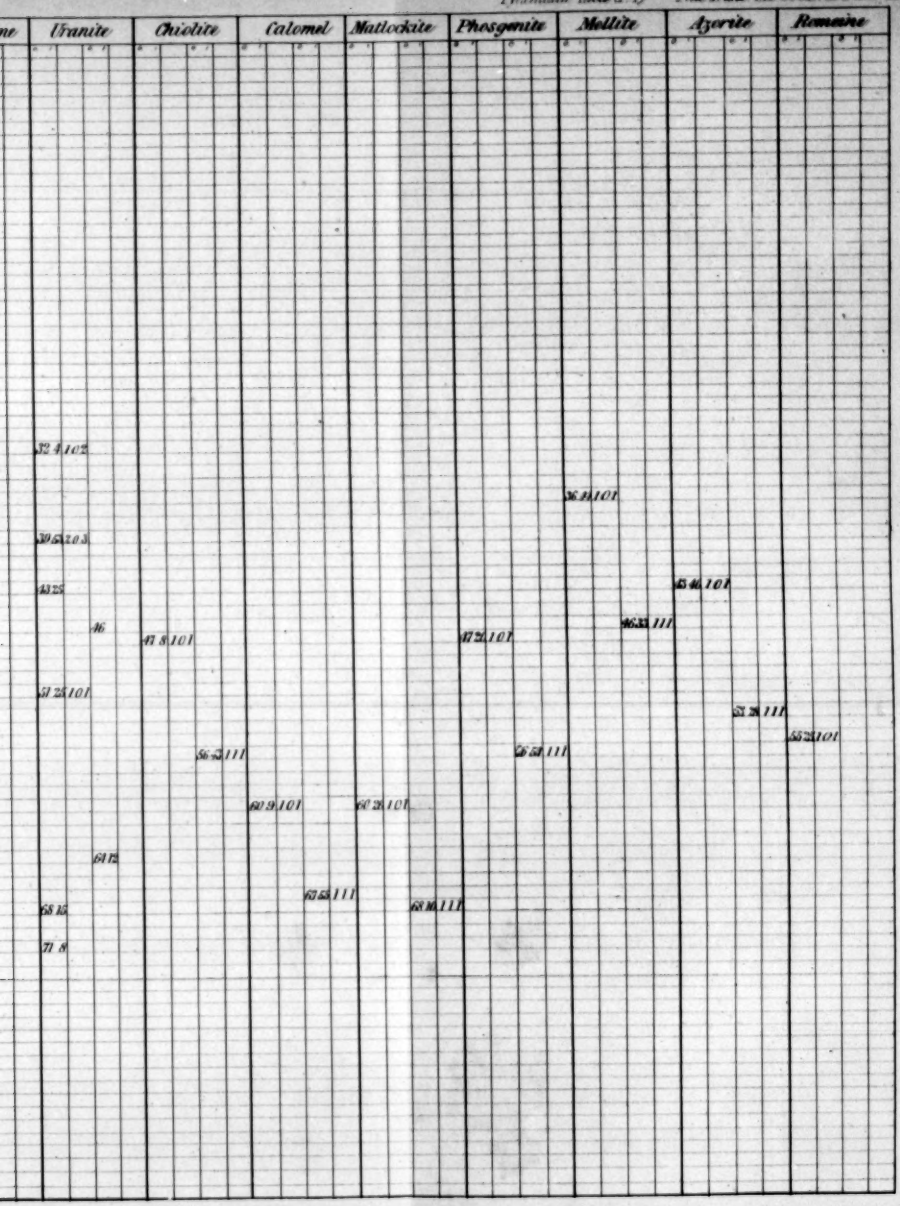
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Chiolite
Calomel
Matlockite
Phosgenite
Mellite
Azerite
Romeine
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39 65 2 0 3
43 23
46
47 8 1 0 1
51 25 1 0 1
56 2 1 1 1
60 9 1 0 1
60 28 1 0 1
64 12
67 68 1 1 1
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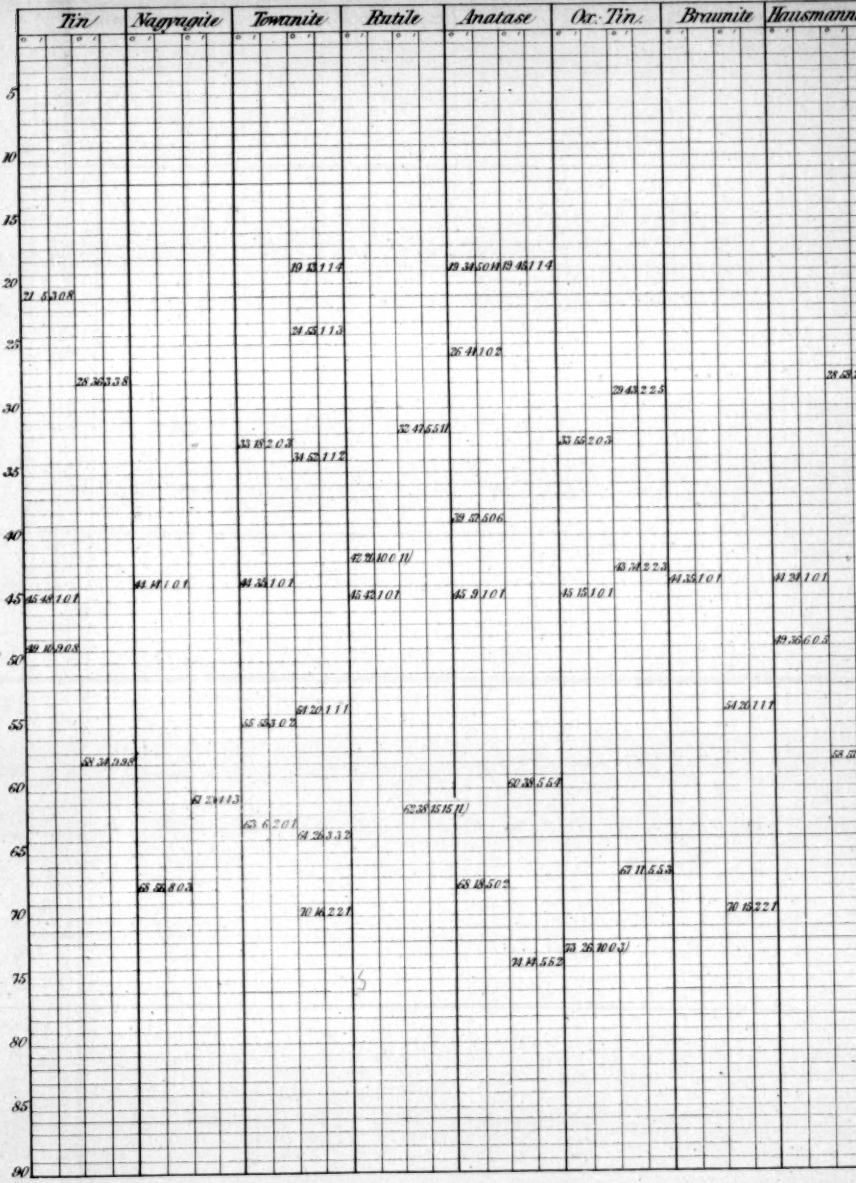
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Tennanite
Rutile
Anatase
Ox: Tin/
Braunite
Hausmann
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19 33 1 1 4
19 36 50 11
19 45 1 1 4
24 53 1 1 3
26 41 1 0 2
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29 43 2 2 8
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49 36 6 0 8
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68 18 5 0 2
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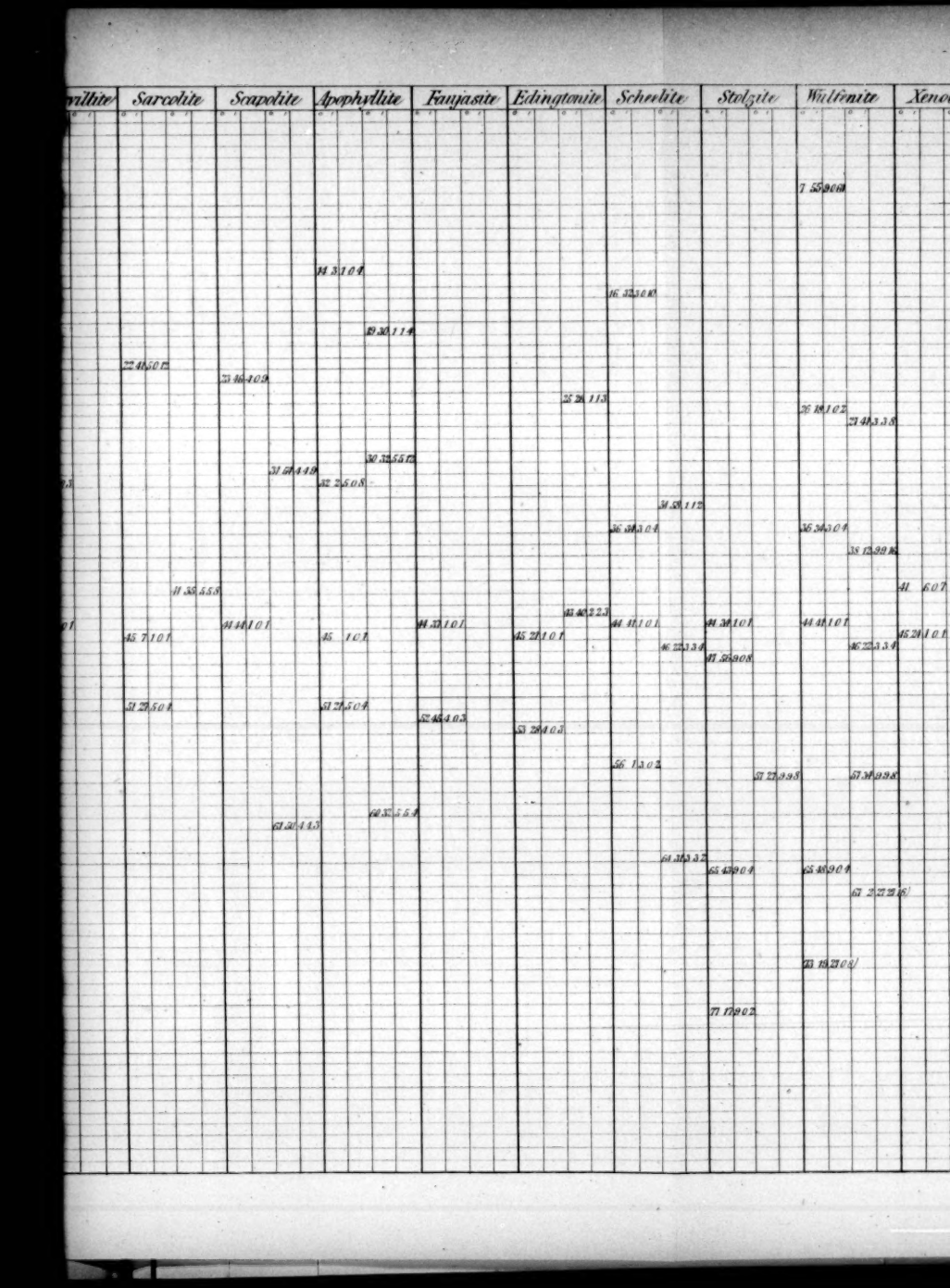
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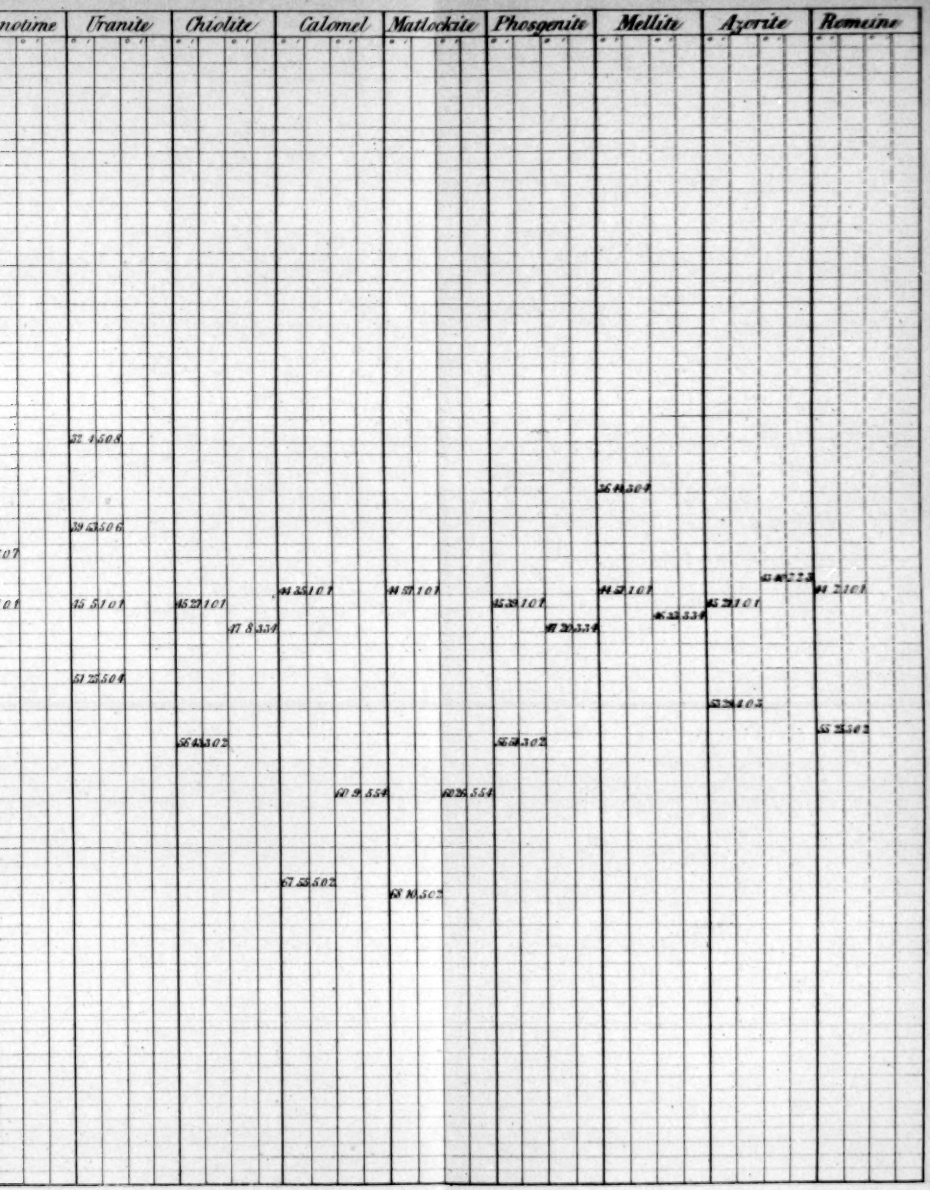
Fanjasite
Edingtonite
Scheelite
Stolzite
Wulfenite
Xeno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Chiolite
Calomel
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Phosgenite
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Azerite
Remeine
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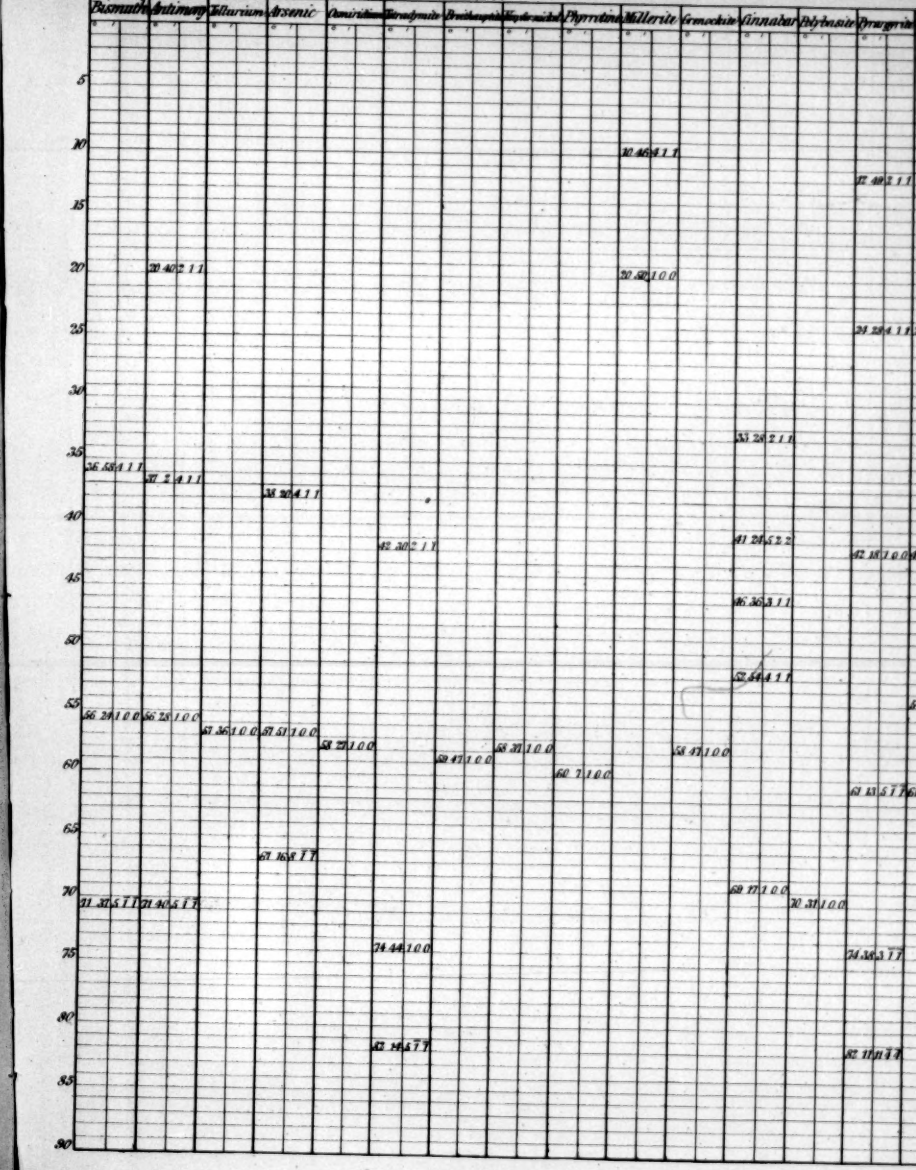
67 55 50 2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Tellurium
Arsenic
Cinnabar
Pyrrotine
Millerite
Greenockite
Cinnabar
Pyrargyrite
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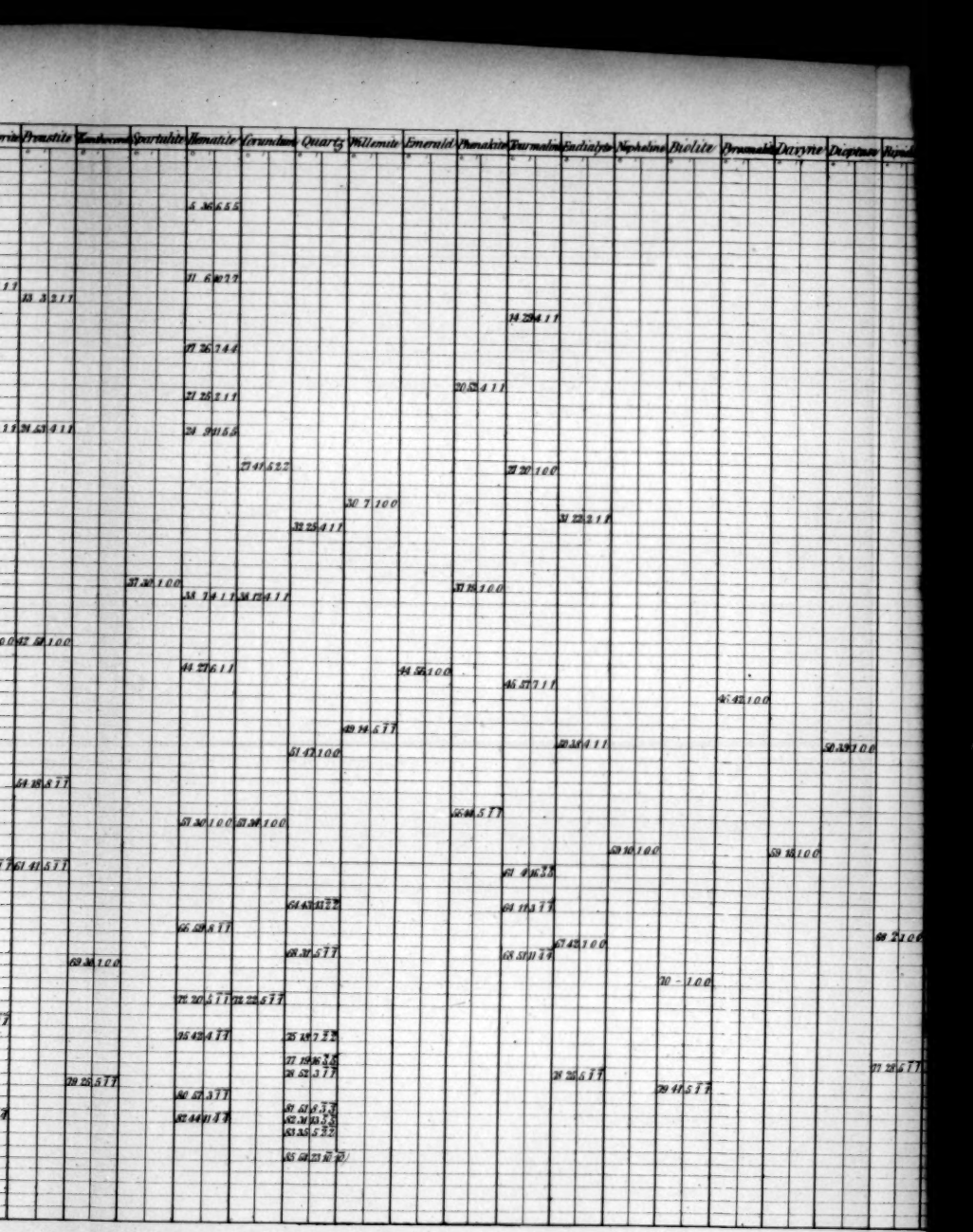
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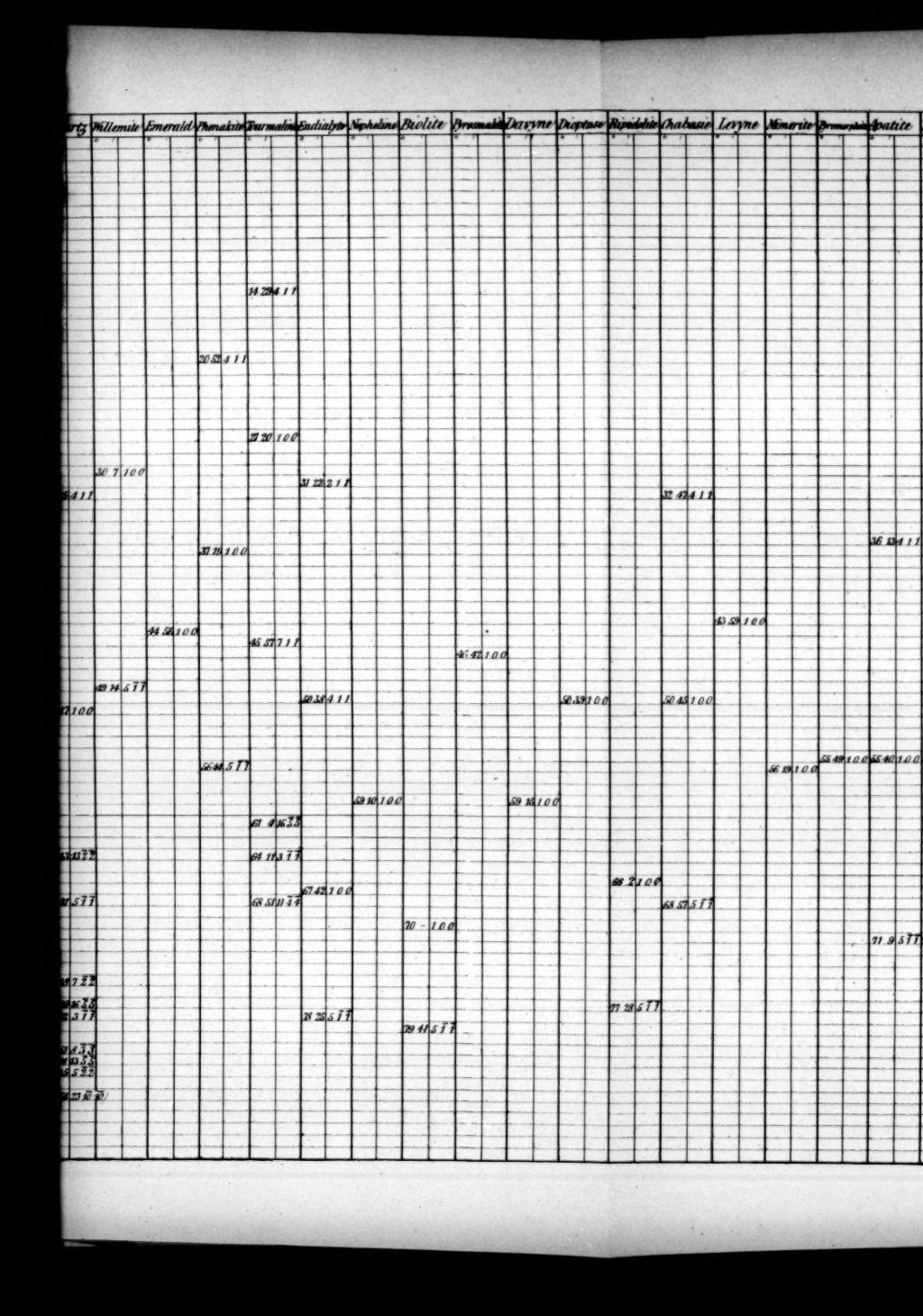
Tourmaline
Quartz
Willemite
Emerald
Phenakite
Tourmaline
Epidialyte
Nepheline
Dioptase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Briolite
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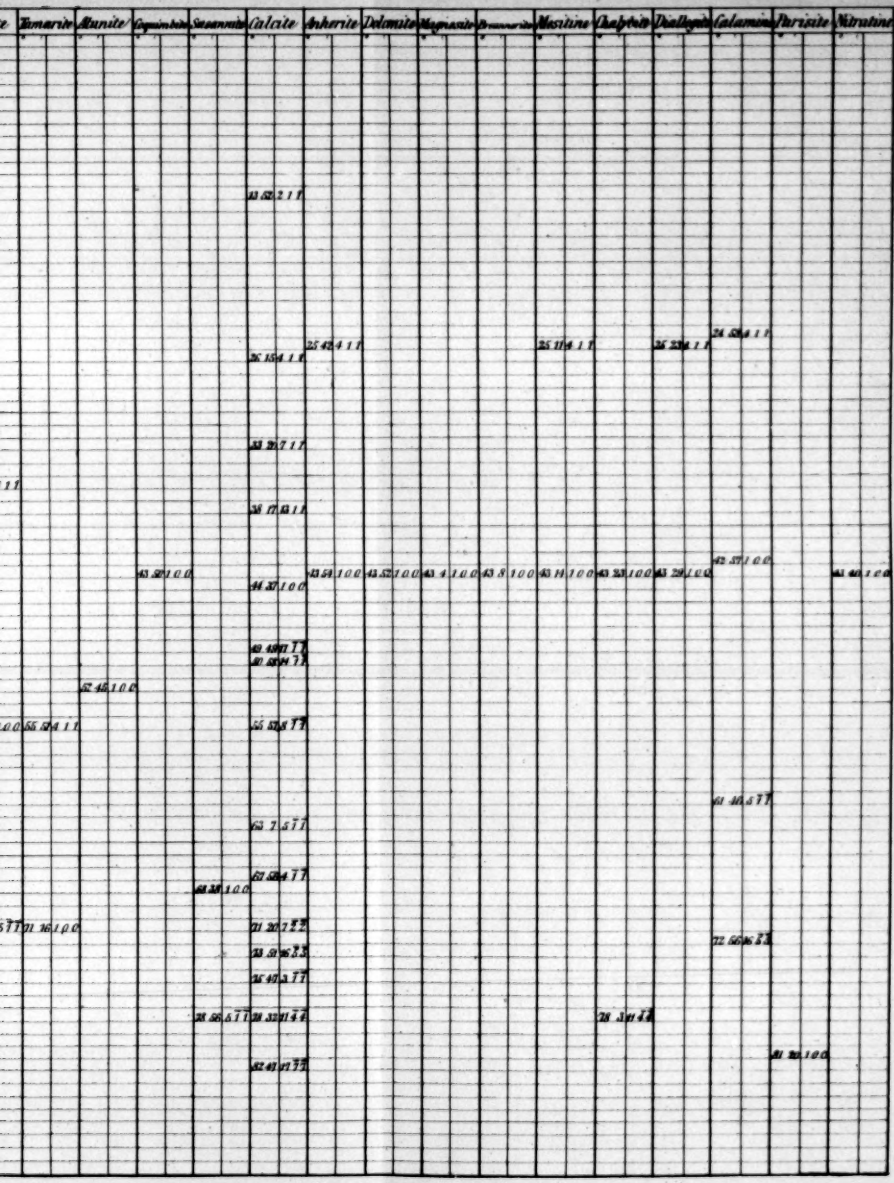
Rhodonite
Chabasie
Levyne
Mimerite
Apatite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Titanite
Cyprine
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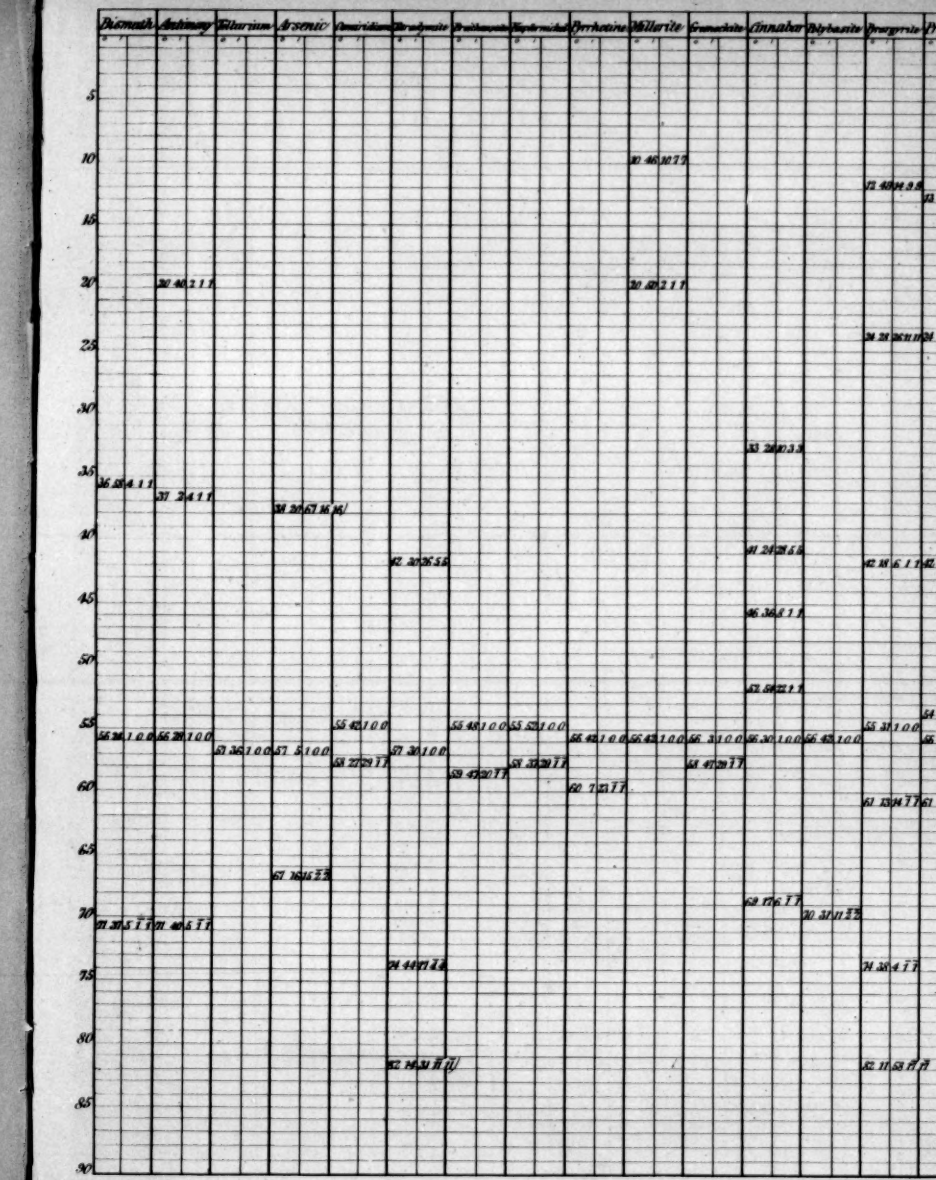
Magnesite
Diamond
Mesitine
Chalybite
Diallogite
Calamine
Parisite
Nitratine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Antimony
Tellurium
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Cinnabar
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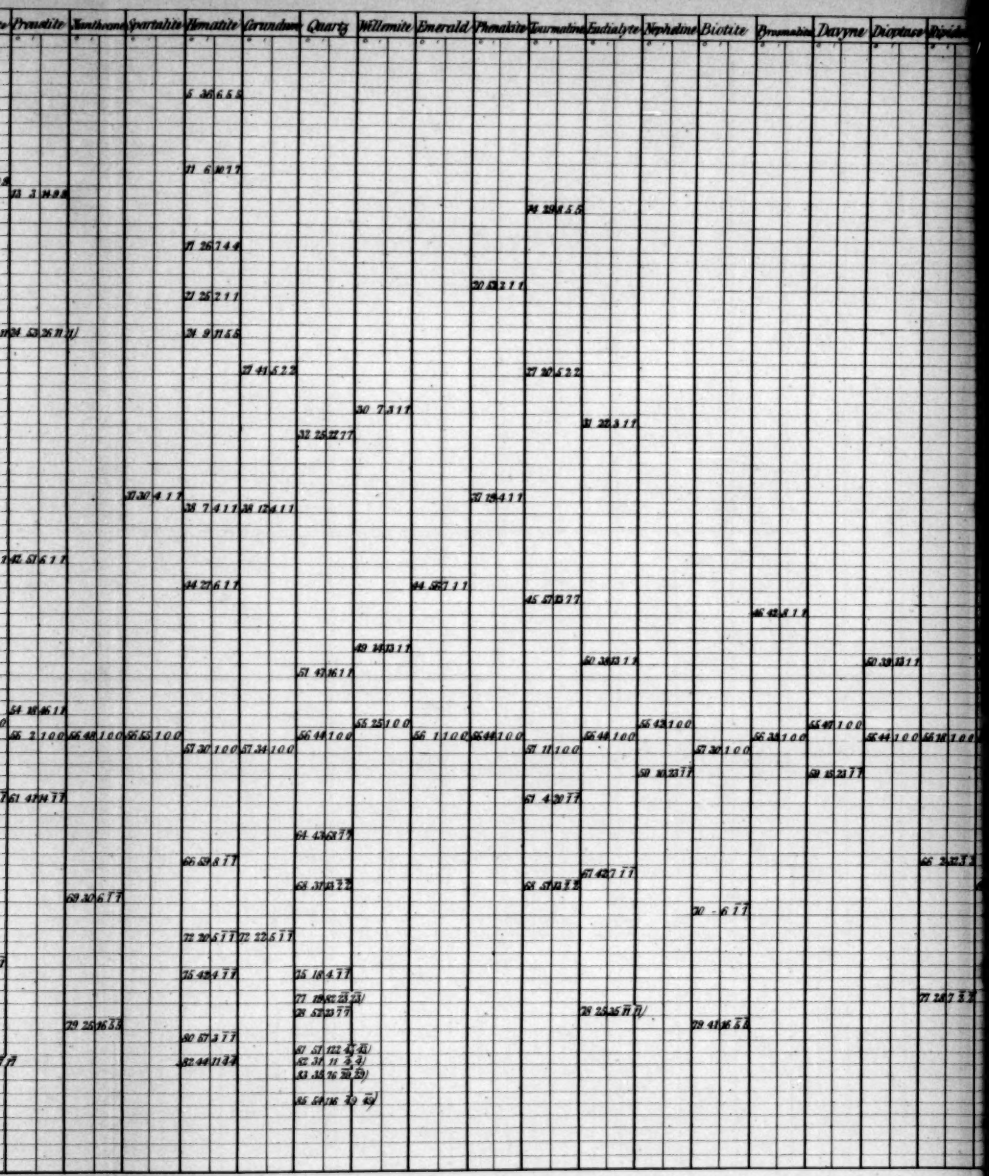
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Xanthocone
Spartalite
Hematite
Corundum
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Phenakite
Tourmaline
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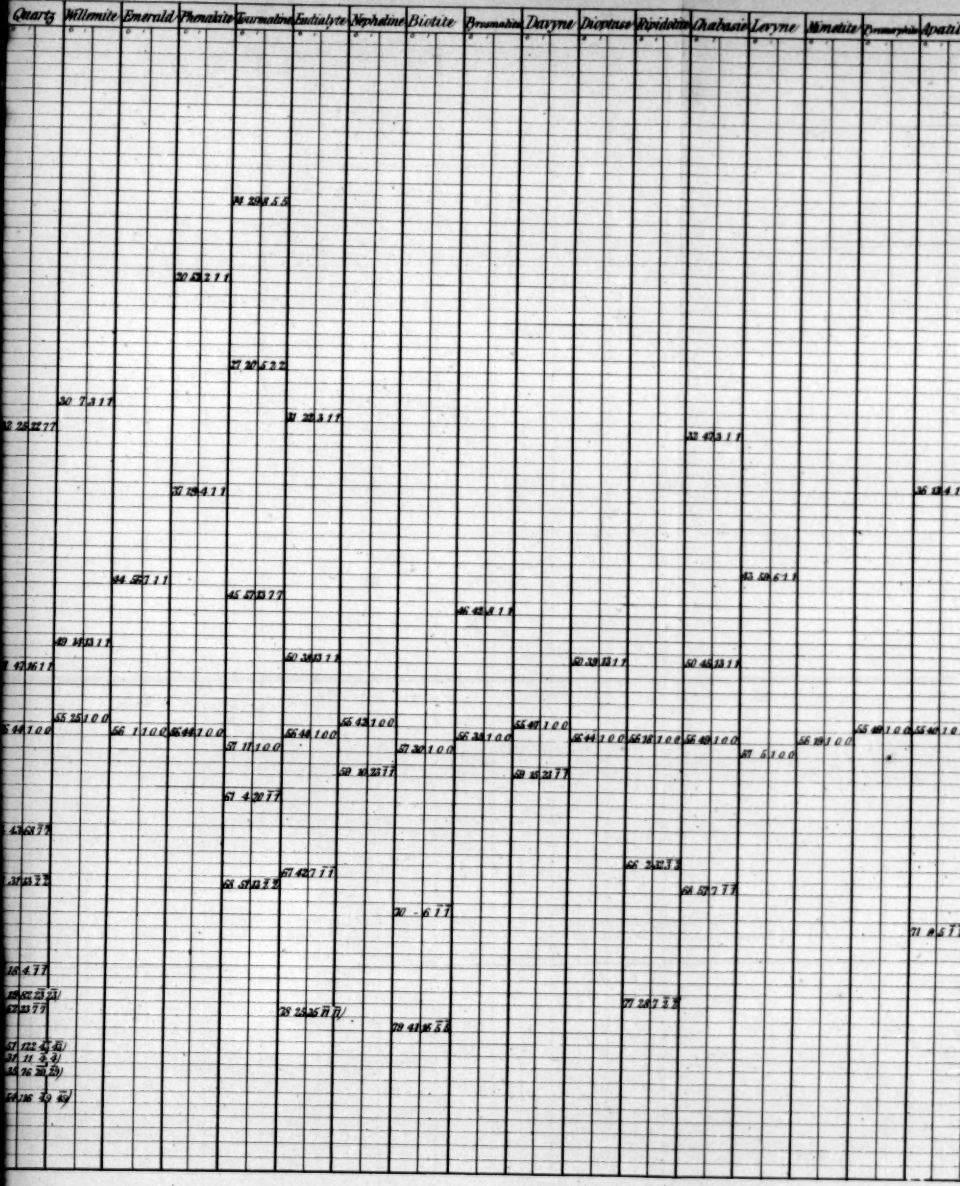
Nephaline
Biotite
Pyrenabite
Davyne
Dioptase
Diopside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Emerald
Phenakite
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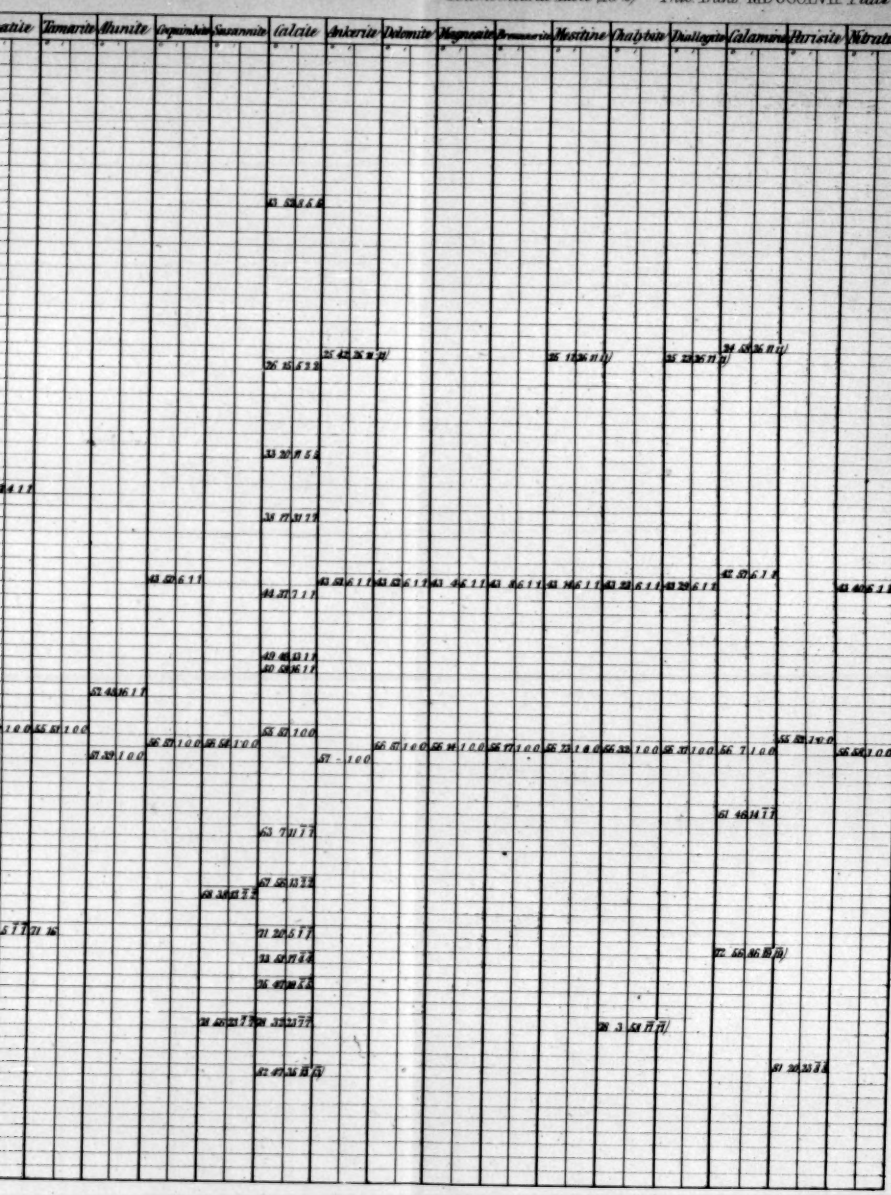
atite
Tamarite
Munite
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Suanite
Calcite
Ankerite
Dolomite
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Mesitine
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Calamine
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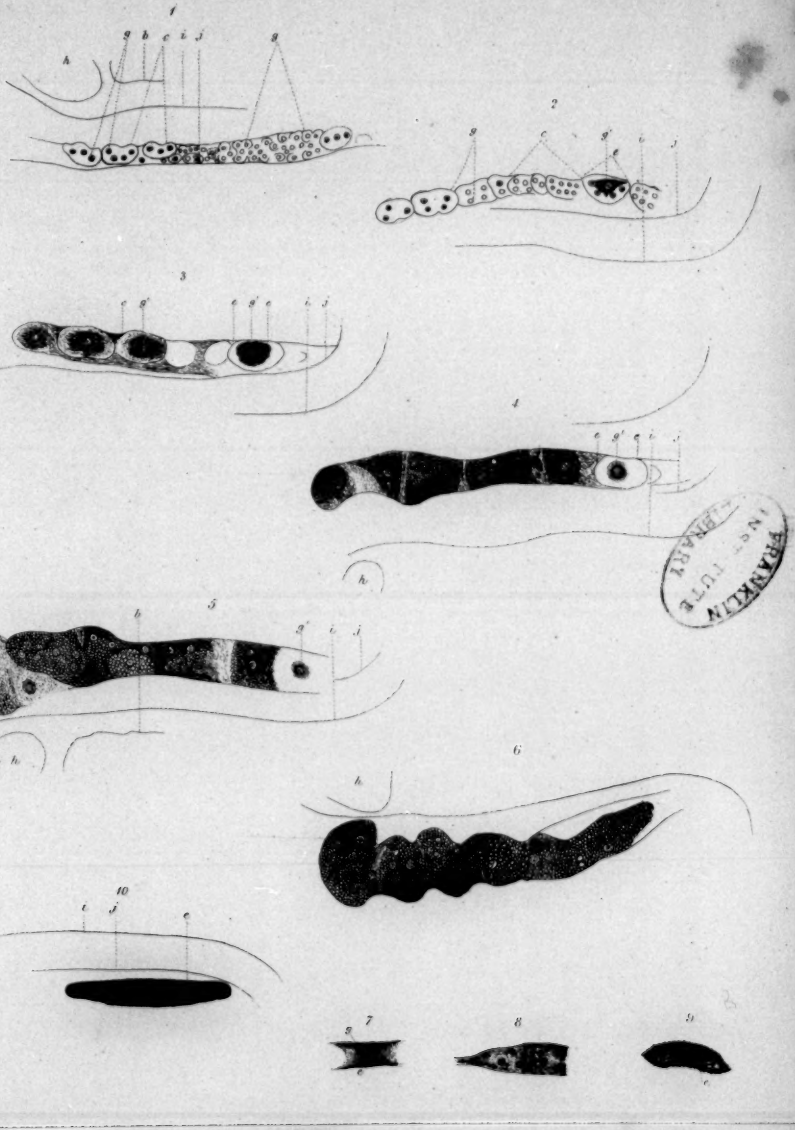

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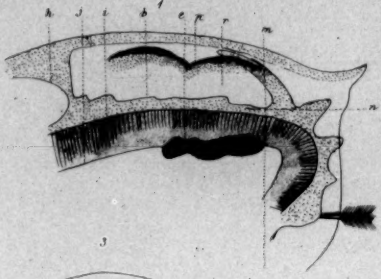

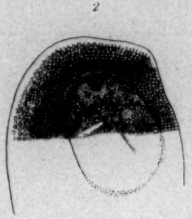

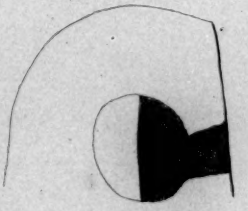

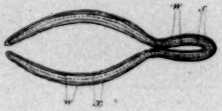

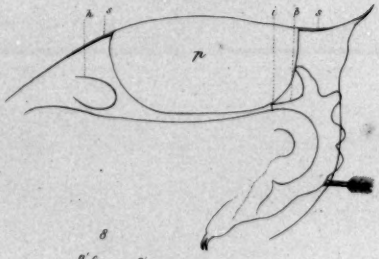

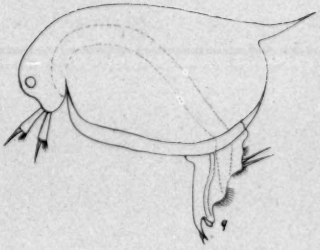

J. Lubbock del.

J. Basire, Lith.

Fig. 1.

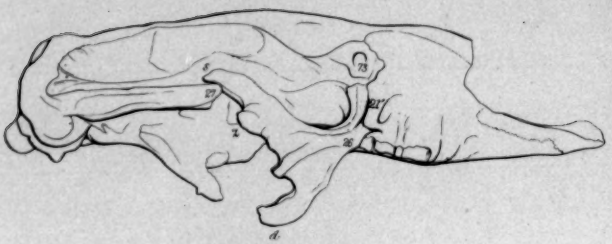

Fig. 2.

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Fig. 3.

Fig. 4.

Fig. 5.

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SCELIDOTHERIUM LEPTOCEPHALUM. 1/4

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Fig. 2.

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Fig. 3.

SCELIDOTHERIUM LEPTOCEPHALUM.

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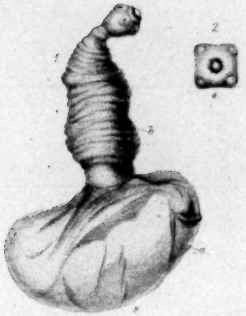

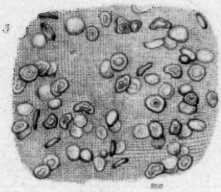

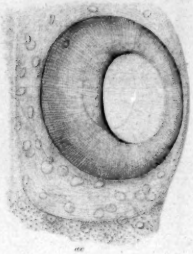

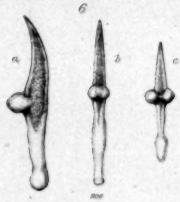

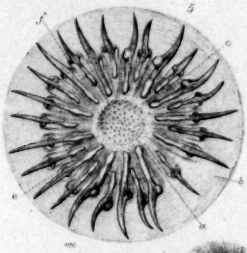

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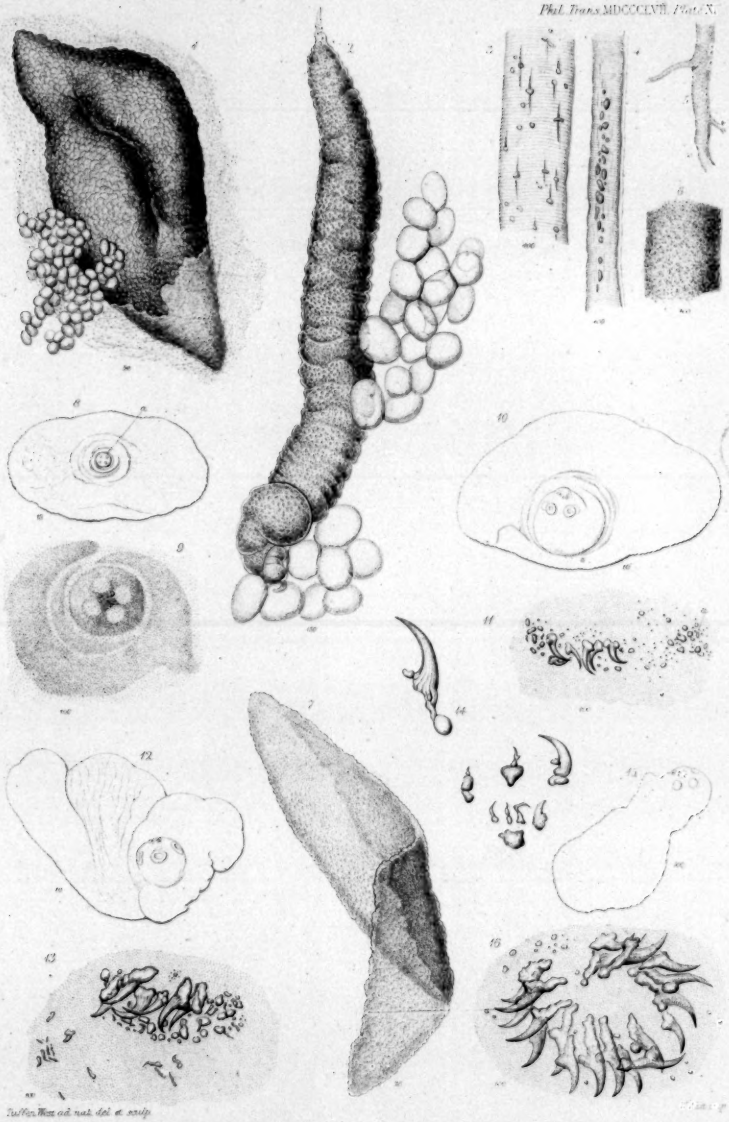

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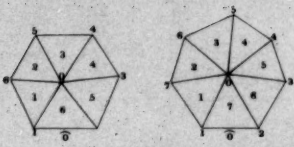

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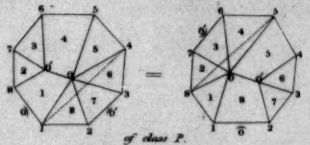

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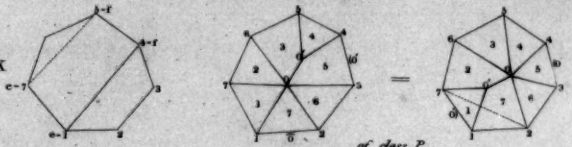

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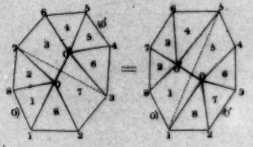

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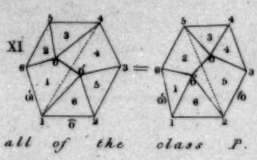

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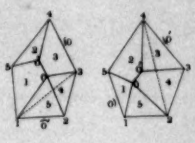

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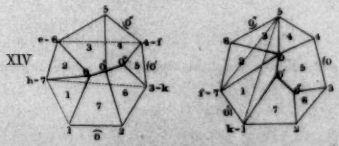

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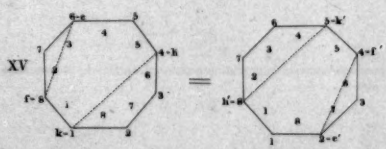

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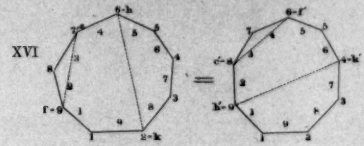

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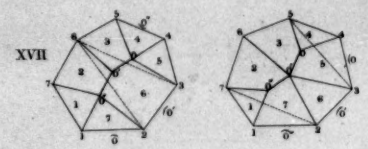

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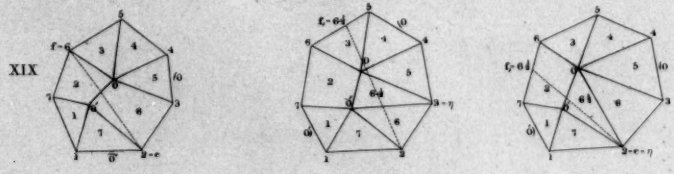

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J. Basire, sc.

II

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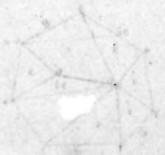

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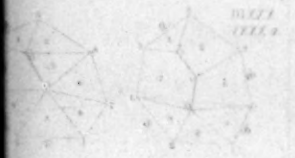

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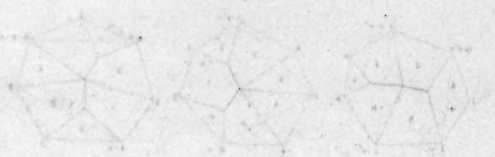
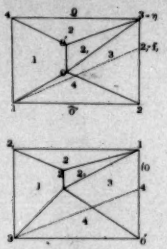
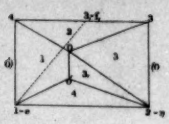


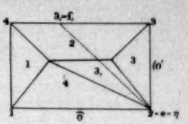
IIIII

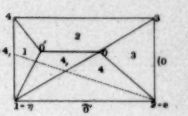
XX
&c.



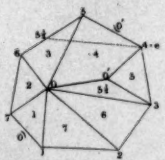


This is the S rejected in XXI.



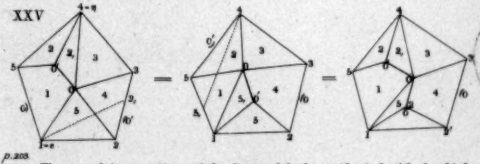


XXI



This is rejected from the class S in XXI because of the unavoidable pair, 05-34 and 34-05.

XXV



p.205.

The second is a repetition of the first, and both are identical with the third.

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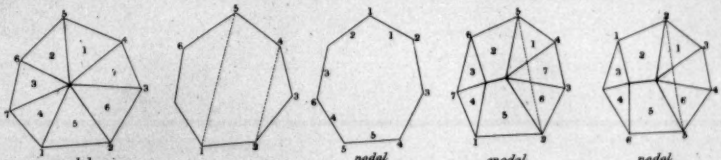
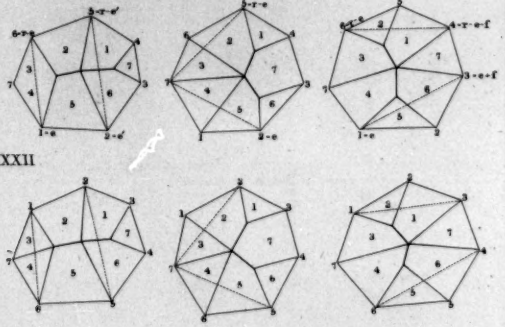
XXXI



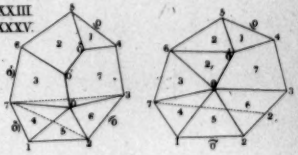
enodal nodal enodal nodal

XXXII



all utrally autopolar.

XXXIII & XXXV.

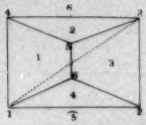


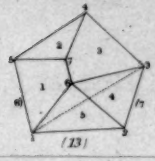
purely enodal.

J. Basire sc.

The autopolar 6-edron and 7 edra.

XXXVI





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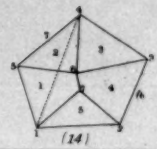

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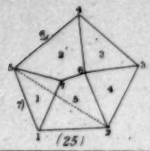

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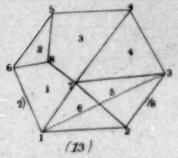

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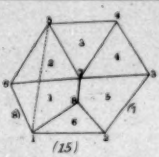

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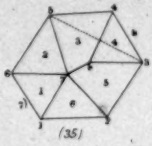

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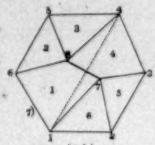
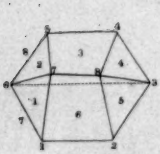


(14)

The 15 autopolar 8 edra.; XXXVI &c.



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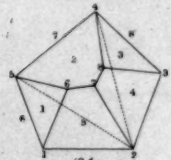
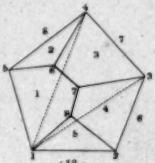


$\frac{(24}{25)}$



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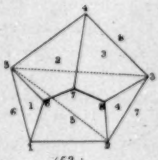

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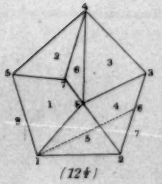
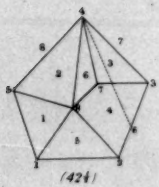
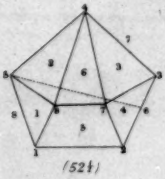


(12$\frac{4}{7}$)



(42$\frac{1}{7}$)



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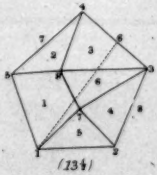

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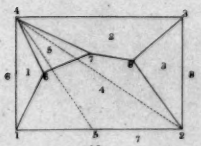
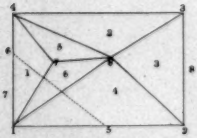


$\frac{(42}{41\frac{1}{7}})$



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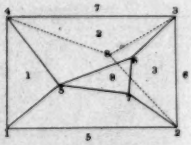

(84, 82, 83.)

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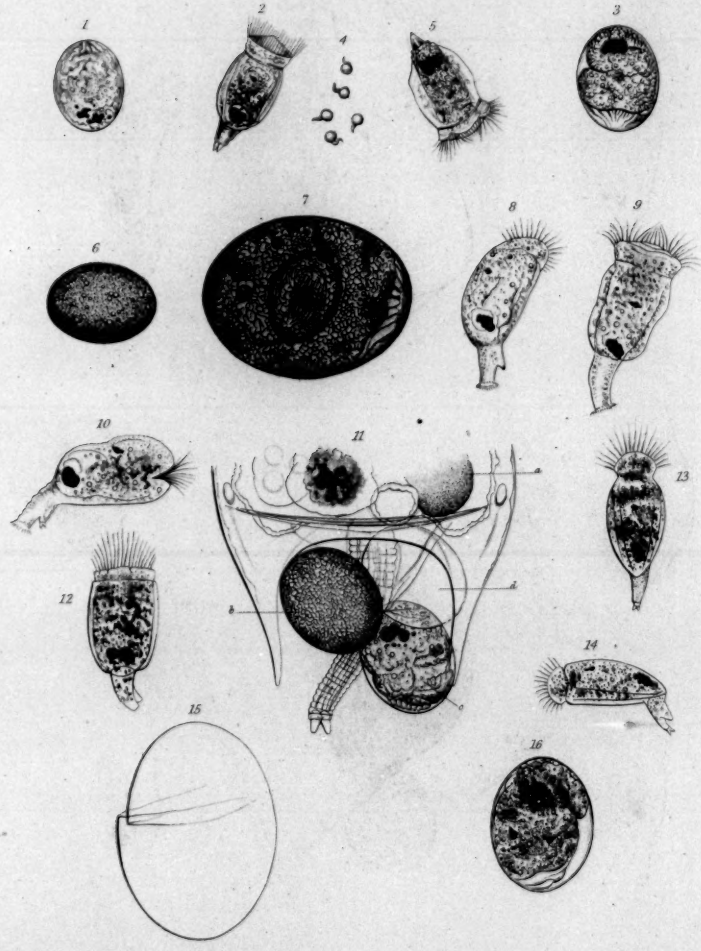

P.H. Gosse, ad. viv. del.

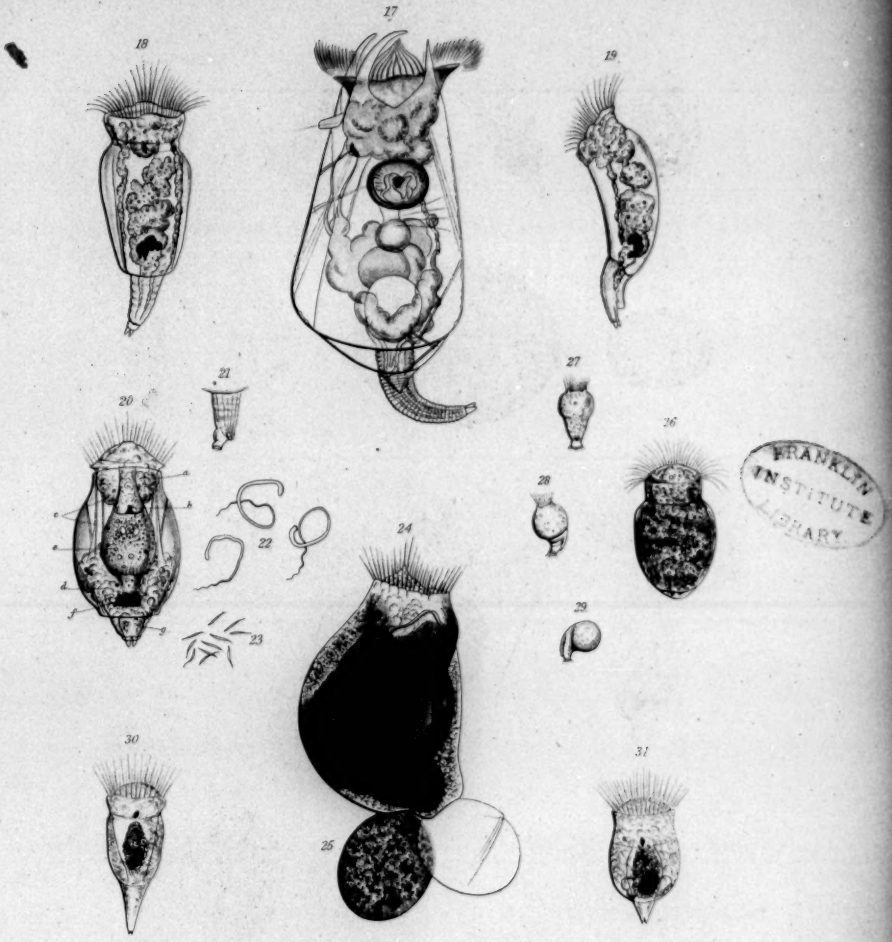

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Fig. 1.

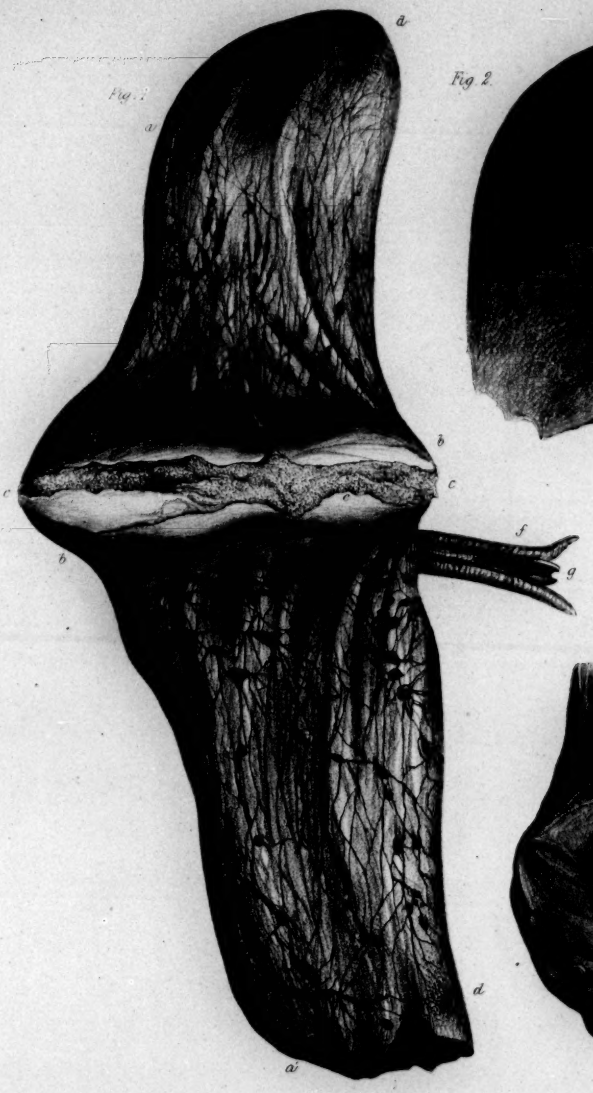

Fig. 2.

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Fig. 3.

From nat. on Stone by J. Erxleben.

Printed by J. Basire

FOETAL MEMBRANES & PLACENTA OF THE ELEPHANT (EL. INDICUS).

Fig. 1.

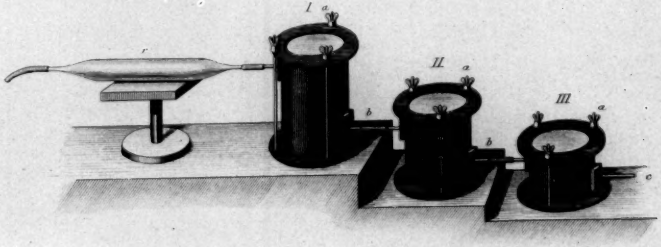

Fig. 2.

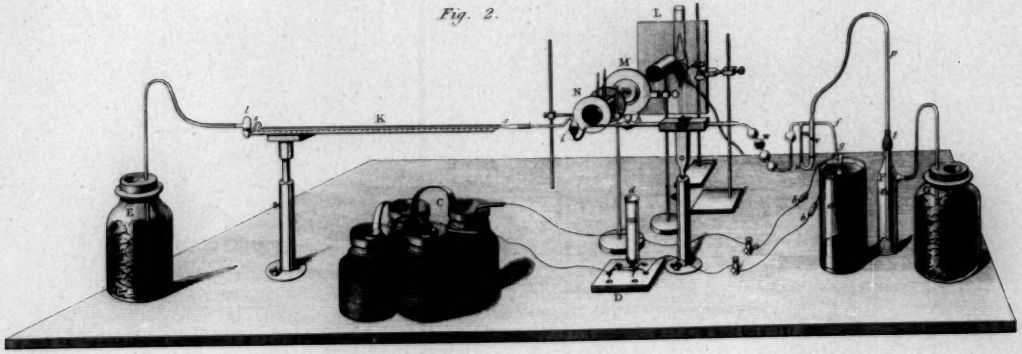

Fig. 3.

Fig. 4

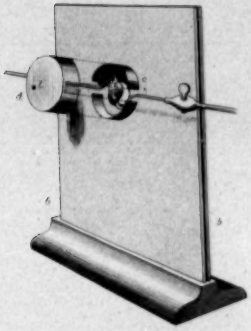

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Fig. 5.

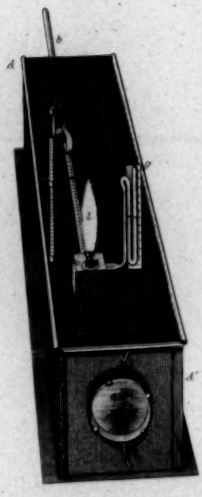

Fig. 6

Fig. I.

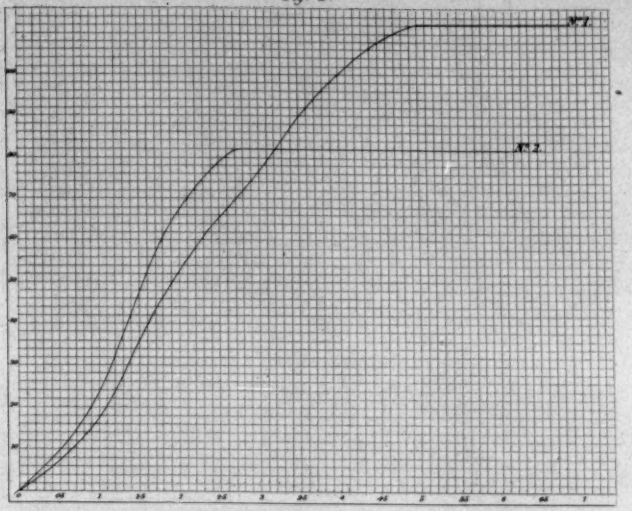

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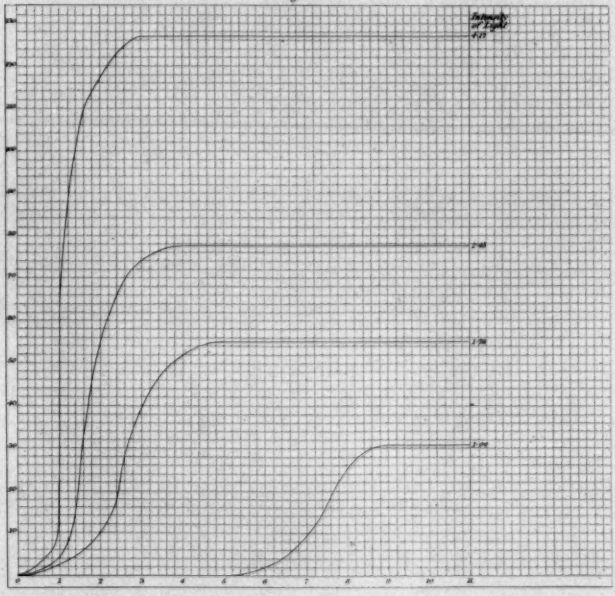
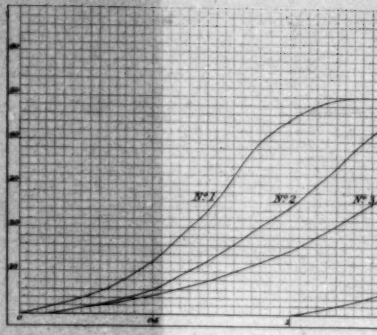
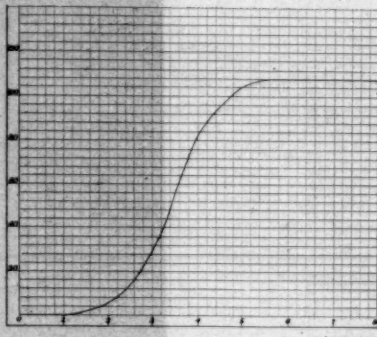
Fig. II.




N°1
N°2



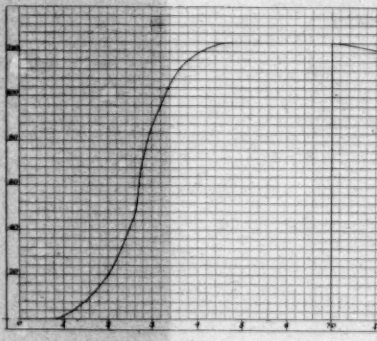


Fig. III.

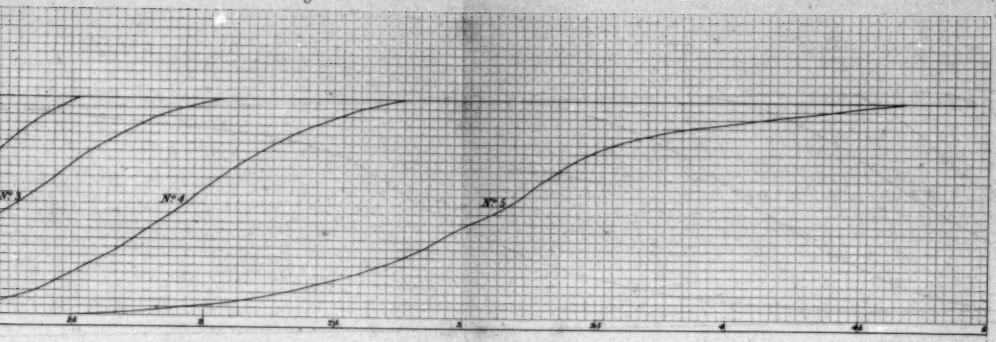


Fig. IV. A.

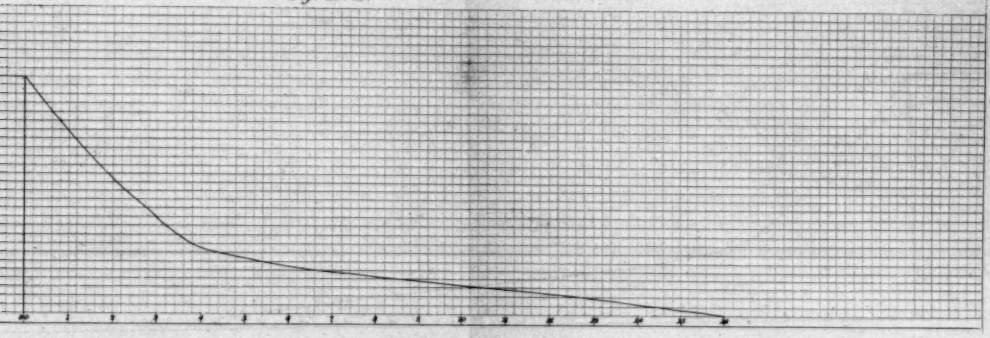


Fig. IV. B.

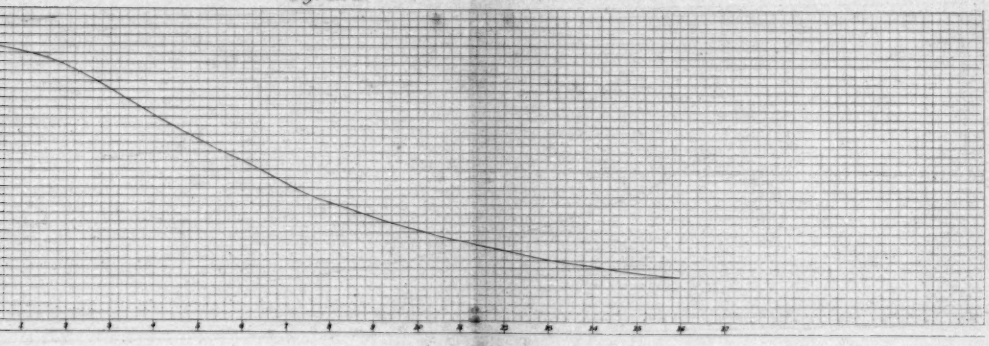
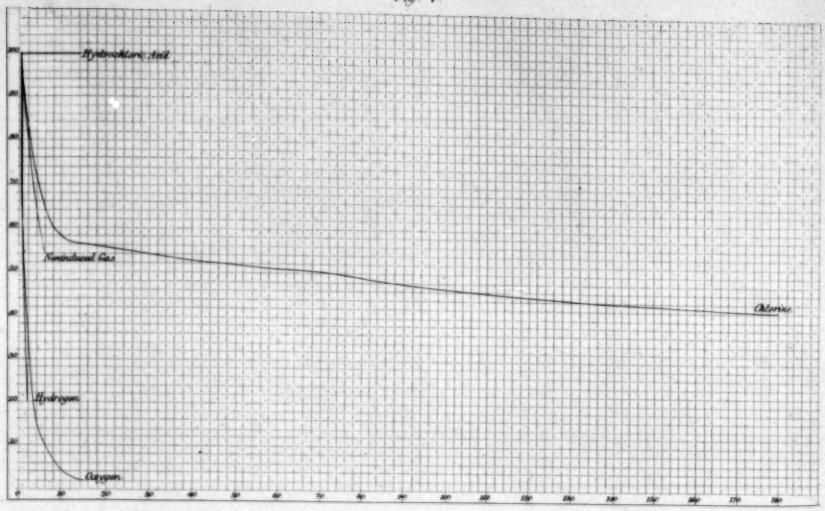


Fig. V.



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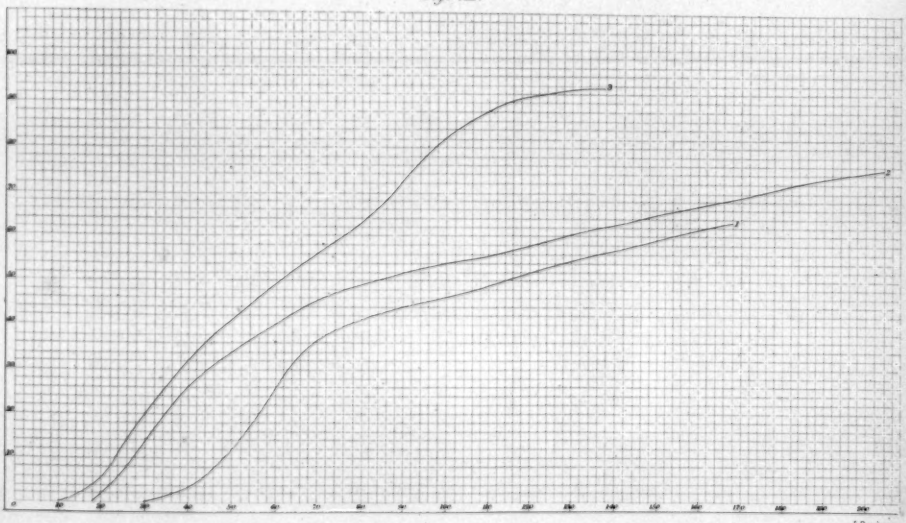
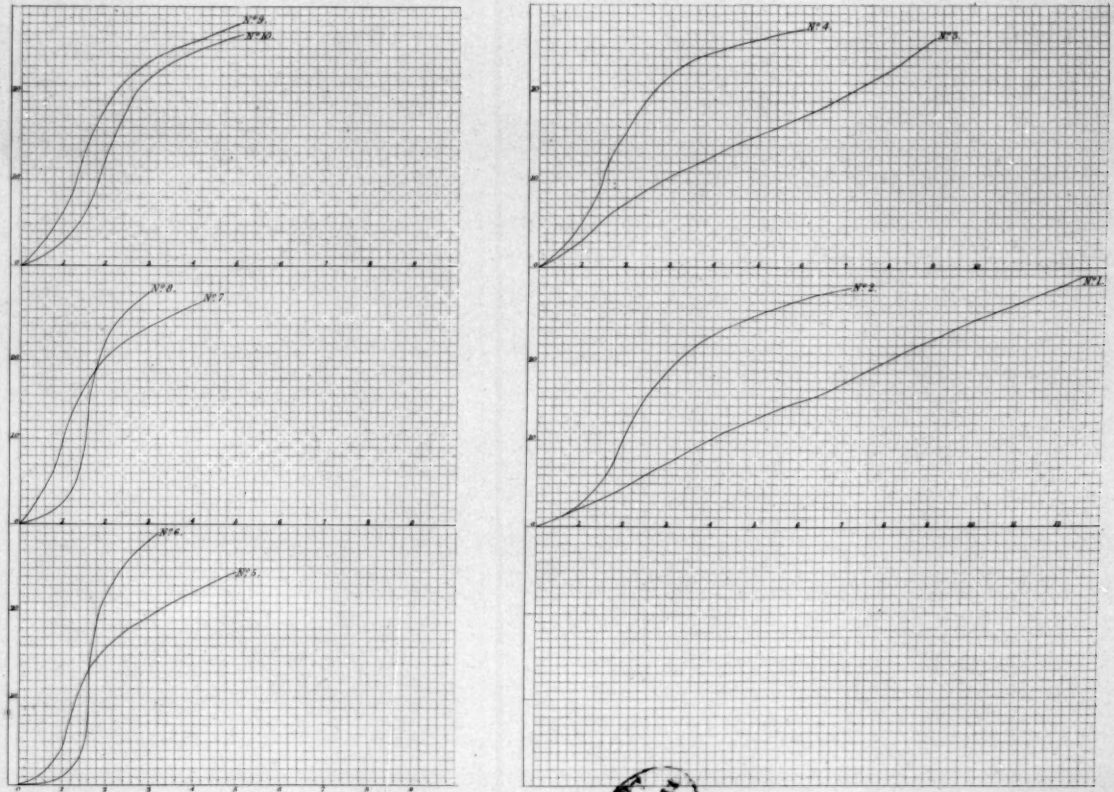


J. Basire, sc.

Fig. VI.



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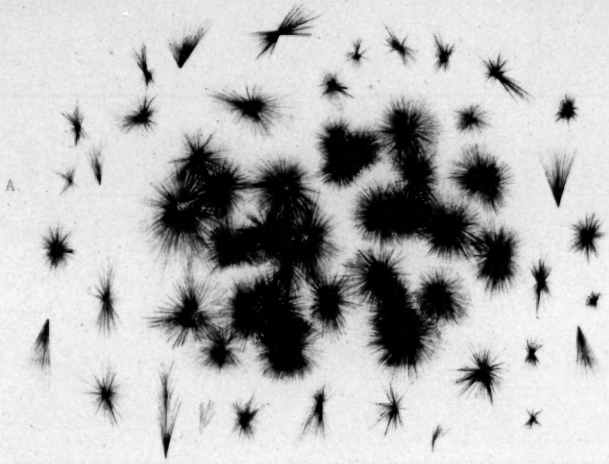

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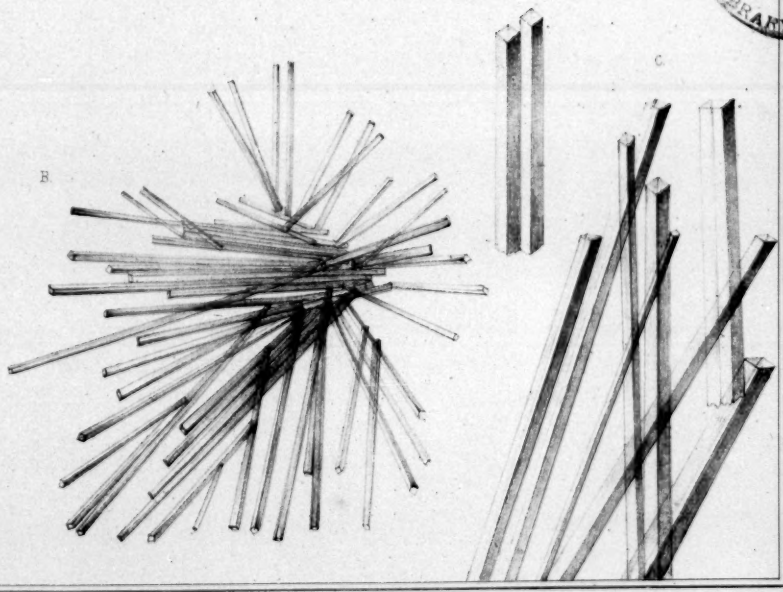

W. S. Shearson, del. ad nat. J. Basire, lithr.

A.—GROUPS OF CRYSTALS OF EXCRETINE SEEN WITH THE NAKED EYE.
B. & C.—GROUPS OF CRYSTALS OF EXCRETINE SEEN UNDER DIFFERENT MAGNIFYING POWERS.

Fig. 1.

Fig. 3.

Fig. 4.

Fig. 5.

Fig. 6.

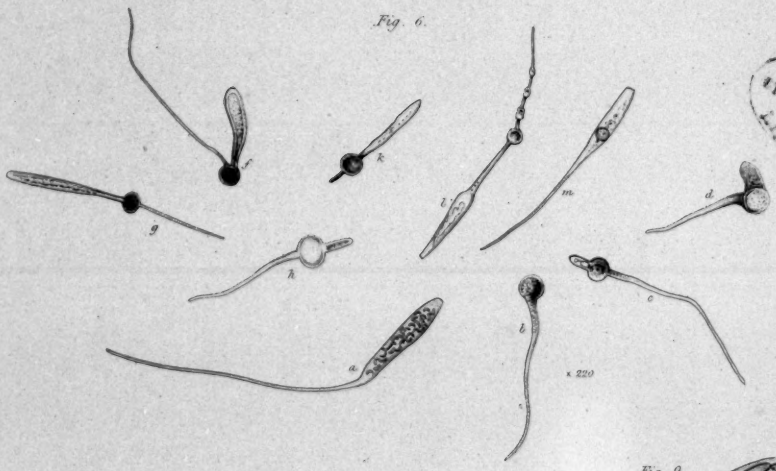

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Fig. 8.

Fig. 7.

Fig. 11.

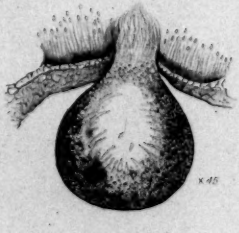

Fig. 9.

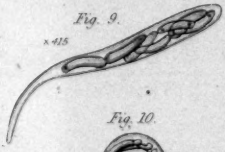

Fig. 10.

F. Durrey. del.

J. Basire. sc.

Fig. 23ª

Fig. 12.

Fig. 23ᵇ

Fig. 14.

Fig. 17.

Fig. 15.

Fig. 16.

Fig. 18.

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Fig. 20.

Fig. 19.

Fig. 21.

Fig. 22.

F. Currey del.ᵗ

J. Bassire, sc.

Fig. 23.

Fig. 26.

Fig. 28.

Fig. 24.

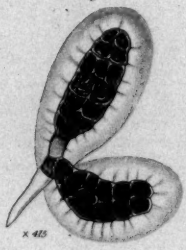

Fig. 27.

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Fig. 29.

Fig. 25.

Fig. 30.

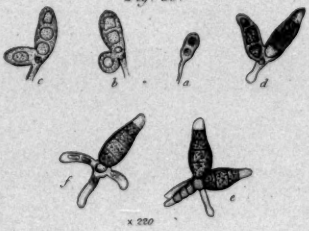
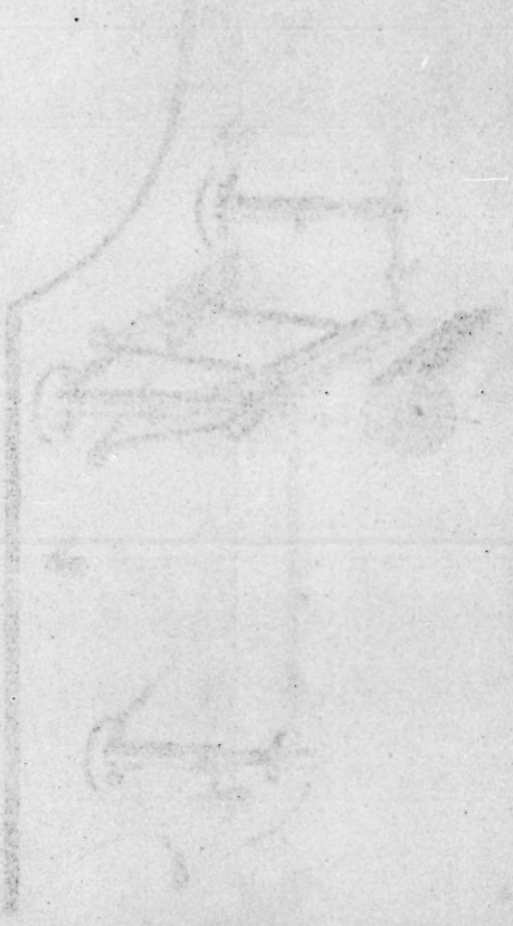
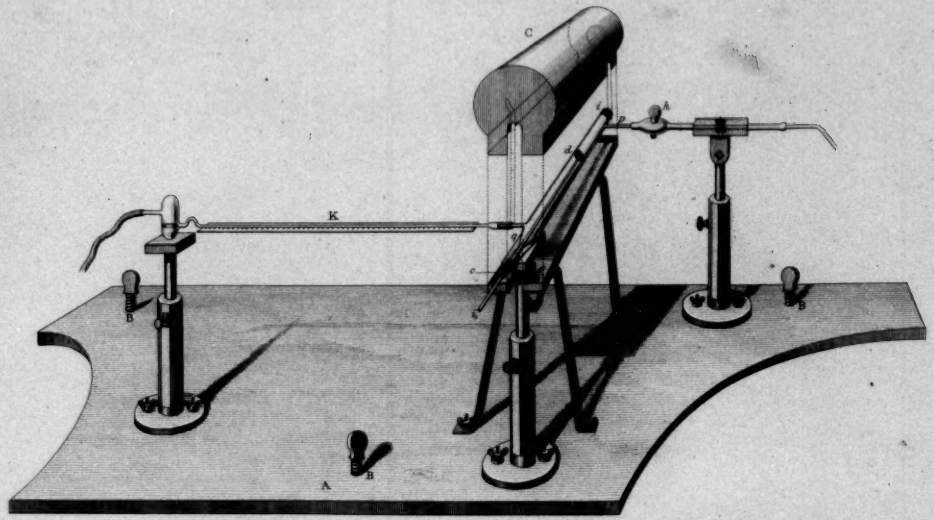


Fig. 31.

F. Carrey, del.

J. Basire, sc.





F.Veith. del.

J.Basire. sc.

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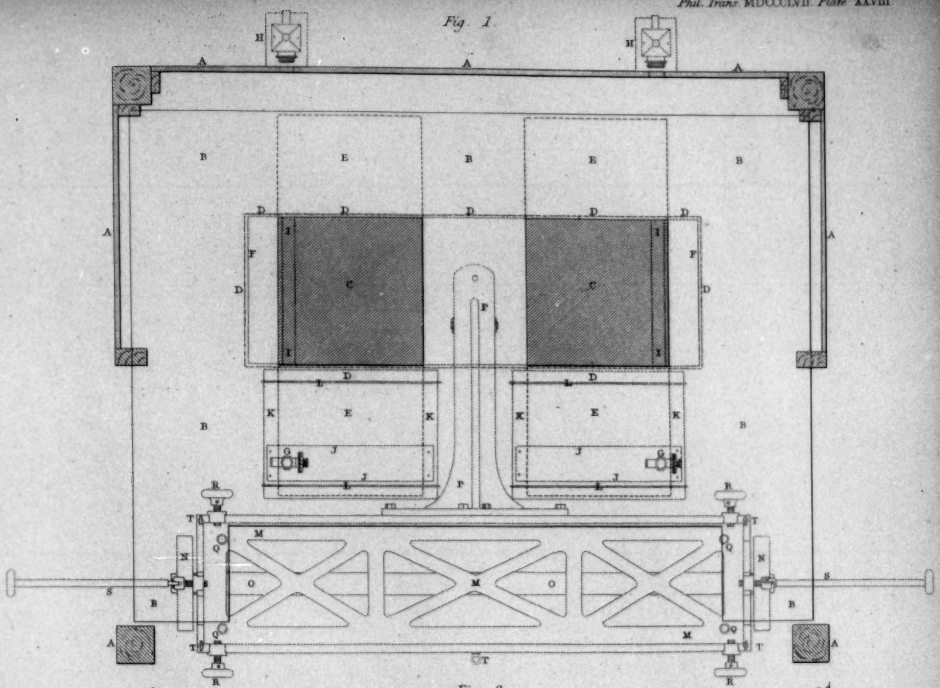
Fig. 1.



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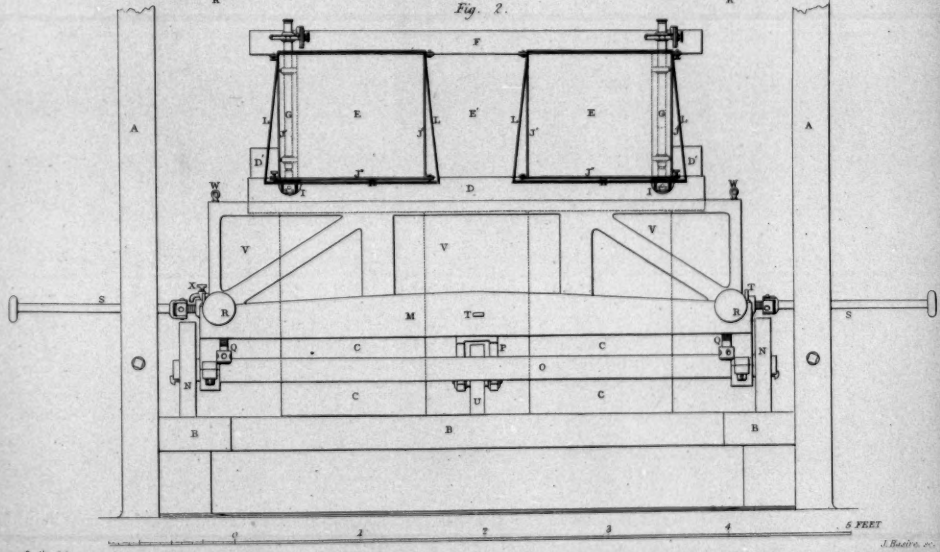

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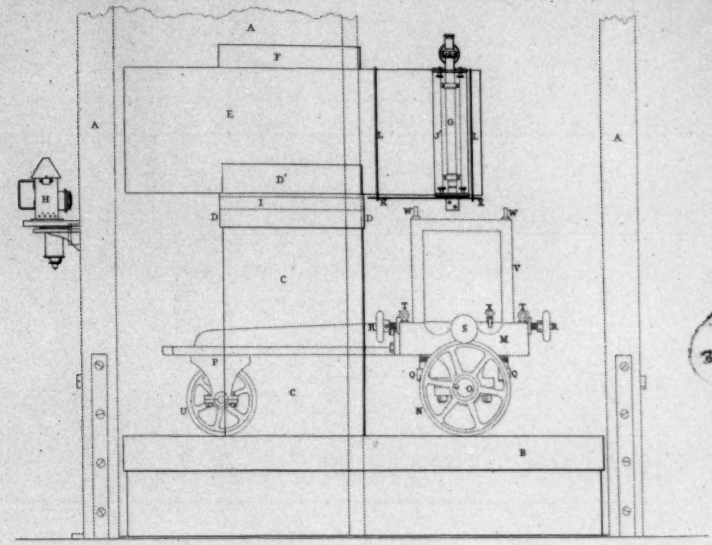

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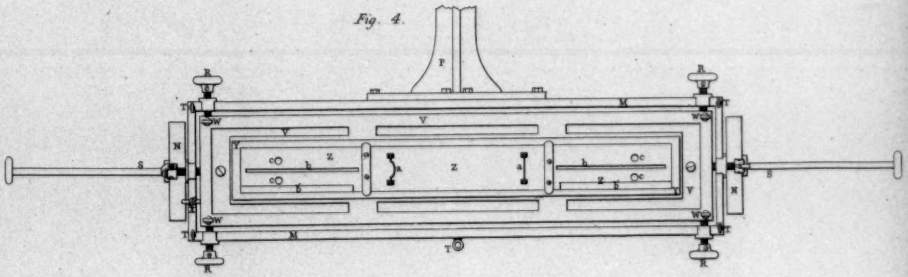

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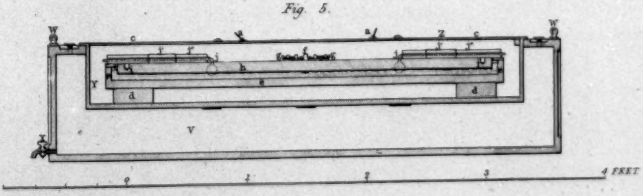

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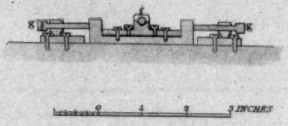

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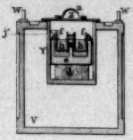

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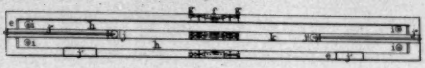
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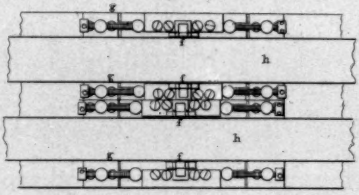

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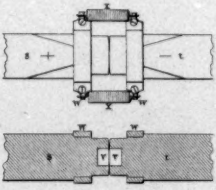
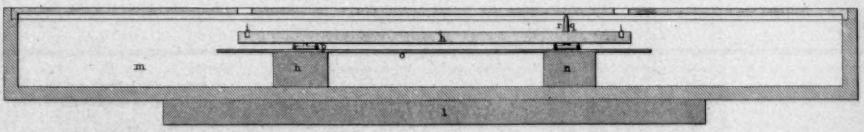


Fig. 14.

Fig. 10.



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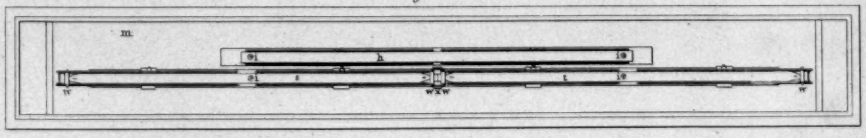

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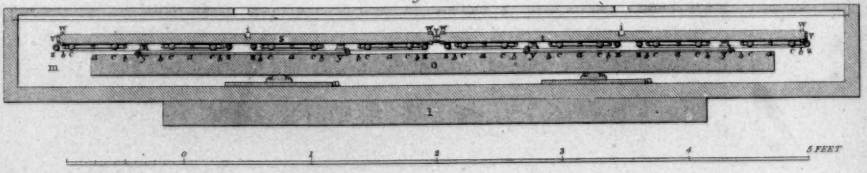
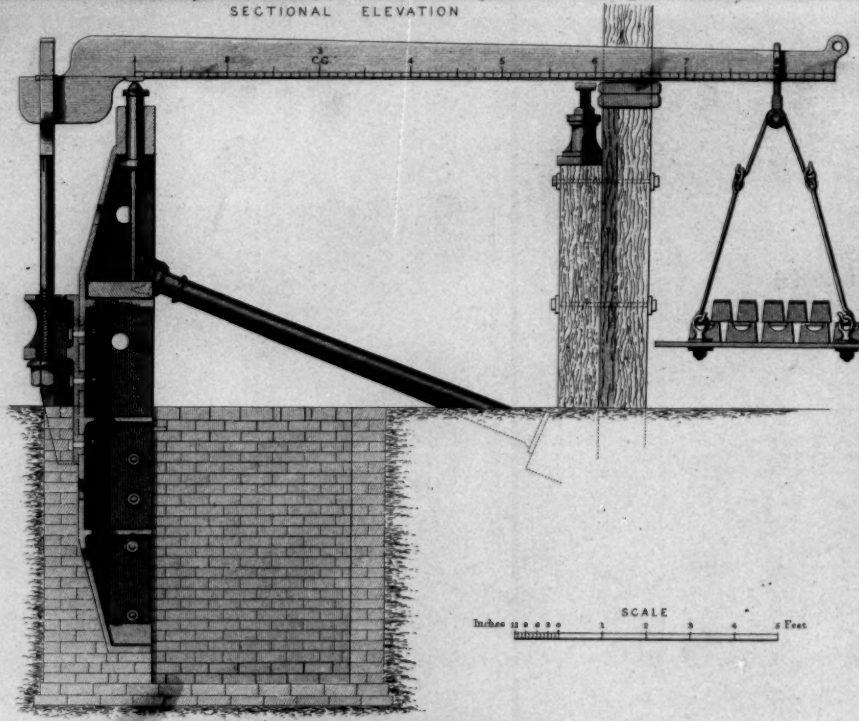
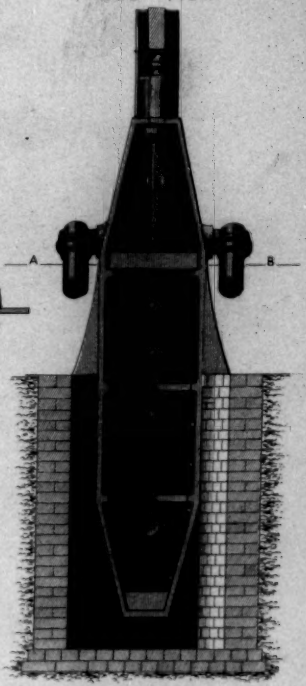


SECTIONAL ELEVATION



END VIEW



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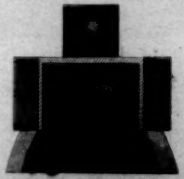
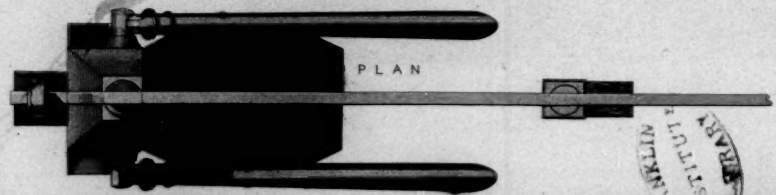


PLAN



APPARATUS USED IN Mr HODGKINSON'S EXPERIMENTS ON THE STRENGTH OF PILLARS.

J. Basire, sc.

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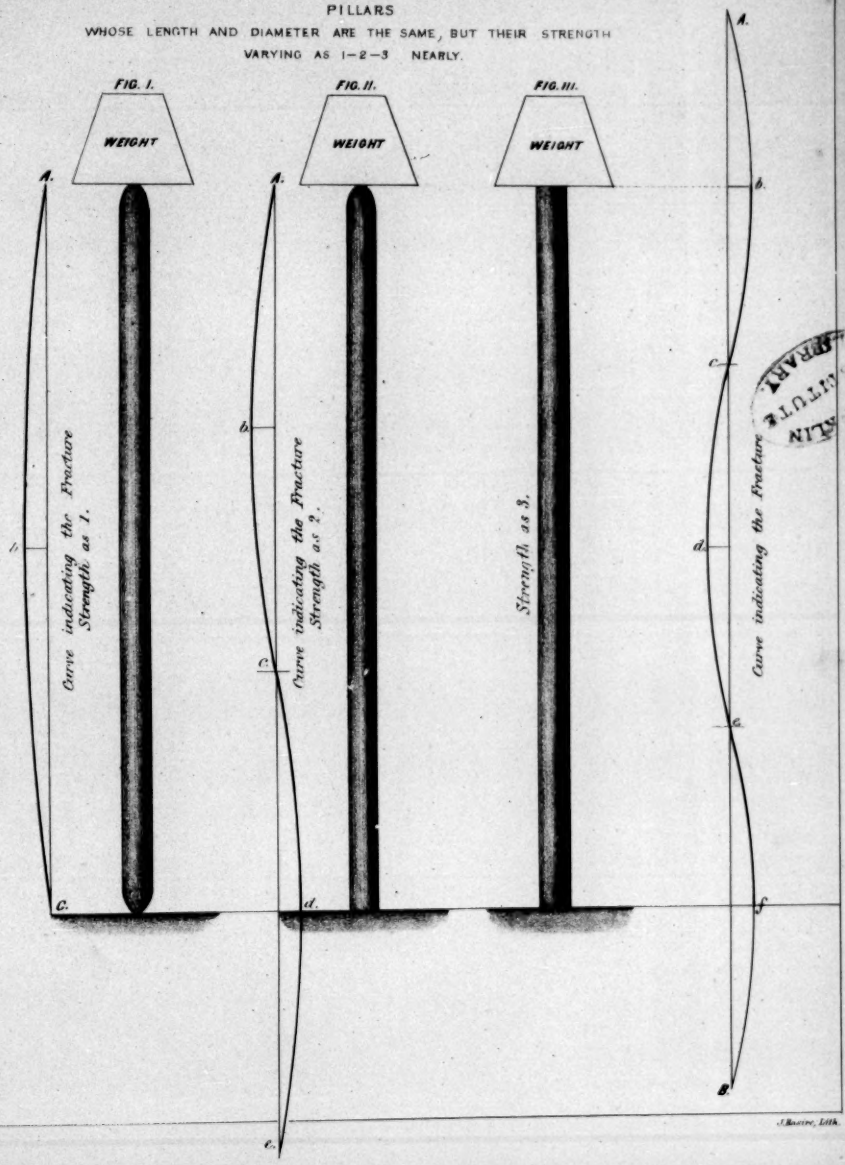
PILLARS

WHOSE LENGTH AND DIAMETER ARE THE SAME, BUT THEIR STRENGTH VARYING AS 1—2—3 NEARLY.



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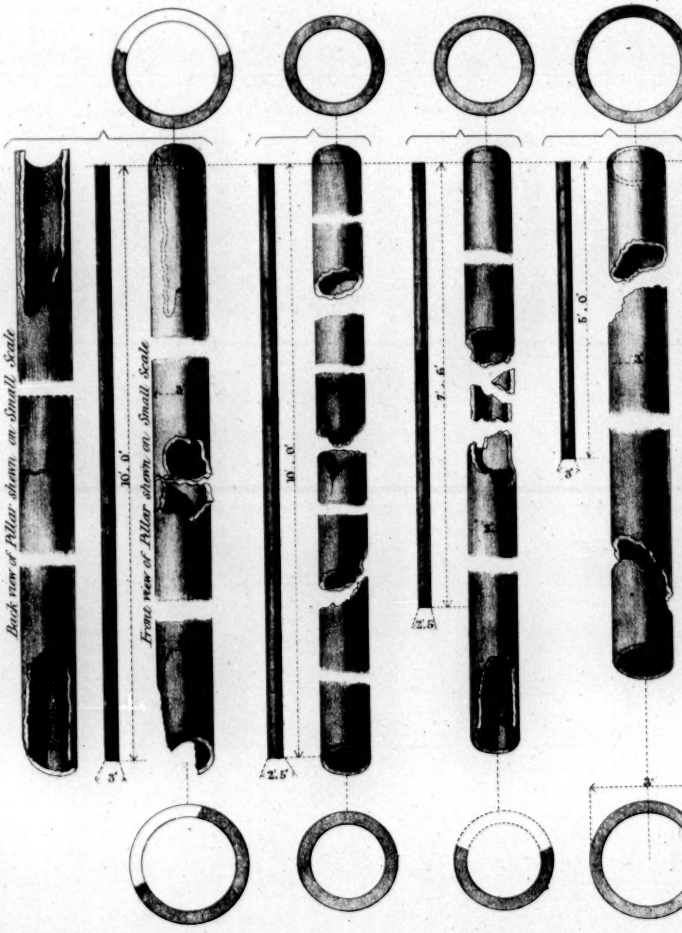
J. Basire, Lith.

HOLLOW PILLARS

TABLE I. TABLE IV.

EXPT 4. EXPTS 1 & 2. EXPT 13.

LOW MOOR, Nº 2.



J. Bazire. Lithog.

S O L I D

TABLE IV.

TABLE V.

EXPT 6.
LONDON MIXTURE.

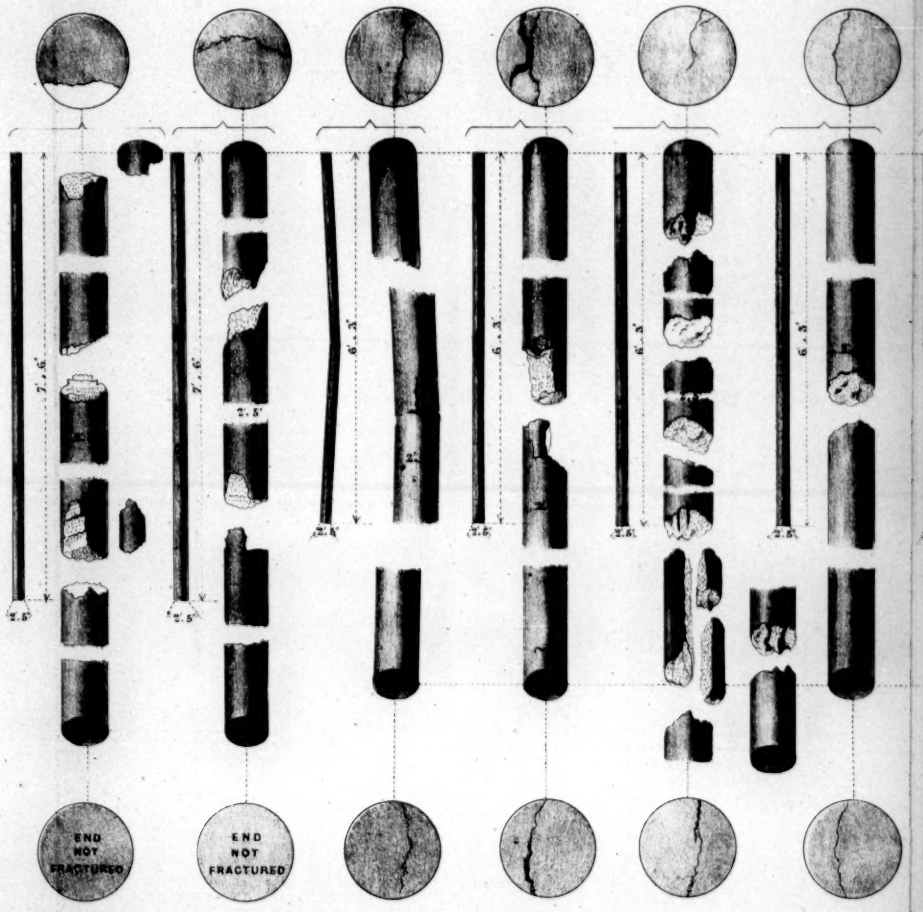
EXPT 2.
OLD PARK.

EXPT 1.
DERWENT Nº1.

EXPT 2.
LONDON MIXTURE.

EXPT 3.
LEVEL Nº1.

EXPT 4.
OLD HILL



E IV.

S O L I D

TABLE V.

EXPT 2.
OLD PARK.

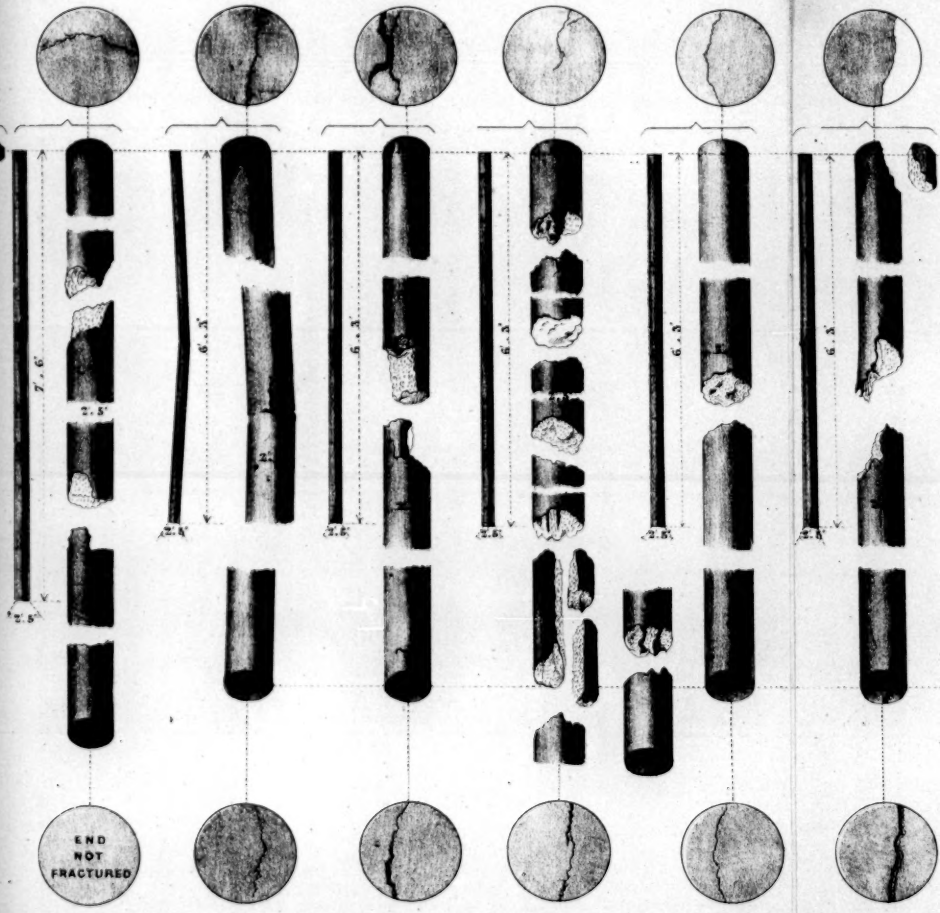
EXPT 1.
DERWENT Nº1.

EXPT 2.
LONDON MIXTURE.

EXPT 3.
LEVEL Nº1

EXPT 4.
OLD HILL

EXPT 5.
SE



PILLARS

TABLE VI.

TABLE V

EXPT 6. SECOND LONDON MIXTURE.

EXPTS 7 TO 9.

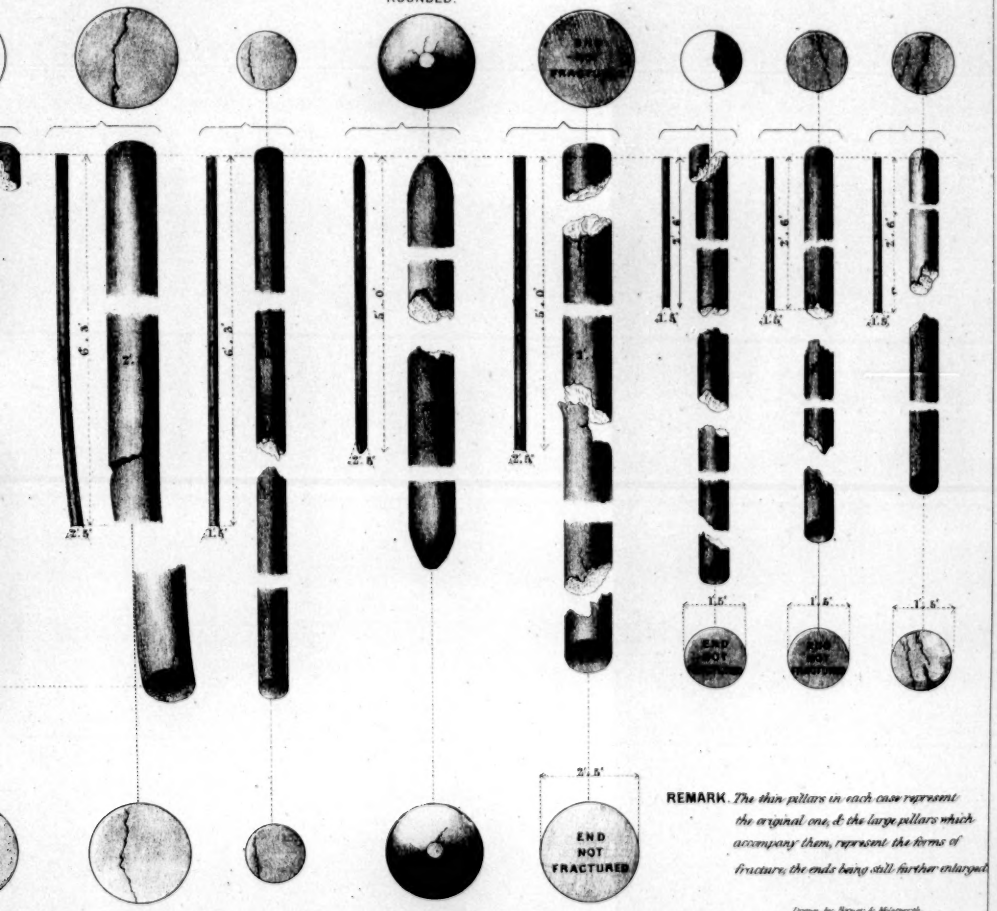
EXPTS 4 TO 6. PILLAR WITH ENDS ROUNDED.

EXPT 1. OLD HILL.

EXPT 13. SECOND LONDON MIXTURE.

EXPT 14.

EXPT 15.



REMARK. *The thin pillars in each case represent the original one, & the large pillars which accompany them, represent the forms of fracture, the ends being still further enlarged.*

Drawn by Turner & Molsworth, 41 Bucklersbury Buildings, Strand.